Recovery and regularization of initial temperature distribution in a two-layer cylinder with perfect thermal contact at the interface

By Khalid MASOOD^{†)}

Hafr Al-Batin Community College, King Fahd University of Petroleum and Minerals, Hafr, Saudi Arabia

(Edited by Takeo YOKOBORI, M.J.A.)

Abstract: We investigate the inverse problem associated with the heat equation involving recovery of initial temperature distribution in a two-layer cylinder with perfect thermal contact at the interface. The heat equation is solved backward in time to obtain a relationship between the final temperature distribution and the initial temperature profile. An integral representation for the problem is found, from which a formula for initial temperature is derived using Picard's criterion and the singular system of the associated operators. The known final temperature profile can be used to recover the initial temperature distribution from the formula derived in this paper. A robust method to regularize the outcome by introducing a small parameter in the governing equation is also presented. It is demonstrated with the help of a numerical example that the hyperbolic model gives better results as compared to the parabolic heat conduction model.

Key words: Inverse problem; layered medium; heat equation; regularization.

Introduction. The classical direct problem in heat conduction is to determine the temperature distribution of a body as the time progresses. The task of determining the initial temperature distribution from the final distribution is distinctly different from the direct problem and it is identified as the initial inverse heat conduction problem. This type of inverse problem is extremely ill-posed, see e.g. Engl.¹⁾ There is another approach to this inverse problem that consists of a complete reformulation of the governing equation. The inverse problem based upon the parabolic heat equation is closely approximated by a hyperbolic heat equation; see e.g. Weber,²⁾ Elden³⁾ and Masood and Zaman.^{4)–6)} This alternate formulation gives rise to an inverse problem, which is stable and well-posed and thus gives more reliable results. The numerical methods for hyperbolic problems are efficient and accurate. Moreover, this alternative approach is more attractive in practical engineering problems such as the non-homogenous solids conduction process, the slow conduction process, and the short-pulse laser applications; see Vedavarz *et al.*⁷⁾ and Gratzke *et al.*⁸⁾ among others. Furthermore, as we see later, the parabolic heat con-

duction model can be treated as a limiting case of the hyperbolic model.

The transient-temperature distribution in a composite medium consisting of several layers in contact has numerous applications in engineering, see e.g. Özişik.⁹⁾ In this paper, the mathematical formulation of the determination of the initial temperature distribution from the final temperature distribution in a composite medium consisting of two-layer cylinder with perfect thermal contact at the interface as shown in the figure is presented. This problem can be transformed to an integral equation of the first kind, from that a formula for initial temperature distribution can be derived by using Picard's theorem and singular system of the associated operator, see e.g. Groetsch.¹⁰⁾

In the second section the heat conduction problem in a two-layer cylinder is formulated. In the third section an outline of the direct solution is presented. The inverse problem of recovering the initial profile from the final data is considered in the fourth section. Regularization of the inverse problem in the parabolic heat equation by an alternate approach is presented in the fifth section. Comparison of the recovered initial profile by parabolic and hyperbolic models is demonstrated by some numerical experiments in the sixth section. Finally, in the

^{†)} Hafr Al-Batin Community College, King Fahd University of Petroleum and Minerals, P.O. Box 1803, Hafr Al-Batin 31991, Saudi Arabia (e-mail: masood@kfupm.edu.sa).

Glossary

Symbol	Definition
i	$i = 1$ corresponds to the layer $0 \leq r \leq a$, and $i = 2$ corresponds to the layer $a \leq r \leq b$
k_i	Thermal conductivity $i = 1, 2$
α_i	Thermal diffusivity $i = 1, 2$
$f_i(x)$	Final temperature distribution $i = 1, 2$
f_{in}	Fourier coefficients $i = 1, 2$
$g_i(x)$	Initial temperature distribution $i = 1, 2$
$J_0(r)$	Bessel functions of the first kind
$Y_0(r)$	Bessel functions of the second kind
M_{1n}	$\frac{k_1}{\alpha_1} \int_0^a \zeta \phi_{1n}^2(\zeta) d\zeta$
M_{2n}	$\frac{k_2}{\alpha_2} \int_a^b \zeta \phi_{2n}^2(\zeta) d\zeta$
N_n	Normalizing constants
$\ \cdot\ $	Norm
$\phi_{in}(x)$	Eigenfunctions $i = 1, 2$
λ_n	Eigenvalues
SNR	Signal to noise ratio

last section a summary of results is presented.

