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A simplified Katsevich algorithm motivated by the distribution properties of k-lines

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Abstract

The Katsevich algorithm is a breakthrough in the theoretically exact algorithms for helical cone beam CT. For future application in medical and industrial CT, determining how to implement it efficiently and accurately is the main task. We analyzed the slope law and intersection law of the k-lines, finding that the k-lines are not intersecting if the half maximal fan angle (HMFA) is less than 21° (numerical solution, so it is approximate) and that the helical pitch and HMFA determine the depth of parallelism of k-lines. Using an appropriate pitch and a HWFA that is less than 21° , one can use a simplified Katsevich algorithm, whose filtration process can be done on the rows of the detector panel so that the pre-weighting, pre-rebinning, post-rebinning and post-weighting steps are all canceled. Simulation experiments show that the simplified algorithm can obtain highly precise images at a faster speed. Our results are intended to be valuable to those who are working on efficient implementations of the Katsevich-type algorithms.

Keywords

simplified algorithm; Katsevich algorithm; k-lines; fan angle; helical cone beam CT

Introduction

Helical cone beam reconstruction is one of the most important subjects in X-ray computed tomography (CT) [6][13]. From 2002 to 2004, Katsevich proposed and improved a FBP-type helical cone beam (HCB) reconstruction algorithm [2], which is named, naturally, the Katsevich algorithm by the CT scientists.

In 2004, Yu Zou and Xiaochuan Pan proposed a new FBP algorithm for HCB CT [16]. The main difference from the Katsevich algorithm is that filtration is by the cone beam projection line of the PI line. In the same year, Yu Zou and Xiaochuan Pan also proposed a more important BPF (back projection filtration) algorithm [15] in which the filtration is by the PI-line.

Subsequently, many researchers have generalized these algorithms or improved them.

In the algorithm development process, there are several main subjects. One is to improve it to adapt variable helical radius or variable pitch or more flexible scanning curves [10][3][11][14]. Another is acceleration techniques [12][7]. Development from 1PI method to 3PI method or fractional-PI method is one as well [1]. And finally, the other subject is concerned with how to remove the artifacts [5].

The theoretical basis of the above algorithms are all based on PI-line. The Katsevich algorithm can be regarded as the classical algorithm that is not only an exact and practical algorithm, but also a key to understanding the other algorithms in the family of exact HCB algorithms.

The Katsevich algorithm is a beautiful but complicated algorithm. Fast and precise implementation is the main goal that must be achieved to meet application demands. There are two important implementation papers in 2003[5] and 2004[9].

Noo [5] proposed implementation steps of the Katsevich algorithm, simplified weighting factor for the back projection process and designed a method to handle the issue of the limits of integration.

Yu and Wang [9] proposed the implementation steps other than the steps of Noo [5], determined the impact of the number of k-lines on reconstruction quality, determined the impact of windows of the unit pulse response of the Hilbert transform on reconstruction quality and determined how to process the endpoints problem of the PI line.

In practice, for the Katsevich algorithm, we find that the k-lines have comparatively high depth of parallelism in most imaging geometries. A naturally idea is whether we can filter the derivative of the projections on the rows of the detector panel other than on the k-lines so as to simplify and speedup the Katsevich algorithm. To investigate the subject, we must research the moving law of k-lines according to the variation of the imaging parameters and conduct experiments to test the simplified algorithm.

In section 1, we give a detailed illustration for the Katsevich formula using charts and explanations. In section 2, we give detailed implementation steps with explanations. In section 3, we research the k-lines' moving law and intersection properties. In section 4, we propose the simplified algorithm and test it using two simulation experiments. In section 5, we give a short conclusion.

1. The Katsevich formula

1.1 Geometry and notions

The Katsevich formula is given without the use of special detector geometry. There are some notions used in the formulae that we will go through here.

(1) Helix—A helix can be decided by the radius R and the pitch h , however we must put it in a fixed 3 dimensional (3D) Cartesian coordinate system (see Fig 1). Then each vertex point can be defined by a 3D space vector.

$$\vec{a}(\lambda) = [R \cos \lambda, R \sin \lambda, \frac{h\lambda}{2\pi}] \quad (1)$$

This is a standard helix whose radius R is a constant, whose pitch h is a constant and has the ray source at the positive x axis when $\lambda = 0$.

