Characterization of Fourier Transforms of Hardy Spaces

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ABSTRACT Characterizations of Fourier transforms of boundary distributions of functions in $H^p(\mathbb{R})$ or $H^p(\mathbb{T})$, $0 < p \leq 1$, are given. These results are applied to obtain Fourier multiplier theorems on H^p .

The purpose of this note is to describe a characterization of distributions arising as Fourier transforms of boundary distributions of functions in H^p of the upper half plane, $0 < p \leq 1$.

We will say that an entire function $A(z)$ of exponential type σ is p -normalized if its restriction to the real line is in $L^2(\mathbb{R})$,

$$\int_{-\infty}^{\infty} |A(x)|^2 dx \leq \sigma^{(1-2/p)}$$

and A , as well as all its derivatives up to order $\left[\frac{1}{p}\right] - 1$, vanish at $z = 0$. We now can state:

THEOREM I. $\hat{f}$ is the Fourier transform of the boundary distribution of an H^p function if and only if it can be represented (in the distribution sense) as

$$\hat{f} = \sum_{i=1}^{\infty} \lambda_i e^{i\alpha_i x} A_i(x),$$

where α_i are real, A_i are p -normalized functions of exponential type

$$\lambda_i > 0 \text{ and } \sum_{i=1}^{\infty} \lambda_i^p < \infty.$$

This statement is a translation, via the Paley-Wiener theorem, of the characterization obtained in ref. 1 of boundary distributions of H^p .

We would like to illustrate the use of *Theorem I* by obtaining certain multiplier theorems. For simplicity we restrict our attention to the case $1 \geq p > 1/2$:

THEOREM II. Let $m(s)$ be bounded and satisfy

$$(a) \quad |m(s+u) + m(s-u) - 2m(s)| \leq \left| \frac{u}{s} \right|^\alpha |s|^\delta \text{ for } |u| < \frac{|s|}{2}$$

$$(b) \quad |m(s)| \leq C|s|^{-\beta}, \beta \geq \frac{\delta}{2\alpha + 1 - 2/p},$$

where

$$\frac{3}{2} > (\alpha - \delta) > \frac{1}{p} - \frac{1}{2}, \delta \geq 0;$$

then $(m(s)\hat{f})^\vee$ is a bounded operator on $H^p(\mathbb{R})$ $1 \geq p > 1/2$.

COROLLARY: For $1/2 < p < \infty$ let $m(s)$ be bounded and satisfy

$$|m(s-u) - m(s)| \leq \left| \frac{u}{s} \right|^\alpha, |u| < \frac{|s|}{2} \text{ with } \alpha > \left| \frac{1}{p} - \frac{1}{2} \right|,$$

then $(m(s)\hat{f})^\vee$ is bounded on H^p .

For $p \leq 1$ the corollary is a restatement of *Theorem II*; for $p > 1$ the former can be obtained by approximating m by multipliers satisfying the condition corresponding to $p = 1$ and then using the Riesz-Thorin interpolation theorem. Other results can be obtained by interpolation as in ref. 2.

The proof of *Theorem II* is based on the following easy estimate for p -normalized functions of type σ .

$$(*) \quad \int_{-\infty}^{\infty} |A(x)|^2 \frac{dx}{|x|^\delta} \leq \sigma^{1-2/p+\delta}.$$

We need only show that for such a function, $m(s)A(s)$ admits a decomposition as in *Theorem I*. This is achieved by approximation $m(s)$ by entire functions in the following way:

$$\text{Fix } 0 < \gamma \leq 1 \text{ and write } 1 = \sum_{n \leq m_0} \varphi(2^{n\gamma}x) \quad x \neq 0$$

for some even C^∞ function φ which vanishes near 0 and ∞ . Then letting

$$\eta(x) = \sum_{n \geq 0} \varphi(2^{n\gamma}x),$$

we obtain

$$m(s) = 2^{-m_0\gamma} \int \hat{\eta}(2^{-m_0\gamma}(s-t))m(t)dt + \sum_{n < m_0} 2^{-n\gamma} \int \hat{\varphi}(2^{-n\gamma}(s-t))m(t)dt.$$