Formulation of the problem. We consider a two-layer cylinder consists of an inner region $0 \leq r \leq a$ and an outer region $a \leq r \leq b$ that are in perfect thermal contact at $r = a$ as illustrated in the Fig. 1. Suppose k_1 and k_2 are the thermal conductivities, and α_1 and α_2 are the thermal diffusivities for the inner and outer regions, respectively. The temperature distribution at the point r and t in the inner region is given by $u_1(r, t)$ and in the outer region by $u_2(r, t)$. These temperature distributions satisfy the following governing equations in the two regions

$$\frac{\alpha_1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_1(r, t)}{\partial r} \right) = \frac{\partial u_1(r, t)}{\partial t} \quad \text{in } 0 \leq r \leq a, \quad t > 0, \quad [1]$$

$$\frac{\alpha_2}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_2(r, t)}{\partial r} \right) = \frac{\partial u_2(r, t)}{\partial t} \quad \text{in } a \leq r \leq b, \quad t > 0, \quad [2]$$

subject to the boundary conditions

$$u_2(b, t) = 0, \quad t > 0, \quad [3]$$

$$u_1(0, t) = \text{finite}, \quad t > 0, \quad [4]$$

$$u_1(a, t) = u_2(a, t), \quad t > 0, \quad [5]$$

Fig. 1. A two-layer cylinder with perfect thermal contact at the interface.

$$k_1 \frac{\partial u_1(a, t)}{\partial r} = k_2 \frac{\partial u_2(a, t)}{\partial r}, \quad t > 0. \quad [6]$$

The boundary condition [3] can be replaced by an insulated boundary, i.e. $\partial_r u_2(b, t) = 0$, or by a radiating type boundary condition depending upon how the boundary of the outer region is kept in a given situation. If the energy dissipates at the boundary then the condition [3] can be replaced by convective type boundary conditions. The analysis for these boundary conditions can be carried out in a manner similar to that described in this paper. We assume the final temperature distribution of two regions at time $t = T$ is given by

$$f_i(r) = u_i(r, T), \quad i = 1, 2. \quad [7]$$

Our aim is to recover initial temperature profiles of the inner and the outer regions given by

$$g_i(r) = u_i(r, 0), \quad i = 1, 2. \quad [8]$$

The direct problem. In this section only an outline of the direct solution is presented, for further details the reader is referred to see Özişik,⁹⁾ Carslaw and Jaeger.¹¹⁾ We assume the solution of the direct problem in the form

$$u_i(r, t) = \sum_{n=1}^{\infty} v_n(t) \phi_{in}(r), \quad i = 1, 2. \quad [9]$$

The eigenfunctions $\phi_{in}(r)$ for the two regions are given by the following expressions:

$$\phi_{1n}(r) = J_0 \left(\frac{\lambda_n r}{\sqrt{\alpha_1}} \right), \quad \text{in } 0 \leq r < a, \quad [10]$$

$$\phi_{2n}(r) = A_{2n}J_0\left(\frac{\lambda_n r}{\sqrt{\alpha_2}}\right) + B_{2n}Y_0\left(\frac{\lambda_n r}{\sqrt{\alpha_2}}\right),$$

in $a < r < b$, [11]

where the coefficients A_{2n} and B_{2n} are of the form

$$A_{2n} = \frac{1}{\Delta} \left[J_0\left(\frac{\lambda_n a}{\sqrt{\alpha_1}}\right) Y_1\left(\frac{\lambda_n a}{\sqrt{\alpha_2}}\right) - \frac{k_1}{k_2} \sqrt{\frac{\alpha_2}{\alpha_1}} J_1\left(\frac{\lambda_n a}{\sqrt{\alpha_1}}\right) Y_0\left(\frac{\lambda_n a}{\sqrt{\alpha_2}}\right) \right],$$

[12]

$$B_{2n} = \frac{1}{\Delta} \left[\frac{k_1}{k_2} \sqrt{\frac{\alpha_2}{\alpha_1}} J_1\left(\frac{\lambda_n a}{\sqrt{\alpha_1}}\right) J_0\left(\frac{\lambda_n a}{\sqrt{\alpha_2}}\right) - J_0\left(\frac{\lambda_n a}{\sqrt{\alpha_1}}\right) J_1\left(\frac{\lambda_n a}{\sqrt{\alpha_2}}\right) \right],$$