(2) Object support—In 3D space, a point of the reconstructed object can be described with a space vector $\vec{r} = (x, y, z)$. Therefore the object function is $f(\vec{r})$, and represents the special distribution of X-ray attenuation coefficients within the object.

For every imaging geometry, there is an object support in which a point can be covered at any scanning angle, whereas outside of the support a point cannot always be covered at any angle.

For helical cone beam reconstruction, the object support is a cylinder whose radius r is less than the radius of the helix.

$$f(\vec{r}) \begin{cases} = 0 & x^2 + y^2 > r^2 \\ \neq 0 & x^2 + y^2 \leq r^2 \end{cases} \quad (2)$$

(3) The cone beam projection of the object—The cone beam projection, also named cone beam transform, is a 2 dimensional (2D) signal, so we can get the signal using a flat detector panel or a curved one. For convenience in developing the mathematical reconstruction formula, a general description was given.

The cone beam projection from the vertex point $\vec{a}(\lambda)$ and by a special direction $\vec{\beta}$ is

$$p(\vec{a}(\lambda), \vec{\beta}) = \int_0^\infty dt f(\vec{a}(\lambda) + t\vec{\beta}) \quad (3)$$

Where $\vec{\beta}$ is a unit direction vector. For a given point $\vec{r}$ and a given rotation angle λ , we can define the direction of the specific X-ray that emitted from the vertex $\vec{a}(\lambda)$ and passes through $\vec{r}$ as

$$\vec{\beta}(\lambda, \vec{r}) = \frac{\vec{r} - \vec{a}(\lambda)}{|\vec{r} - \vec{a}(\lambda)|} \quad (4)$$

(4) PI-line—PI-line takes a very important role during the development of exact HCB CT.

PI-line is a straight line that connects two points $\vec{a}(\lambda_1)$ and $\vec{a}(\lambda_2)$ on the helix, which are in one helical turn, i.e. the two rotation angles satisfy $|\lambda_2 - \lambda_1| < 2\pi$. PI-line segment is the portion of the PI-line within the helix cylinder (See Fig 2).

PI-line has an important property that a point within the helix cylinder belongs to one and only one PI-line.

(5) k-plane and k-line—A k-plane is any plane that has three intersections with the helix, in which one intersection is at the middle of the others. We denote the k-plane as $K(\lambda, \psi)$, which passes through three points: $\vec{a}(\lambda)$, $\vec{a}(\lambda+\psi)$ and $\vec{a}(\lambda+2\psi)$ (see Fig 3).

For a given point $\vec{r}$, there exists a k-plane that is specified by $\lambda \in [\lambda_1 \lambda_2]$ and can pass through the point. But we must note that the k-plane is not unique.

K-line is the intersecting line of the k-plane and the detector panel. If the detector is a flat panel, then the k-line is a straight line.

(6) T-D window—For a flat detector panel, the upper turn and the lower turn are projected to form two curves. The region bounded by the two curves is named the T-D window [4].

It has been proven that the object can be reconstructed exactly just by using the projections within the T-D windows [4].

1.2 The Katsevich formula

According to the notions above, we can get the Katsevich formula

$$f(\vec{r}) = -\frac{1}{2\pi^2} \int_{\lambda_1}^{\lambda_2} d\lambda \frac{1}{|\vec{r} - \vec{a}(\lambda)|} \int_0^{2\pi} \frac{\partial}{\partial q} p(\vec{a}(q), \vec{\Theta}(\lambda, \vec{r}, \gamma)) \Big|_{q=\lambda} \frac{d\gamma}{\sin \gamma} \quad (5)$$

Where

$$\vec{\Theta}(\lambda, \vec{r}, \gamma) = \cos \gamma \cdot \vec{\beta}(\lambda, \vec{r}) + \sin \gamma \cdot \vec{e}(\lambda, \vec{r}) \quad (6)$$

And

$$\vec{e}(\lambda, \vec{r}) = \vec{\beta}(\lambda, \vec{r}) \times \vec{u}(\lambda, \vec{r}) \text{ and } \vec{u}(\lambda, \vec{r}) = \vec{u}(\lambda, \psi(\lambda, \vec{r})) \quad (7)$$

In order to understand the formula clearly, one should know what the four space vectors are and their relations (see Fig 3). There are several illustrations below.

