Dedekind's η -function and the cohomology of infinite dimensional Lie algebras

(Macdonald's identities/Euler–Poincaré principle/affine root system/loop space/automorphic forms)

HOWARD GARLAND

Yale University, New Haven, Connecticut 06520

Communicated by G. D. Mostow, April 3, 1975

ABSTRACT We compute the cohomology of certain infinite dimensional Lie algebras which are subalgebras of Lie algebras introduced by Moody and Kac. We note a relation between our results and the cohomology of loop spaces of compact groups. Finally, we derive, by Euler–Poincaré, identities of Macdonald for powers of the Dedekind η -function.

We announce new results on the cohomology groups of certain infinite dimensional Lie algebras (*Theorems 3.2 and 3.4*, below). These Lie algebras are subalgebras of algebras introduced by Moody and Kac (see refs. 1 and 2). We also obtain an empirical relation between these cohomology groups and the cohomology of loop spaces of compact groups (see *Corollary 3.6*, below). Finally, we derive Macdonald's identities for the powers of the Dedekind η -function (see ref. 3), by applying the Euler–Poincaré principle and our cohomology results. In ref. 4, Kac obtained a generalization of Macdonald's identities, using infinite dimensional Lie algebras, though his methods were different from ours, and he did not discuss the cohomology groups which we study here. If, in [3.5], one sets $e^{\alpha_i} = 1$, $i = 1, \dots, l$, one obtains identities for powers of Dedekind's η -function. For example, if $\mathfrak{G} = \mathfrak{sl}_2(\mathbb{C})$, the Lie algebra of all 2×2 complex matrices with trace zero, then one obtains Jacobi's identity:

$$\eta(X)^3 = \sum_{n \equiv 1(4)} n X^{n^2/8}$$

(see ref. 3).

We remark that *Theorems 3.2 and 3.4* are analogues of results in Kostant (5), for certain finite dimensional Lie algebras. Moreover, the results in Kostant (6) suggest that *Corollary 3.6* is not simply an accident.

Detailed proofs will appear elsewhere.

§1. We let $\mathfrak{G}$ be a complex, simple Lie algebra (we denote the complex numbers by $\mathbb{C}$). We let $\mathbb{C}\langle t \rangle$ be the ring of all finite Laurent series $a_{-m}t^{-m} + \dots + a_0 + \dots + a_nt^n$, $a_i \in \mathbb{C}$. We set $\tilde{\mathfrak{G}} = \mathbb{C}\langle t \rangle \otimes_{\mathbb{C}} \mathfrak{G}$, and set $[\alpha \otimes g, \alpha' \otimes g'] = \alpha \alpha' \otimes [g, g']$, for $\alpha, \alpha' \in \mathbb{C}\langle t \rangle$, and $g, g' \in \mathfrak{G}$. With this bracket operation, the space $\tilde{\mathfrak{G}}$ is an infinite dimensional Lie algebra over $\mathbb{C}$. The algebra $\tilde{\mathfrak{G}}$ was first introduced by Moody and Kac (see e.g., ref. 2). We let $\mathfrak{G}_{(n)} = t^n \otimes \mathfrak{G}$, and we identify $\mathfrak{G}$ with

the subalgebra $\mathfrak{G}_{(0)}$ of $\tilde{\mathfrak{G}}$. We let

$$\mathfrak{U}^- = \bigoplus_{n \leq -1} \mathfrak{G}_{(n)}, \quad \mathfrak{U}^+ = \bigoplus_{n \geq 1} \mathfrak{G}_{(n)}$$

We introduce a derivation $D: \tilde{\mathfrak{G}} \rightarrow \tilde{\mathfrak{G}}$ (which we call the degree operator). By definition, $D|_{\mathfrak{G}_{(n)}}$ is multiplication by the integer n . We let $\mathfrak{G}(A)$, the affine algebra associated with $\mathfrak{G}$, be the semi-direct product $\tilde{\mathfrak{G}} \oplus \mathbb{C} \cdot D$ (thus $[D, g] = D(g)$, for $g \in \tilde{\mathfrak{G}}$). We fix a Cartan subalgebra $\mathfrak{H} \subseteq \mathfrak{G}$, and we let $\mathfrak{H}$ be the subalgebra of $\mathfrak{G}(A)$, $\mathfrak{H} = \mathfrak{H} \oplus \mathbb{C} \cdot D$. A non-zero linear functional α on $\mathfrak{H}$ is called an (affine) root of $\mathfrak{G}(A)$ (relative to $\mathfrak{H}$) in case there exists $y \neq 0$ in $\mathfrak{G}(A)$ so that $ad x(y) = \alpha(x)y$, for all $x \in \mathfrak{H}$. We let $\mathcal{A}$ denote the set of all affine roots, and for $\alpha \in \mathcal{A}$, we set $\mathfrak{G}_\alpha = \{y \in \mathfrak{G}(A) \mid ad x(y) = \alpha(x)y, \text{ for all } x \in \mathfrak{H}\}$. A non-zero element of $\mathfrak{G}_\alpha$ will be called a root vector (corresponding to α). We then have a direct sum decomposition