[13]

where

$$\Delta = J_0\left(\frac{\lambda_n a}{\sqrt{\alpha_2}}\right) Y_1\left(\frac{\lambda_n a}{\sqrt{\alpha_2}}\right) - J_1\left(\frac{\lambda_n a}{\sqrt{\alpha_2}}\right) Y_0\left(\frac{\lambda_n a}{\sqrt{\alpha_2}}\right).$$

The eigenvalues λ_n are solution of the following transcendental equation

$$\begin{vmatrix} J_0\left(\frac{\lambda_n a}{\sqrt{\alpha_1}}\right) & -J_0\left(\frac{\lambda_n a}{\sqrt{\alpha_2}}\right) & -Y_0\left(\frac{\lambda_n a}{\sqrt{\alpha_2}}\right) \\ \frac{k_1}{k_2} \sqrt{\frac{\alpha_2}{\alpha_1}} J_1\left(\frac{\lambda_n a}{\sqrt{\alpha_1}}\right) & -J_1\left(\frac{\lambda_n a}{\sqrt{\alpha_2}}\right) & -Y_1\left(\frac{\lambda_n a}{\sqrt{\alpha_2}}\right) \\ 0 & J_0\left(\frac{\lambda_n b}{\sqrt{\alpha_2}}\right) & Y_0\left(\frac{\lambda_n b}{\sqrt{\alpha_2}}\right) \end{vmatrix} = 0.$$

[14]

The direct solution of the problem, that is, the temperature distribution $u_i(x, t)$, $i = 1, 2$ in any one of the two regions is given by

$$u_i(r, t) = \sum_{n=1}^{\infty} \exp(-\lambda_n^2 t) \frac{1}{N_n} \phi_{in}(r) \times \left[\frac{k_1}{\alpha_1} \int_0^a \zeta \phi_{1n}(\zeta) g_1(\zeta) d\zeta + \frac{k_2}{\alpha_2} \int_a^b \zeta \phi_{2n}(\zeta) g_2(\zeta) d\zeta \right],$$

$i = 1, 2$, [15]

where N_n are the normalizing constants and are of the following form:

$$N_n = \frac{k_1}{\alpha_1} \int_0^a \zeta \phi_{1n}^2(\zeta) d\zeta + \frac{k_2}{\alpha_2} \int_a^b \zeta \phi_{2n}^2(\zeta) d\zeta.$$

[16]

The inverse solution. The method we use to solve the inverse problem is based on the reduction of the direct problem to an integral equation of the first kind. The expression [15] together with condition [7] leads to an integral equation of the first kind. That integral equation can be inverted by the application of Picard's theorem using the singular system of the integral operator involved.

Case 1. Consider one of $g_i(x) = 0$, for instance $g_1(x) \neq 0$ and $g_2(x) = 0$. The expression [15] in this case is

$$u_i(r, t) = \sum_{n=1}^{\infty} \exp(-\lambda_n^2 t) \frac{1}{N_n} \phi_{in}(r) \times \left[\frac{k_1}{\alpha_1} \int_0^a \zeta \phi_{1n}(\zeta) g_1(\zeta) d\zeta \right], \quad i = 1, 2.$$

[17]

Using condition [7] in the expression [17] leads to

$$f_i(r) = \int_0^a K_i(r, \zeta) g_1(\zeta) d\zeta, \quad i = 1, 2, \quad [18]$$

where

$$K_i(r, \zeta) = \frac{k_1}{\alpha_1} \sum_{n=1}^{\infty} \zeta \exp(-\lambda_n^2 T) \frac{1}{N_n} \phi_{in}(r) \phi_{1n}(\zeta),$$

$i = 1, 2$. [19]

Our aim is to solve the integral equation [18] for the unknown initial temperature distribution $g_1(r)$. To accomplish this goal, we record the final profile in the first layer, that is $i = 1$. Therefore expressions [18] and [19] reduce to

$$f_1(r) = \int_0^a K_1(r, \zeta) g_1(\zeta) d\zeta, \quad [20]$$

where

$$K_1(r, \zeta) = \frac{k_1}{\alpha_1} \sum_{n=1}^{\infty} \zeta \exp(-\lambda_n^2 T) \frac{1}{N_n} \phi_{1n}(r) \phi_{1n}(\zeta).$$