- 1) The 4 vectors $\vec{\Theta}(\lambda, \vec{r}, \gamma)$, $\vec{\beta}(\lambda, \vec{r})$, $\vec{e}(\lambda, \vec{r})$ and $\vec{u}(\lambda, \vec{r})$ are all unit direction vector.
- 2) $\vec{\Theta}(\lambda, \vec{r}, \gamma)$, $\vec{\beta}(\lambda, \vec{r})$, $\vec{e}(\lambda, \vec{r})$ are all in the k-plane.
- 3) $\vec{u}(\lambda, \vec{r})$ is the normal vector of the k-plane.
- 4) If λ and $\vec{r}$ are fixed, then $\vec{\beta}(\lambda, \vec{r})$, $\vec{e}(\lambda, \vec{r})$ and $\vec{u}(\lambda, \vec{r})$ are fixed vectors.
- 5) If λ and $\vec{r}$ are fixed and γ varies from 0 to 2π , then $\vec{\Theta}(\lambda, \vec{r}, \gamma)$ rotates to form a circle in the k-plane, with the source vertex being the rotation center. We can refer to the angle γ as the k-angle and call a point on the k-line a k-point.

Clearly, a k-point corresponds to a k-angle, i. e. $\vec{\Theta}(\lambda, \vec{r}, \gamma)$ can sweep over each k-point in a certain k-line by varying γ from 0 to 2π .

- 6) $\frac{\partial}{\partial q} p(\vec{a}(\lambda), \vec{\Theta}(\lambda, \vec{r}, \gamma))|_{q=\lambda}$ is a one dimensional (1-D) projection signal on the k-line of the detector plane.
- 7) The filtration is a 1-D filtration of the k-line, so it can be computed efficiently.
- 8) The k-plane is not unique, so the k-lines may intersect on the detector. To simplify the filtration, this paper will discuss how to avoid these k-line intersections on the detector panel.

2. The Katsevich algorithm for a flat detector panel

For implementation of the Katsevich algorithm, we should use a special detector and re-express it to be a more practical form named Katsevich algorithm.

There are two main detector forms: curved detector and flat detector panels. In medical CT, the two forms are both often used, whereas the flat detector is the main form in industrial CT. So, we will focus on the flat detector panel.

As shown in Fig 4, we can define

$$\begin{cases} \vec{d}_1 = (-\sin\lambda, \cos\lambda, 0) \\ \vec{d}_2 = (0, 0, 1) \\ \vec{d}_3 = (-\cos\lambda, -\sin\lambda, 0) \end{cases} \quad (8)$$

The three space vectors are all unit direction vectors. They are orthogonal reciprocally and varied according to λ , so we name them rotating orthogonal vectors system.

If we draw a line that passes through the source vertex and is vertical to the z-axis, we get the central X-ray. Putting a flat detector that is vertical to the central X-ray and is at a distance S from source vertex, we get the flat detector. Then we construct a 2D coordinate system uov on the detector whose origin is the intersection of the central X-ray and the detector, with u axis along $\vec{d}_1$ and v axis along $\vec{d}_2$.

To re-express the Katsevich formula in a special form representing the flat detector, the projection formula can be rewritten from the general form to the flat detector form.

The general form $p(\vec{a}(\lambda), \vec{\beta})$ is expressed in (3) and (4). Let $p(\lambda, u, v)$ denote the cone beam projection on the flat detector.

The formulae relating $p(\vec{a}(\lambda), \vec{\beta})$ to $p(\lambda, u, v)$ are

$$u = S \frac{(\vec{r} - \vec{a}(\lambda)) \cdot \vec{d}_1}{(\vec{r} - \vec{a}(\lambda)) \cdot \vec{d}_3} \quad (9)$$

$$v = S \frac{(\vec{r} - \vec{a}(\lambda)) \cdot \vec{d}_2}{(\vec{r} - \vec{a}(\lambda)) \cdot \vec{d}_3} \quad (10)$$

The formula relating $p(\lambda, u, v)$ to $p(\vec{a}(\lambda), \vec{\beta})$ is

$$\vec{\beta} = \frac{1}{\sqrt{u^2 + v^2 + S^2}} (u\vec{d}_1 + v\vec{d}_2 + S\vec{d}_3) \quad (11)$$

Now we can summarize the steps for implementing the Katsevich algorithm.