Here we choose m_0 so that $2^{-m_0} \leq \sigma < 2^{-m_0+1}$ and

$$\gamma = \frac{2(\alpha - \delta) + 1 - 2/p}{2\alpha + 1 - 2/p} \text{ for } m_0 \geq 0, \gamma = 1 \text{ if } m_0 < 0.$$

It is clear that the terms in the sum are entire functions; simple calculations using condition (a), (b), and (*) yield the desired estimates for the decomposition of $m(s)A(s)$.

Remarks: For $p = 1$ *Theorem I* is equivalent to Fefferman's result on the duality between H^1 and B.M.O.₂; see ref. 2. This permits us to formulate the following result in $\mathbb{R}^n$.

THEOREM I¹. $\hat{f}$ is the Fourier transform of an $H^1(\mathbb{R}^n)$ -function if and only if

$$\hat{f}(x) = \sum_{i=0}^{\infty} \lambda_i e^{i\alpha_i x} A_i(x)$$

where $\sum_0^\infty \lambda_i < \infty$, A_i are restrictions to $\mathbb{R}^n$ of entire functions of exponential type σ_i , vanishing at 0 and normalized by

$$\int_{\mathbb{R}^n} |A_i(x)|^2 dx \leq \frac{1}{\sigma_i^n}.$$

We observe that it follows from the normalization of A_i that the series converges absolutely and uniformly. Multiplier theorems in this setting as well as in the more abstract one, involving singular integrals with mixed homogeneity, can again be proved as above.

A more flexible characterization of Fourier transforms of H^p functions, which does not involve entire functions of exponential type, can be obtained by introducing *Fourier molecules*:

A function $M(x)$ is said to be a *Fourier molecule* for H^1 if

$$\left(\int |M(x)|^2 dx \right) \left(\int |M'(x)|^2 dx \right) \leq 1$$

and

$$|M(x)| \leq \sigma|x| \text{ where } \frac{1}{\sigma} = \int |M(x)|^2 dx.$$

It is obvious that $\check{M}$ and $(\operatorname{sgn} x M)^\vee$ belong to L^1 ; thus, $M \in H^1$. Also obvious is the fact that a 1-normalized function of exponential type is a Fourier molecule.

For $2/3 < p \leq 1$ we say that M is a *Fourier molecule* for H^p if $\sigma^{(1/p-1)} M$ is a molecule for H^1 , where

$$\frac{1}{\sigma} = \int |\sigma^{1/p-1} M|^2 dx$$

(similar definitions can be given for $p < 2/3$). *Theorem I* can now be rewritten with Fourier molecules M_i replacing the A_i 's. Some advantages of this formulation can be seen from the following remarks:

(a) Multiplication by $m(x)$ preserves Fourier molecules if and only if m is bounded and $\sup_R \int_R^{2R} |m'(x)|^2 dx < \infty$.

In other words a Fourier multiplier satisfying this condition is bounded on H^p , $2/3 < p < \infty$.

(b) If M is a Fourier molecule, $|M(x)|$ is also a Fourier molecule. This is clear since $||M(x)|'| \leq |M'(x)|$.

(c) We can also define the notion of a Fourier molecule for H^1 of the circle. Such a molecule is a sequence $M(k)$ such that

$$\left(\sum_{-\infty}^{\infty} |M(k)|^2 \right) \left(\sum_{-\infty}^{\infty} |M(k+1) - M(k)|^2 \right) \leq 1$$

$$|M(k)| \leq \sigma|k| \text{ with } \frac{1}{\sigma} = \sum_{-\infty}^{\infty} |M(k)|^2$$

and *Theorem I* remains valid.

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1. Coifman, R. R. (1974) "A real variable characterization of H^p ," *Studia Math.*, in press.
2. Fefferman, C. & Stein, E. M. (1972) " H^p spaces of several variables," *Acta Math.* 129, 137-193.