[21]

Thus the inverse problem is reduced to solving integral equation of the first kind. The singular system in the first layer of the integral operator in [20] is

$$\left\{ \frac{M_{1n}}{N_n} \exp(-\lambda_n^2 T); \sqrt{\frac{k_1}{M_{1n}\alpha_1}} \phi_{1n}(r), \sqrt{\frac{k_1}{M_{1n}\alpha_1}} \phi_{1n}(\zeta) \right\},$$

[22]

where

$$M_{1n} = \frac{k_1}{\alpha_1} \int_0^a \zeta \phi_{1n}^2(\zeta) d\zeta. \quad [23]$$

In the expression [22], the first term in the braces is the singular values and the next two terms correspond to the singular functions. Such a decomposition of an operator is called the singular value decomposition, see Engl.¹⁾

Now by application of Picard's theorem (see ref. 1)) the inverse problem is solvable if and only if

$$\sum_{n=1}^{\infty} \left(\frac{N_n}{M_{1n}} \right)^2 \exp[2\lambda_n^2 T] |f_{1n}|^2 < \infty, \quad [24]$$

where

$$f_{1n} = \sqrt{\frac{k_1}{M_{1n}\alpha_1}} \int_0^a \zeta \phi_{1n}(\zeta) f_1(\zeta) d\zeta, \quad [25]$$

are the classical Fourier coefficients of f_1 . In this case by Picard's theorem, we can recover the initial profile by the following expression

$$g_1(r) = \sqrt{\frac{k_1}{\alpha_1}} \sum_{n=1}^{\infty} \frac{N_n}{M_{1n}^{3/2}} \exp[\lambda_n^2 T] f_{1n} \phi_{1n}(r). \quad [26]$$

The case $g_1(x) = 0$ and $g_2(x) \neq 0$ can be carried out on same lines as the case $g_1(x) \neq 0$ and $g_2(x) = 0$. Picard's theorem demonstrates the ill-posed nature of the problem considered. If we perturb the data by setting $f^\delta = f + \delta\phi_n$ we obtain a perturbed solution $g^\delta = g + \delta\phi_n \exp[\lambda_n^2 T]$. Hence the ratio $\|g^\delta - g\|/\|f^\delta - f\| = \exp[\lambda_n^2 T]$ can be made arbitrarily large due to the fact that the singular values $\exp[-\lambda_n^2 T]$ decay exponentially. This rate of decay depends on the size of the eigenvalues and on the size of the time displacement. It is also intuitively clear that for large values of T , the influence of the initial condition on the solution reduces and thus initial condition may not be recoverable, see ref. 4) for the effect of T on recovery of the initial profile.

Case 2. Consider the case $g_1(x) \neq 0$ and $g_2(x) \neq 0$. The expression [15] in this case is

$$u_i(r, t) = \sum_{n=1}^{\infty} \exp(-\lambda_n^2 t) \frac{1}{N_n} \phi_{in}(r) \times \left[\frac{k_1}{\alpha_1} \int_0^a \zeta \phi_{1n}(\zeta) g_1(\zeta) d\zeta + \frac{k_2}{\alpha_2} \int_a^b \zeta \phi_{2n}(\zeta) g_2(\zeta) d\zeta \right],$$

$$i = 1, 2. \quad [27]$$

Using condition [7] in the expression [17] leads to

$$f_i(r) - \tau_i(r) = \int_0^a K_i(r, \zeta) g_1(\zeta) d\zeta, \quad i = 1, 2, \quad [28]$$

where

$$K_i(r, \zeta) = \frac{k_1}{\alpha_1} \sum_{n=1}^{\infty} \zeta \exp(-\lambda_n^2 T) \frac{1}{N_n} \phi_{in}(r) \phi_{1n}(\zeta), \quad i = 1, 2, \quad [29]$$

and

$$\tau_i(r) = \sum_{n=1}^{\infty} \exp(-\lambda_n^2 t) \frac{1}{N_n} \phi_{in}(r) \times \left[\frac{k_2}{\alpha_2} \int_a^b \zeta \phi_{2n}(\zeta) g_2(\zeta) d\zeta \right], \quad i = 1, 2. \quad [30]$$