Step 1: derivative of angle λ .

$$g_1(\lambda, u, v) = \frac{d}{d\lambda} p(\lambda, u, v) = \frac{\partial p}{\partial \lambda} + \frac{\partial p}{\partial u} \cdot \frac{\partial u}{\partial \lambda} + \frac{\partial p}{\partial v} \cdot \frac{\partial v}{\partial \lambda} \quad (12)$$

Where

$$\frac{\partial u}{\partial \lambda} = \frac{S^2 + u^2}{S} \text{ and } \frac{\partial v}{\partial \lambda} = \frac{uv}{S} \quad (13)$$

Step 2: pre-weighting

$$g_2(\lambda, u, v) = g_1(\lambda, u, v) \frac{S}{\sqrt{u^2 + v^2 + S^2}} \quad (14)$$

Step 3: pre-rebinning

$$g_3(\lambda, u, \psi) = g_2(\lambda, u, v_k(u, \psi)) \quad (15)$$

Where

$$v_k(u, \psi) = \frac{Sh}{2\pi R} \left(\psi + \frac{\psi}{\tan \psi} \frac{u}{S} \right) \text{ and } \psi \in \left[-\frac{\pi}{2} - \alpha_{max}, \frac{\pi}{2} + \alpha_{max} \right] \quad (16)$$

Where

$$\alpha_{max} = \arcsin \frac{r}{R} \quad (17)$$

Clearly, α_{max} is the half maximal fan angle (HMFA). Fixing ψ , we can see that the function $v_k(u, \psi)$ is the equation for a straight line. Varying ψ in the domain of definition (see (16)), we can see a family of k-lines, which cover a parallel quadrangle region.

Step 4: Filtration of the k-lines

$$g_4(\lambda, u, \psi) = g_3(\lambda, u, \psi) * \frac{1}{\pi u} \quad (18)$$

Fixing λ and ψ , we can see $g_3(\lambda, u, \psi)$ is a 1D projection signal with respect to u . Naturally, $g_4(\lambda, u, \psi)$ is the Hilbert transform of the signal $g_3(\lambda, u, \psi)$.

Step 5: post-rebinning

$$g_5(\lambda, u, v) = g_4(\lambda, u, \psi_k(u, v)) \quad (19)$$

$\psi_k(u, v)$ is the inverse function of $v_k(u, \psi)$. Clearly, $\psi_k(u, v)$ is not a rigorous function for there may be more than one value for a certain (u, v) . When there is more than one $\psi_k(u, v)$, we can use the value whose absolute value is the smallest in the values.

Step 6: post-weighting

$$g(\lambda, u, v) = \frac{\sqrt{u^2 + v^2 + S^2}}{S} g_5(\lambda, u, v) \quad (20)$$

This is the filtered projection.

Step 7: back projection

$$f(\vec{r}) = \frac{1}{2\pi} \int_{\lambda_1}^{\lambda_2} d\lambda \frac{1}{|\vec{r} - \vec{a}(\lambda)|} g(\lambda, u, v) \quad (21)$$

The most important property is that the integral limits are varying according to the given point $\vec{r}$. The relationship between $\vec{r}$ and λ_1 and λ_2 is given by:

$$x = R[(1-t)\cos\lambda_1 + t\cos\lambda_2]$$

$$y = R[(1-t)\sin\lambda_1 + t\sin\lambda_2] \quad (22)$$

$$z = \frac{h}{2\pi} [(1-t)\lambda_1 + t\lambda_2]$$

Clearly, it is easy to compute (x, y, z) from $(\lambda_1, \lambda_2, t)$. But it is more complicated to compute $(\lambda_1, \lambda_2, t)$ from (x, y, z) , for there is no analytical expression. So we must use numerical methods to compute them.