$$\mathfrak{G}(A) = \mathfrak{H} \oplus \bigoplus_{\alpha \in \mathcal{A}} \tilde{\mathfrak{G}}_\alpha$$

We can then choose an order on $\mathcal{A}$ so that there are $l + 1$ simple roots $\alpha_1, \dots, \alpha_{l+1}$ ($l = \dim \mathfrak{H}$); i.e., every root $\alpha \in \mathcal{A}$ has a unique expression

$$\alpha = \sum_{i=1}^{l+1} n_i \alpha_i, \quad [1.1]$$

where the n_i are either all nonnegative or all nonpositive integers. Moreover, we can (and do) choose the α_i so that (i) $\alpha_1, \dots, \alpha_l$ restricted to $\mathfrak{H}$ are a set of simple roots for $\mathfrak{G}$ (relative to some order) and (ii) if α_0 denotes the corresponding highest root of $\mathfrak{G}$, then $\alpha_{l+1}|_{\mathfrak{H}} = -\alpha_0$ and $\alpha_{l+1}(D) = 1$.

If, in [1.1], all the n_i are nonnegative, we say α is positive. If $-\alpha$ is positive, we say α is negative. We let $\mathcal{A}^\pm$ denote the $\pm$ roots of $\mathcal{A}$, and note that $\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^-$ (disjoint union). Now $\alpha_1, \dots, \alpha_{l+1}$ give a basis for $\mathfrak{H}^*$, the dual space of $\mathfrak{H}$. We define an inner product on $\mathfrak{H}^*$ by

$$(\alpha_i, \alpha_j) = (\alpha_i|_{\mathfrak{H}}, \alpha_j|_{\mathfrak{H}}), 1 \leq i, j \leq l + 1,$$

where the inner product on the right is that induced by

the Killing form of $\mathfrak{G}$. We define $S_i = S_{\alpha_i}: \mathfrak{R}^* \rightarrow \mathfrak{R}^*$ by

$$S_i(\lambda) = \lambda - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i, i = 1, \dots, l + 1,$$

and we let W_a denote the group generated by the S_i . The group W_a is the affine Weyl group, in the sense of ref. 7, Chap. 6, corresponding to the root system of $\mathfrak{G}$.

§2. We let $\mathfrak{R} \subseteq \mathfrak{G}$ be a compact form and we let $(\ , \)$ denote the Killing form on $\mathfrak{G}$. We let $\mathfrak{Q} = i\mathfrak{R}$, and note that $\mathfrak{G} = \mathfrak{R} + i\mathfrak{R}$ (direct sum over the real numbers $\mathbf{R}$). For $z = x + iy$ in $\mathfrak{G}$ ($x, y \in \mathfrak{Q}$), we set $z^* = x - iy$. As in ref. 5, we define a positive-definite, Hermitian inner product $\{ \ , \ }$ on $\mathfrak{G}$ by $\{z_1, z_2\} = (z_1, z_2^*)$. We then define a positive-definite, Hermitian inner product $\{ \ , \ }$, on $\mathfrak{G}(A)$, by stipulating that D and $\mathfrak{G}$ be orthogonal, that $\mathfrak{G}_{(n)}$ and $\mathfrak{G}_{(m)}$ be orthogonal for $n \neq m$, that $\{D, D\} = 1$, and that $\{t^n \otimes g_1, t^n \otimes g_2\} = \{g_1, g_2\}$, for $g_1, g_2 \in \mathfrak{G}$. For a subspace $X \subseteq \mathfrak{G}(A)$, we then obtain an induced Hermitian structure $\{ \ , \ }$ on ΛX , the exterior algebra of X , by setting $\{x_1 \wedge \dots \wedge x_s, x'_1 \wedge \dots \wedge x'_s\} = \det \{x_i, x'_j\}$, for $x_1, \dots, x_s, x'_1, \dots, x'_s \in X$.