Our aim is to solve the integral equation [27] for the unknown initial temperature distribution $g_1(r)$. To accomplish this goal, we record the final profile in the first layer, that is $i = 1$. Therefore expressions [18] and [19] reduce to

$$f_1(r) - \tau_1(r) = \int_0^a K_1(r, \zeta) g_1(\zeta) d\zeta. \quad [31]$$

Thus the inverse problem is reduced to solving integral equation of the first kind. The singular system in the first layer of the integral operator in [31] is

$$\left\{ \frac{M_{1n}}{N_n} \exp(-\lambda_n^2 T); \sqrt{\frac{k_1}{M_{1n}\alpha_1}} \phi_{1n}(r), \sqrt{\frac{k_1}{M_{1n}\alpha_1}} \phi_{1n}(r) \right\}, \quad [32]$$

Now by application of Picard's theorem the inverse problem is solvable if and only if

$$\sum_{n=1}^{\infty} \left(\frac{N_n}{M_{1n}} \right)^2 \exp[2\lambda_n^2 T] |f_{1n}|^2 < \infty, \quad [33]$$

where

$$f_{1n} = \sqrt{\frac{k_1}{M_{1n}\alpha_1}} \int_0^a \zeta \phi_{1n}(\zeta) [f_1(\zeta) - \tau_1(\zeta)] d\zeta, \quad [34]$$

are the classical Fourier coefficients. In this case by Picard's theorem, we can recover the initial profile by the following expression

$$g_1(r) = \sqrt{\frac{k_1}{\alpha_1}} \sum_{n=1}^{\infty} \frac{N_n}{M_{1n}^{3/2}} \exp[\lambda_n^2 T] f_{1n} \phi_{1n}(r). \quad [35]$$

In order to recover $g_2(r)$, the expression [28] can be written as

$$f_i(r) - \eta_i(r) = \int_a^b \tilde{K}_i(r, \zeta) g_2(\zeta) d\zeta, \quad i = 1, 2, \quad [36]$$

where

$$\tilde{K}_i(r, \zeta) = \frac{k_2}{\alpha_2} \sum_{n=1}^{\infty} \zeta \exp(-\lambda_n^2 T) \frac{1}{N_n} \phi_{in}(r) \phi_{2n}(\zeta), \quad i = 1, 2, \quad [37]$$

and

$$\eta_i(r) = \sum_{n=1}^{\infty} \exp(-\lambda_n^2 T) \frac{1}{N_n} \phi_{in}(r) \times \left[\frac{k_1}{\alpha_1} \int_0^a \zeta \phi_{1n}(\zeta) g_1(\zeta) d\zeta \right], \quad i = 1, 2. \quad [38]$$

In the second layer, the expression [36] can be written as

$$f_2(r) - \eta_2(r) = \int_a^b \tilde{K}_2(r, \zeta) g_2(\zeta) d\zeta. \quad [39]$$

Thus the inverse problem is reduced to solving integral equation of the first kind. The singular system in the first layer of the integral operator in [31] is

$$\left\{ \frac{M_{2n}}{N_n} \exp(-\lambda_n^2 T); \sqrt{\frac{k_2}{M_{2n}\alpha_2}} \phi_{2n}(r), \sqrt{\frac{k_2}{M_{2n}\alpha_2}} \phi_{2n}(r) \right\}, \quad [40]$$

where

$$M_{2n} = \frac{k_2}{\alpha_2} \int_a^b \zeta \phi_{2n}^2(\zeta) d\zeta. \quad [41]$$

Now by application of Picard's theorem the inverse problem is solvable if and only if

$$\sum_{n=1}^{\infty} \left(\frac{N_n}{M_{2n}} \right)^2 \exp[2\lambda_n^2 T] |f_{2n}|^2 < \infty, \quad [42]$$

where

$$f_{2n} = \sqrt{\frac{k_2}{M_{2n}\alpha_2}} \int_a^b \zeta \phi_{2n}(\zeta) [f_2(r) - \eta_2(\zeta)] d\zeta, \quad [43]$$

are the classical Fourier coefficients. In this case by Picard's theorem, we can recover the initial profile by the following expression

$$g_2(r) = \sqrt{\frac{k_2}{\alpha_2}} \sum_{n=1}^{\infty} \frac{N_n}{M_{2n}^{3/2}} \exp[\lambda_n^2 T] f_{2n} \phi_{2n}(r). \quad [44]$$