3. On k-lines

K-lines play a very important role in the Katsevich algorithm. Now we will discuss how we can avoid the intersections of k-lines and whether or not we can filter the projection derivative on the rows of the detector panel.

3.1 On slope of k-lines

The k-line equation can be re-written as

$$v = ku + b = \frac{\psi}{\tan\psi} \frac{h}{2\pi R} u + \frac{Sh}{2\pi R} \psi \quad \psi \in \left[-\frac{\pi}{2} - \alpha_{max}, \frac{\pi}{2} + \alpha_{max}\right] \quad (23)$$

k is the slope of the straight line and b is the intercept on the v axis. In order to determine the depth of parallelism of k-lines, we will analyze how k varies with different selections of parameters.

Firstly, we fix h and R and vary ψ to see the variation of k , i.e. to research the lean law of the k-lines.

Without loss of generality, we draw a family of k-lines (see Fig 5) and the wave of function

$\frac{\psi}{\tan\psi}$ (see Fig 6) with the parameters $[R, r, h, S] = [20, 4.85, 3, 30]$.

From Fig 5 and Fig 6, we can get the following properties.

- 1) $\frac{\psi}{\tan\psi}$ is an even function, with a maximal value at $\psi = 0$.
- 2) The area covered by the k-lines is a parallel quadrangle.
- 3) From 0 to $\frac{\pi}{2} + \alpha_{max}$, the lean law of k-lines states that the k-line is an increasing function in $\left[0, \frac{\pi}{2}\right)$, is a decreasing function in $\left(\frac{\pi}{2}, \frac{\pi}{2} + \alpha_{max}\right]$, and is a constant function at $\psi = \frac{\pi}{2}$.
- 4) The k-line is parallel to the u axis at $\psi = \frac{\pi}{2}$. The k-lines are more and more slanted with ψ varying from $\frac{\pi}{2}$ to $\frac{\pi}{2} + \alpha_{max}$ or from $\frac{\pi}{2}$ to 0.
- 5) The lean law of k-lines for $\psi \in \left[-\frac{\pi}{2} - \alpha_{max}, 0\right]$ is symmetrical to that for $\psi \in \left[0, \frac{\pi}{2} + \alpha_{max}\right]$.

Secondly, we fix λ and vary h and R . It is easy to see that the k-lines become closer to parallel to the u axis as R enlarges and h decreases. (see Fig 7, which is obtained using the parameters of $[R, r, h, S] = [40, 4.96, 1, 50]$).

The properties suggest that making the k-lines have the smallest slope so as to filter the projection on the detector rows can replace the filtration on k-lines. In other words, one should:

- 1) Make R the maximal value
- 2) Make h the minimum value
- 3) Make r the minimum value

3.2 On intersection of k-lines

From Fig 5 and Fig 7, we see that the k-lines are not intersecting, but one cannot assume that they are always this way (see Fig 8, Fig 9 and Fig 10)

Fig 8 is obtained by fixing R , r , and S and varying h .

It can be seen that selecting different values of h cannot avoid this intersection. The three k-line distribution charts are very similar.

Fig 9 is obtained by fixing h , S and the width of the detector u_m and varying R .

It can be seen that there are many intersections with $R = 20$. There are less intersections with $R = 23$. When $R = 30$ there are no intersections in the detector panel.

It should be noted that u_m and r have both been decided based on the horizontal size of the object support, i. e. r and u_m should be big enough to cover the object. The relation between u_m and r is

$$\alpha_{max} = \sin^{-1} \frac{r}{R} = \tan^{-1} \frac{u_m}{S} \quad u_m = S \tan \alpha_{max} = S \tan \sin^{-1} \frac{r}{R} \quad (24)$$

So, varying R and fixing u_m is similar to varying R and fixing r for they both mean varying HMFA.

Fig 9 illustrates that decreasing HMFA to a certain value can avoid the intersection of k-lines.

Fig 10 is obtained by fixing R , S , and h and varying r .

It can be seen that there are many intersections with $r = 8.944$. There are less intersections with $r = 7.8$. When $r = 6.6$, there are no intersections in the detector panel.

Fig 10 illustrates that decreasing HMFA to a certain value can avoid the intersection of k-lines.