For a subalgebra $\mathfrak{B} \subseteq \mathfrak{G}(A)$, we let $\partial: \Lambda^{i+1}\mathfrak{B} \rightarrow \Lambda^i\mathfrak{B}$ be the usual boundary operator defined by

$$\partial(x_1 \wedge \dots \wedge x_r) = \sum_{a < b} (-1)^{a+b+1} [x_a, x_b] \wedge x_1 \wedge \dots \wedge x_r.$$

We let $D: \Lambda \mathfrak{U}^\pm \rightarrow \Lambda \mathfrak{U}^\pm$ denote the unique extension of $D: \mathfrak{U}^\pm \rightarrow \mathfrak{U}^\pm$, to a derivation of degree 0. For $n \leq -1$, we let $C(n) \subseteq \Lambda \mathfrak{U}^-$ be the eigenspace of D corresponding to the eigenvalue n . We note that each subspace $C(n)$ is finite-dimensional, and that we have an orthogonal direct sum decomposition:

$$\Lambda \mathfrak{U}^- = \bigoplus_{n \leq -1} C(n). \quad [2.1]$$

Moreover, $\text{ad } \xi, \xi \in \mathfrak{G}$, leaves $\mathfrak{U}^-$ invariant. For $\xi \in \mathfrak{G}$, we let $\theta(\xi): \Lambda \mathfrak{U}^- \rightarrow \Lambda \mathfrak{U}^-$ be the unique derivation of degree 0, corresponding to $\text{ad } \xi$. We then have

$$(i) D\partial = \partial D, (ii) D\theta(\xi) = \theta(\xi)D, \xi \in \mathfrak{G}, \\ (iii) \partial \circ \theta(\xi) = \theta(\xi) \circ \partial, \xi \in \mathfrak{G}. \quad [2.2]$$

Thus, each $C(n)$ is a subcomplex of $(\Lambda \mathfrak{U}^-, \partial)$ and is a $\mathfrak{G}(A)_0$ -submodule, where $\mathfrak{G}(A)_0$ is, by definition, the subalgebra $\mathfrak{G} \oplus \mathbf{C} \cdot D \subseteq \mathfrak{G}(A)$. We let $d: \Lambda \mathfrak{U}^- \rightarrow \Lambda \mathfrak{U}^-$ be the adjoint of ∂ :

$$\{dx, y\} = \{x, \partial y\}, x, y \in \Lambda \mathfrak{U}^-.$$

The operator d is well defined and

$$d(C(n)) \subseteq C(n), n \leq -1, d\theta(\xi) = \theta(\xi)d, \xi \in \mathfrak{G}. \quad [2.3]$$

We let Δ denote the Laplacian $\Delta = d\partial + \partial d$, and we let $\mathbf{H}^i = \text{kernel } \Delta \cap \Lambda^i \mathfrak{U}^-$ denote the space of harmonic i -forms. It follows from [2.2] and [2.3] that $\mathbf{H}^i$ is a $\mathfrak{G}(A)_0$ -submodule of $\Lambda^i \mathfrak{U}^-$. Also, by [2.1], [2.2], and [2.3], we have that $\Lambda \mathfrak{U}^-$ is an orthogonal direct sum of finite-dimensional subspaces $C(n)$, each of which is in-

variant under d, ∂ , and hence, Δ . It follows easily that we have

PROPOSITION 2.4. The space $\Lambda \mathfrak{U}^-$ admits an orthogonal direct sum decomposition into $\mathfrak{G}(A)_0$ submodules

$$\Lambda \mathfrak{U}^- = \mathbf{H} \oplus \text{Im } d \oplus \text{Im } \partial,$$

where $\mathbf{H} = \bigoplus_{i \geq 0} \mathbf{H}^i$. In particular, the spaces $\mathbf{H}_i$ and $H_i(\mathfrak{U}^-)$, the i th homology of the complex $(\Lambda \mathfrak{U}^-, \partial)$, are isomorphic $\mathfrak{G}(A)_0$ -modules.

We let C denote the Casimir operator of $\mathfrak{G}$ corresponding to the Killing form. The representation $\theta: \mathfrak{G} \rightarrow \text{End } \Lambda \mathfrak{U}^-$ induces a representation θ of the universal enveloping algebra of $\mathfrak{G}$. For our purposes, a central result is

THEOREM 2.5. The operator Δ is equal to $-D/2 - \theta(C)/2$.