Regularizing the inverse solution. In order to overcome the ill-posedness of the inverse problem, we may model the problem by introducing a hyperbolic term with a small parameter in the classical heat equation. It is well established that this new model regularizes the problem in classical heat model, see Masood *et al.*^{(4)–(6)} The hyperbolic heat conduction model in two regions has the following form

$$\frac{\alpha_1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_1(r, t)}{\partial r} \right) = \frac{\partial u_1(r, t)}{\partial t} + \epsilon \frac{\partial^2 u_1(r, t)}{\partial t^2} \quad \text{in } 0 \leq r \leq a, \quad t > 0, \quad [45]$$

$$\frac{\alpha_2}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_2(r, t)}{\partial r} \right) = \frac{\partial u_2(r, t)}{\partial t} + \epsilon \frac{\partial^2 u_2(r, t)}{\partial t^2} \quad \text{in } a \leq r \leq b, \quad t > 0, \quad [46]$$

where the parameter ϵ is assumed to be small and $\epsilon \rightarrow 0^+$. Together with conditions [3]–[8] and one additional condition given below

$$\frac{\partial u_i}{\partial t}(r, 0) = 0, \quad i = 1, 2. \quad [47]$$

In this case, we get the following ordinary differential equation

$$\epsilon \frac{d^2 v_n(t)}{dt^2} + \frac{dv_n(t)}{dt} + \lambda_n^2 v_n(t) = 0, \quad [48]$$

together with

$$v_n(0) = k_n, \quad [49]$$

and

$$\frac{\partial v_n(0)}{\partial t} = 0. \quad [50]$$

Since $\epsilon \rightarrow 0^+$, this is a singular perturbation problem. We apply the WKBJ method⁽¹²⁾ to obtain an asymptotic representation for the solution of [48] containing parameter ϵ ; the representation is to be valid for small values of the parameter. It is demonstrated in ref. 12) that the solution stays closer to the exact solution for large values such as $\epsilon = 0.5$. The solution of [48] is given by

$$v_n(t) = \left(\frac{\epsilon \lambda_n^2 - 1}{2\epsilon \lambda_n^2 - 1} \right) k_n \exp[-\lambda_n^2 t] + \left(\frac{\epsilon \lambda_n^2 k_n}{2\epsilon \lambda_n^2 - 1} \right) \exp \left[\lambda_n^2 t - \frac{t}{\epsilon} \right]. \quad [51]$$

The remaining procedure of finding the inverse solution is same as in the previous section. The inverse solutions [35] and [44] for the hyperbolic heat conduction model can be written as

$$g_1(r) = \sqrt{\frac{k_1}{\alpha_1}} \sum_{n=1}^{\infty} \frac{N_n}{M_{1n}^{3/2}} \times \frac{f_{1n} \phi_{1n}(r)}{\left\{ \begin{aligned} &\left(\frac{\epsilon \lambda_n^2 - 1}{2\epsilon \lambda_n^2 - 1} \right) \exp[-\lambda_n^2 T] \\ &+ \left(\frac{\epsilon \lambda_n^2}{2\epsilon \lambda_n^2 - 1} \right) \exp \left[\lambda_n^2 T - \frac{T}{\epsilon} \right] \end{aligned} \right\}}. \quad [52]$$

$$g_2(r) = \sqrt{\frac{k_2}{\alpha_2}} \sum_{n=1}^{\infty} \frac{N_n}{M_{2n}^{3/2}} \times \frac{f_{2n} \phi_{2n}(r)}{\left\{ \begin{aligned} &\left(\frac{\epsilon \lambda_n^2 - 1}{2\epsilon \lambda_n^2 - 1} \right) \exp[-\lambda_n^2 T] \\ &+ \left(\frac{\epsilon \lambda_n^2}{2\epsilon \lambda_n^2 - 1} \right) \exp \left[\lambda_n^2 T - \frac{T}{\epsilon} \right] \end{aligned} \right\}}. \quad [53]$$

The solutions given by [35] and [44] can be recovered by letting $\epsilon \rightarrow 0^+$ in equation [52] and [53] respectively. This shows that the parabolic heat conduction model can be treated as a limiting case of the hyperbolic heat conduction model. It is shown in ref.4) that by choosing an appropriate value of the parameter ϵ the hyperbolic heat conduction model behaves much better than the parabolic heat conduction model.