From these three numerical analyses, we have determined that the intersection issue of k-lines is closely associated with HMFA.

Going through Fig 8 to Fig 10 to estimate the intersections' locations, it can be found that the first intersection from 0 to positive infinity on the u axis is always the intersection of the

k-line of $\psi = \frac{\pi}{2} + \alpha_{max}$ and its nearest k-line. If the u axis coordinate of the intersection is greater than u_m , there is not intersection on the detector panel.

Suppose that the number of the k-lines is $N_k + 1$, then the interval of ψ is $\Delta = \frac{\pi + 2\alpha_{max}}{N_k}$.

The equation of the k-line of $\psi = \frac{\pi}{2} + \alpha_{max}$ is

$$v = \frac{\frac{\pi}{2} + \alpha_{max}}{\tan(\frac{\pi}{2} + \alpha_{max})} \frac{h}{2\pi R} u + \frac{Sh}{2\pi R} (\frac{\pi}{2} + \alpha_{max}) \quad (25)$$

The equation of the k-line of $\psi = \frac{\pi}{2} + \alpha_{max} - \Delta$ is

$$v = \frac{\frac{\pi}{2} + \alpha_{max} - \Delta}{\tan(\frac{\pi}{2} + \alpha_{max} - \Delta)} \frac{h}{2\pi R} u + \frac{Sh}{2\pi R} (\frac{\pi}{2} + \alpha_{max} - \Delta) \quad (26)$$

Using equations (26) and (25), one obtains

$$u = \frac{S\Delta \tan(\frac{\pi}{2} + \alpha_{max} - \Delta) \tan(\frac{\pi}{2} + \alpha_{max})}{(\frac{\pi}{2} + \alpha_{max} - \Delta) \tan(\frac{\pi}{2} + \alpha_{max}) - (\frac{\pi}{2} + \alpha_{max}) \tan(\frac{\pi}{2} + \alpha_{max} - \Delta)} \quad (27)$$

Clearly, u in (27) denotes the u axis coordinate of the intersection of the two k-lines given by equations (25) and (26) respectively.

It can be seen from Fig 11 that the function (27) is a decreasing function with respect to α_{max} , which implies a basic law that the easier the k-lines intersect at a lower u , the bigger α_{max} is.

A natural idea is that α_{max} cannot be too big. However, in medical CT and industrial CT, a very small α_{max} is not practical. Therefore, it is important to search for a critical α_{max} so as to avoid intersection on the detector panel.

Recalling that the width of the detector u_m is related to α_{max} in (24), we get an equation set (28).

$$\begin{cases} u = S \tan \alpha_{max} \\ u = \frac{S\Delta \tan(\frac{\pi}{2} + \alpha_{max} - \Delta) \tan(\frac{\pi}{2} + \alpha_{max})}{(\frac{\pi}{2} + \alpha_{max} - \Delta) \tan(\frac{\pi}{2} + \alpha_{max}) - (\frac{\pi}{2} + \alpha_{max}) \tan(\frac{\pi}{2} + \alpha_{max} - \Delta)} \end{cases} \quad (28 \text{ a}) \quad (28 \text{ b})$$

Solving the equation by fixing N_k , we can get an α_{max} , which is the critical value. But it is difficult to calculate the root analytically. So, we solve it using the numerical method, i.e. searching the intersection of the two curves of the two equations in (28).

From Fig 11, we can see that the root is 22.92° , with $N_k = 20$. If HMFA is less than 22.92° , there is no intersection in detector panel. Oppositely, there will be intersections on the panel if you select a HMFA, that is greater than 22.92° .

We can use the parameters of Fig 8,9, 10 to verify the law stating that their N_k values are all 20 (see Table 1).

From the parameters in Fig 10 (2), it can be seen that HMFA is greater than but very nearly the critical value. According to the intersection law that we have summarized, the k-lines should have intersections but they should be located at the edge of detector, which is consistent with the distribution status of the k-lines in Fig 10 (2).

Now, we can arrive at the conclusion that HMFA determines the intersection of k-lines and that N_k determines the critical value of HMFA.

Using some special N_k for equation set (28) and using numerical method to solve this equation set, we get Table 2 and Fig 12.