Roughly, our proof of Theorem 2.5 goes as follows: Let G denote the connected, simply connected (compact) Lie group with Lie algebra $\mathfrak{R}$. We express both Δ and $-D/2 - \theta(C)/2$ in terms of the curvature of G , regarded as a symmetric space. Using identities of the type used in ref. 8, we observe that the expression for Δ collapses to that for $-D/2 - \theta(C)/2$ (the collapsing is considerable).

§3. Using Proposition 2.4 and Theorem 2.5, we can now describe $H_i(\mathfrak{U}^-)$ as a $\mathfrak{G}(A)_0$ -module. In fact, it suffices, by Proposition 2.4, to describe the isomorphic $\mathfrak{G}(A)_0$ -module $\mathbf{H}^i$. Thus, for $\sigma \in W_a$, we set $\alpha_\sigma = \sigma \cdot \alpha^- \cap \alpha^+$, and we let $\alpha(\mathfrak{U}^\pm)$ be the set of all $\alpha \in \alpha^\pm$ so that $\mathfrak{G}_\alpha \subseteq \mathfrak{U}^\pm$. We let W^1 be the set of all $\sigma \in W_a$ such that $\alpha_\sigma \in \alpha(\mathfrak{U}^+)$. If $\psi \subset \alpha^+$, is a finite subset, we set $\langle \psi \rangle = \sum_{\alpha \in \psi} \alpha$. We note that each α_σ is finite and has cardinality equal to $l(\sigma)$, the minimal integer k , such that σ can be written as a product of k S_i 's. One can show

[3.1] for $\sigma \in W^1$, each element $-\langle \alpha_\sigma \rangle \in \mathfrak{R}^*$ is dominant in the sense that $(-\langle \alpha_\sigma \rangle, \alpha_i) \geq 0, i = 1, \dots, l$.

We let $(\Lambda \mathfrak{U}^-)^{-\langle \alpha_\sigma \rangle}, \sigma \in W^1$, denote the sum of all irreducible $\mathfrak{G}(A)_0$ submodules with highest weight $-\langle \alpha_\sigma \rangle$. We say $\alpha \in \alpha$ is a real root, in case $\alpha = \sigma(\alpha_i)$ for some $\sigma \in W_a, i = 1, \dots, l + 1$. One can show that for each real root α , the space $\mathfrak{G}_\alpha$ is one-dimensional, and we then fix $e_\alpha \neq 0$ in $\mathfrak{G}_\alpha$. Let ψ be a finite subset of real roots in α^+ . We let $e_{-\langle \psi \rangle} = \sum_{\alpha \in \psi} e_{-\alpha}$, where we take the product in some fixed order.

We note that for all $\sigma \in W_a$, one can show that $\alpha_\sigma \subseteq \text{real roots}$. One can then prove

THEOREM 3.2. $\mathbf{H}^i = \bigoplus_{\substack{\sigma \in W^1 \\ l(\sigma) = i}} (\Lambda \mathfrak{U}^-)^{-\langle \alpha_\sigma \rangle}$. Moreover, for

each $\sigma \in W^1$, the representation of $\mathfrak{G}(A)_0$ with highest weight $-\langle \alpha_\sigma \rangle$ appears with multiplicity exactly one, and, up to a non-zero scalar, the vector $e_{-\langle \alpha_\sigma \rangle}$ is the unique corresponding highest weight vector.

As we remarked earlier the above decomposition for $\mathbf{H}^i$ as a $\mathfrak{G}(A)_0$ -module gives the same decomposition for $H_i(\mathfrak{U}^-)$, as a $\mathfrak{G}(A)_0$ -module.