Example 1 Consider the initial temperature distributions of the form

$$g_1(r) = \phi_{12}(r) = J_0 \left(\frac{\lambda_2 r}{\sqrt{\alpha_1}} \right), \quad \text{and} \quad g_2(r) = 0. \quad [54]$$

The initial profiles should be of the form so that the integrals appearing in the expression [15] exist and converge to the initial profiles as the time approaches to zero. One such criterion for the integrals to exist is that $g_1(r)$ is continuous in $0 < r < a$ and $g_2(r)$ is continuous in $a < r < b$. More generally, we can choose the initial profiles which are measurable and satisfy a criteria of boundedness, see John.¹³⁾

To see that the initial profile [54] is recovered by the processing formula [26], first we calculate the final data f_{12} given by [25]. The expression [20] yields

$$f_1(r) = \frac{M_{1n}}{N_2} \exp(-\lambda_2^2 T) \phi_{12}(r), \quad [55]$$

and the expression [25] yields

$$f_{12} = \sqrt{\frac{\alpha_1}{k_1}} \frac{M_{1n}^{3/2}}{N_2} \exp(-\lambda_2^2 T). \quad [56]$$

Now the initial profile given by [54] can be recovered if we use the final data given by [56] in the processing formula [26]. The initial profile given by [54] can also be recovered exactly by the processing formula [52].

Example 2 Consider the initial temperature distribution of the form

$$g_1(r) = \phi_{12}(r), \quad \text{and} \quad g_2(r) = \phi_{22}(r). \quad [57]$$

To see that the initial profiles [57] is recovered by the processing formulae [35], [44] and [52]–[53], first we calculate the final profiles f_{12} and f_{22} given by [34] and [43] respectively. The expressions [31] and [39] yield

$$f_1(r) - \tau_1(r) = \frac{M_{1n}}{N_2} \exp(-\lambda_2^2 T) \phi_{12}(r), \quad [58]$$

$$f_2(r) - \eta_2(r) = \frac{M_{2n}}{N_2} \exp(-\lambda_2^2 T) \phi_{22}(r). \quad [59]$$

Substitution of the expressions [58] and [59] in [34] and [43] respectively lead to

$$f_{12} = \sqrt{\frac{\alpha_1}{k_1}} \frac{M_{1n}^{3/2}}{N_2} \exp(-\lambda_2^2 T), \quad [60]$$

$$f_{22} = \sqrt{\frac{\alpha_2}{k_2}} \frac{M_{2n}^{3/2}}{N_2} \exp(-\lambda_2^2 T). \quad [61]$$

Now the initial profiles given by [54] can be recovered if we use the final profiles given by [60] and [61] in the parabolic processing formulae [35], [44] and hyperbolic processing formulae [52]–[53].

Numerical experiments. In this section we present numerical examples intended to demonstrate the usefulness of reformulating the parabolic model into hyperbolic model. In Figs. 2–5, the thick solid line represents the exact initial profile, the thin solid line represents the recovered initial profile by hyperbolic model and the dashed line represents the recovered initial profile by the parabolic model. The

Fig. 2. The recovered initial profile when the noisy signal is used in the parabolic model in the inner layer.

Fig. 3. The recovered initial profile when the noisy signal is used in the hyperbolic model in the inner layer.

parabolic model is highly unstable to the noisy signal, so to see the unstable behavior consider $SNR = 400 \text{ dB}$ in all the experiments, which corresponds to very low level of noise. Consider Example 2, and choose for convenience the constants $\alpha_1 = k_1 = 2$, $\alpha_2 = k_2 = 4$, however for any real application these constants correspond to the material properties. The time displacement is chosen to be $T = 1$, in all the figures.

In any real world experiment the recorded final profile cannot be free from noise, so we add white Gaussian noise in the final data given by [60] and [61]. We use this corrupted data in the parabolic model [35] and [44] and compare it with the exact ini-

Fig. 4. The recovered initial profile when the noisy signal is used in the parabolic model in the outer layer.