From Table 2, it can be seen that the bigger N_k is, the smaller the critical angle of HMFA is. However, the rate of descent of the critical angle of HMFA is smaller and smaller (see Fig 12). 21° can be approximately regarded as the limit of N_k going to infinity.

Generally speaking, there are compromises to consider when selecting N_k . If N_k is too big, then HMFA is small, which means the cone beam cannot cover a large object. On the other hand, if N_k is too small, the interpolation error will become bigger in the pre-rebinning process and post-rebinning process.

To avoid intersections in the detector panel, selecting an appropriate HMFA is a key issue, i. e. we should select an appropriate R and r . According to our derivation, one can basically ensure the avoidance of intersection of k-lines with a HMFA that is less than 21° .

Further, we use the intersection law to explain the k-lines distribution cases in two important papers.

In Katsevich's paper [2], there is an intersection in Fig 9, which is obtained using $\frac{r}{R}=0.7$, i. e. the HMFA is 44.427° . Since the HMFA is greater than 21° , the k-lines are intersecting.

In Noo's paper [5], there is an intersection in Fig 4, which is obtained using $\frac{r}{R}=\frac{25}{57}=0.4386$, i.e. the HMFA is 26.0146° . Since the HMFA is greater than 21° , the k-lines are intersecting.

4. A simplified Katsevich algorithm and simulation experiments

According to the analyses of section 3.1 and 3.2, it is clear that one can get a good k-lines distribution chart on the detector panel by selecting appropriate R , r and h .

If the k-lines are not intersecting and adequately parallel, the Katsevich algorithm can be simplified.

The simplified Katsevich algorithm differs from the regular Katsevich algorithm in the filtration steps. The pre-weighting step, the pre-rebinning step, the post-rebinning step, and the post-weighting step can all be canceled. The filtration on k-lines will be replaced by filtration on rows of the detector.

Two simulation experiments were conducted to test the simplified algorithm. The parameters are in Table 3.

The scanned object is made up of 5 solid spheres, each of which has a special density and location. The parameters are in Table 4.

Fig 13 shows the k-lines distribution of the two experiments. It can be seen that the k-lines are not intersecting and have a good depth of parallelism, so we can use the simplified algorithm.

Fig 14 (left) is the image slice reconstructed using the Katsevich algorithm and the right one is the same slice reconstructed using the simplified algorithm. For a quantitative comparison, we use the data cursor to show five points' values, each of which belongs to a different sphere. From the values, we can see that the Katsevich algorithm is only slightly better than the simplified Katsevich algorithm.

From Fig 15, there is no distinguishable difference between the two reconstructed slices.

From Fig 16, the difference between the profiles at $z=0$ and $y=2$ are nearly indistinguishable. The results show that the simplified algorithm can produce good quality images if the k-lines are not intersecting and have good depth of parallelism so that the simplified algorithm can be performed relatively fast.

By the way, the CT images have some concentric circles artifacts which are related to geometry and can be eliminated by using low-pass filtering or average filtering on the image rows. For it belongs to image post-processing process and to avoid the readers misunderstanding, we just show the initial CT images.

5 conclusions

The Katsevich algorithm is a classical method for helical cone beam CT reconstruction. In order to develop a more practical CT system, the main task is determining how to implement the Katsevich algorithm efficiently and precisely and how to improve it to adapt to a variety of scanning geometries.

In this paper, we researched the k-lines distribution laws, one of which is the slope issue of the k-lines and the other one of which is the intersection issue. Also, we designed a simplified Katsevich algorithm in the case of k-lines having a good parallelism.

21° is the critical value of HMFA. If HMFA is greater than 21° , there may have intersections on the detector panel. If HMFA is less than 21° , there are not intersections on the detector panel. It is easy to design the Katsevich algorithm if the k-lines are not intersecting.

It should be mentioned that the conclusion is consistent with that in [17], in which $\frac{r}{R}=0.358$ is suggested, i.e. $\alpha_{max}=\arcsin\frac{r}{R}=20.9774^\circ$ is suggested. It should be emphasized that the two methods to research the k-lines intersection issue are different. Our method is using the equations of k-lines at $\psi=\frac{\pi}{2}+\alpha_{max}$ and $\psi=\frac{\pi}{2}+\alpha_{max}-\Delta$ to search the intersection, however the method in [17] uses a different equation set about $\frac{r}{R}$ and v_1 to calculate the critical ratio of $\frac{r}{R}$.