Now in ref. 3, Macdonald has obtained certain formal identities for what he defines as an affine root system (roughly, the real roots of $\mathfrak{G}$ form an affine root system in the sense of Macdonald). Thus, if $\Phi \subseteq \mathfrak{G}$ denotes those roots which are linear combinations of $\alpha_1, \dots, \alpha_l$, if $\Phi^\pm = \mathfrak{G}^\pm \cap \Phi$, if $\gamma = \alpha_0 + \alpha_l$, and if $X = e^{-\gamma}$, then if one considers formal alternating sums of exponential characters of $\mathfrak{H}$, for $\Lambda\mathfrak{U}^-$ and for $\bigoplus_i H_i(\mathfrak{U}^-)$, one obtains from *Theorem 3.2* and Euler-Poincaré:

$$\prod_{n \geq 1} \prod_{\alpha \in \Phi^+} (1 - X^n e^\alpha) (1 - X^n e^{-\alpha}) (1 - X^n)^l = \sum_{\sigma \in W^1} \chi(-\langle \alpha_\sigma \rangle) (-1)^{l(\sigma)}. \quad [3.3]$$

Here, $\chi(-\langle \alpha_\sigma \rangle)$ is the sum of the exponential characters of R , which appear in the representation with highest weight $-\langle \alpha_\sigma \rangle$, of $\mathfrak{G}(A)_0$.

There is an alternate way to derive Macdonald's identity as a consequence of Euler-Poincaré. Namely, consider the subalgebra $\mathfrak{N}^- = \bigoplus_{\alpha \in \Phi^+} \mathfrak{G}_\alpha$, and let $\mathfrak{N}_{\mathfrak{G}}^-$ denote the subalgebra of $\mathfrak{G}(A)$, $\mathfrak{N}_{\mathfrak{G}}^- = \mathfrak{N}^- \oplus \mathfrak{U}^-$. Then $H_i(\mathfrak{N}_{\mathfrak{G}}^-)$ is an $\mathfrak{H}$ -module and one can obtain the following result from *Theorem 3.2*:

THEOREM 3.4. $H_i(\mathfrak{N}_{\mathfrak{G}}^-) = \bigoplus_{\substack{\sigma \in W^1 \\ l(\sigma) = i}} H_i(\mathfrak{N}_{\mathfrak{G}}^-) - \langle \alpha_\sigma \rangle$, where

$H_i(\mathfrak{N}_{\mathfrak{G}}^-) - \langle \alpha_\sigma \rangle$ is a one-dimensional subspace which, as an R -module, transforms according to the weight $-\langle \alpha_\sigma \rangle$.

From *Theorem 3.4* and Euler-Poincaré, one then obtains

$$\prod_{n \geq 1} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha} X^{n-1}) (1 - e^\alpha X^n) (1 - X^n)^l = \sum_{\sigma \in W_a} (-1)^{l(\sigma)} e^{-\langle \alpha_\sigma \rangle}, \quad [3.5]$$

which is precisely Macdonald's identity for the affine root system corresponding to the root system of $\mathfrak{G}$. It is not difficult to show that [3.3] and [3.5] are equivalent, without using cohomology theory.

We note that $H_i(\mathfrak{N}_{\mathfrak{G}})$ has finite dimension equal to the number of $\sigma \in W_a$ such that $l(\sigma) = i$. Moreover, if $p(t)$ denotes the Poincaré polynomial $p(t) = \sum_{i \geq 0} t^i \dim H_i(\mathfrak{N}_{\mathfrak{G}})$, then using Bott's result in ref. 9, we obtain

COROLLARY 3.6. $p(t^2)$ is the product of the Poincaré polynomial of $\Omega(G)$, the loop space of G , and the Poincaré polynomial of G/T , where $T \subseteq G$ is a maximal torus.

1. Kac, V. G. (1968) "Simple irreducible graded Lie algebras of finite growth," *Math. USSR-Izvestija* (English translation) 2, 1271-1311.
2. Moody, R. V. (1968) "A new class of Lie algebras," *J. Alg.* 10, 211-230.
3. Macdonald, I. G. (1972) "Affine root systems and Dedekind's η -function," *Inv. Math.* 15, 91-143.
4. Kac, V. G. (1974) "Infinite-dimensional Lie algebras and Dedekind's η -function," *Funktsional Analiz. i Ego Prilozhen.* 8, 77-78.
5. Kostant, B. (1961) "Lie algebra cohomology and the generalized Borel-Weil theorem," *Ann. Math.* 74, 329-387.
6. Kostant, B. (1963) "Lie algebra cohomology and generalized Schubert cells," *Ann. Math.* 77, 72-144.
7. Bourbaki, N. (1968) in *Groupes et Algèbres de Lie* (Hermann, Paris), chap. 4, 5, and 6.
8. Weil, A. (1962) "On discrete subgroups of Lie groups II," *Ann. Math.* 75, 578-602.
9. Bott, R. (1956) "An application of the Morse theory to the topology of Lie-Groups," *Bull. Soc. Math. Fr.* 84, 251-281.