Fig. 5. The recovered initial profile when the noisy signal is used in the hyperbolic model in the outer layer.

tial profile. The parabolic model as discussed before will give unstable results. We also use this corrupted data in the hyperbolic model [52]–[53] and compare it with the exact initial profile. The parameter ϵ is chosen in such a way that the spikes due to the noise appear to be minimum.

The inherent instability of the parabolic model is clear from Fig. 2 and Fig. 4. The exact initial profile appears as a straight line and the recovered initial profile oscillates with large amplitude in both regions. Therefore the parabolic model does not give any information about the initial profile even for the very low level of noise. In contrast, from Fig. 3 and Fig. 5, the recovered initial temperature distri-

bution in both the regions by hyperbolic model is in good agreement with the exact initial profile. It is worth mentioning that for higher level of noise, the parabolic model is more unstable. The hyperbolic model is stable to higher level of noise but the recovered profile deviates from the exact initial profile considerably as the magnitude of noise increases and it may not be suitable for a particular application. Some more regularization techniques are needed to recover the initial profile with more accuracy if the SNR is low.

Conclusions. The recovery of initial temperature distribution from the observation of final temperature distribution in a two-layer cylinder model is presented. It is shown that the initial temperature distribution in either one of the regions can be recovered by the processing formulae of both the parabolic and the hyperbolic models. Furthermore, it is shown that the parabolic model can be treated as a limiting case of the hyperbolic model.

The inverse solution of the heat conduction model is characterized by discontinuous dependence on the data. A small error in the n th Fourier coefficient is amplified by the factor $\exp[\lambda_n^2 T]$. Thus it depends on the rate of decay of singular values and this rate of decay also depends on the size of the parameter T . In order to get some meaningful information, one has to consider first few degrees of freedom in the data and has to filter out everything else depending on the rate of decay of singular values and the size of parameter T . It is shown that a complete reformulation of the heat conduction problem as a hyperbolic equation may produce meaningful results. The hyperbolic model with a small parameter is stable and regularizes the heat conduction equation. The hyperbolic model may not give accurate results for highly noisy data, say for example $SNR = 30\text{ db}$. It is the problem of future investigation to apply some other regularization technique or method to recover

the initial temperature distribution with more accuracy for lower values of SNR.

Acknowledgment. The author wish to acknowledge support provided by the Hafr Al-Batin Community College, King Fahd University of Petroleum and Minerals.

References

- 1) Engl, H. W., Hanke, M., and Neubauer, A. (1996) Regularization of Inverse Problems. Kluwer, Dordrecht, pp. 31–42.
- 2) Weber, C. F. (1981) Int. J. Heat Mass Transfer **24**, 1783–1792.
- 3) Elden, L. (1987) Inverse and Ill-Posed Problems (eds. Engl, H. W., and Groetsch, C. W.). Academic Press, Inc., London, pp. 345–350.
- 4) Masood, K., Messaoudi, S., and Zaman, F. D. (2002) Int. J. Heat Mass Transfer **45**(14), 2959–2965.
- 5) Masood, K., and Zaman, F. D. (2004) J. Heat Transfer **126**(2), 294–296.
- 6) Masood, K., and Zaman, F. D. (2004) Arabian J. Sci. Eng. **29**(1B), 3–12.
- 7) Vedavaz, A., Mitra, K., and Kumar, S. (1994) J. Appl. Phys. **76**(9), 5014–5021.
- 8) Gratzke, U., Kapadia, P. D., and Dowden, J. (1991) J. Phys. D: Appl. Phys. **24**, 2125–2134.
- 9) Özışık, M. N. (1980) Heat Conduction. Wiley Intersc., New York.
- 10) Groetsch, C. W. (1980) Elements of Applicable Functional Analysis. Marcel Dekker, Inc., New York.
- 11) Carslaw, H. S., and Jaeger, J. C. (1959) Conduction of Heat in Solids. Oxford University Press, London.
- 12) Bender, C. M., and Orszag, S. A. (1978) Advanced Mathematical Methods for Scientists and Engineers. McGraw Hill, New York.
- 13) John, F. (1982) Partial Differential Equations. Springer-Verlag, New York.

(Received April 12, 2006; accepted Sept. 12, 2006)