In fact, we do not need so small an HMFA less than 21° to avoid the intersections. Note, that the bigger N_k is, the smaller the critical angle of HMFA is, i.e. a bigger HMFA is available if we do not need too many k-lines. For example, 24.7° is available in the case of just needing 11 k-lines to cover the detector.

If the k-lines are not intersecting and have good depth of parallelism, we can implement the filtration process on the rows of the detector so that the Katsevich algorithm reconstruction can be accelerated for the pre-weighting, pre-rebinning, post-rebinning and post-weighting steps are all canceled.

Our conclusions are useful supplements to the knowledge hierarchy of the Katsevich-type algorithms.

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Fig 1.
Illustration of a helical cone beam geometry.

Fig 2.
Illustration for PI line.

Fig 3.
Illustration for k-plane, k-line and four unit direction vectors.

Fig 4.
Illustration of the three rotation vectors and the flat detector coordinate system.

Fig 5.
Illustration for the distribution of k-lines.

Fig 6.

The wave of the function $\frac{\psi}{\tan\psi}$.

Fig 7. Illustration for the distribution of k-lines, observed that the k-lines are nearly parallel to the rows of the detector.

Fig 8.
Illustration of the impact of h on the intersection of k-lines.

Fig 9.
Illustration of the impact of R on the intersection of k -lines.

Fig 10.
Illustration of the impact of r on the intersection of k-lines.

Fig 11.
Illustration for searching the critical value numerically.

Fig 12.
The relation of critical angle of α_{max} and N_k .

Fig 13.
K-lines distribution in the simulation experiments.

Fig 14.
Reconstructed image of slice $z = 0$. The left one is obtained using the Katsevich algorithm and the right one is obtained using the simplified algorithm.

Fig 15.

Comparison of phatom and reconstruction results at the slice $z = 0$. The left one is the phatom. The middle one is the rusult by using the Katsevich algorithm. The right one is the result by using the simplified algorithm.

Fig 16.

Comparison of the profiles of the three images in Fig 15 at $z=0$ and $y=2$.

Table 1

Verifying that 22.92° is the critical value of HMFA, with $N_k = 20$

	r	R	HMFA (degree)	Intersecting?
Fig 8 (1)	8.944	20	26.56(>22.92)	Y
Fig 8 (2)	8.944	20	26.56(>22.92)	Y
Fig 8 (3)	8.944	20	26.56(>22.92)	Y
Fig 9 (1)	8.944	20	26.56(>22.92)	Y
Fig 9 (2)	9.17	25	23.50(>22.92)	Y
Fig 9 (3)	9.49	30	18.44(<22.92)	N
Fig 10 (1)	8.944	20	26.56(>22.92)	Y
Fig 10 (2)	7.8	20	22.95(>22.92)	Y
Fig 10 (3)	6.6	20	19.26(<22.92)	N

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Table 2

The relation of critical angle of HMFA and N_k

N_k	10	20	30	40	50	60	70	80	100000
Critical angle of HMFA	24.7°	22.92°	22.27°	21.79°	21.72°	21.62°	21.55°	21.43°	21.01°

Table 3
Cone beam imaging parameters for the Katsevich algorithm and the simplified algorithm

	Algorithm	R	h	S	Detector matrix	u	v	Scanning range	λ	Number of k-lines
Experiment 1	Katsevich	20	3	30	201×39	0.075	0.075	Two turn	0.5°	39
Experiment 2	Simplified	20	3	30	201×39	0.075	0.075	Two turn	0.5°	No

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Table 4

The parameters of the scanned object

	Ball 1	Ball 2	Ball 3	Ball 4	Ball 5
Location of ball center	(0,0,0)	(-2,2,0)	(2,2,0)	(0,-2,0)	(0,0,0)
Radius of ball	4	1	1	0.5	0.8
Density of ball	0.8	0.4	0.6	0.2	0.1