Observational validation of the chronometric cosmology: I. Preliminaries and the redshift–magnitude relation

(galaxy/quasar)

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ABSTRACT Results of comprehensive observational tests of the cosmological theory proposed by Segal in 1972 are summarized in this and succeeding notes. Within a framework of natural and general causality and symmetry considerations, the theory is mathematically essentially unique. It implies definite relations between directly observed quantities (redshift, apparent magnitude and diameter, number counts, and the like) which are entirely free of adjustable parameters, such as the q_0 and Λ of general relativistic cosmologies. Rigorous statistical analysis of large (≥ 100) or systematically defined samples of galaxies or quasars accepts these relations at substantial probability levels. Note I establishes uniform statistical procedures and compares the redshift–magnitude relations of major samples with the theoretical relations of the present and expansion theories.

1. Operational cosmology

The chronometric theory (1–3) is one of a general class implying a redshift–distance relation: $z = F(r) + u$, where r is the distance to an extragalactic object of redshift z , F is a function specified by the theory, and u is a statistical variate representing the effect on redshift of the special ('peculiar') motion of the object. However, the distance r is in itself not directly observable.

The operational content of a cosmology resides rather in the corollary relations between directly observable quantities associated with extragalactic objects. Such quantities include the redshift z , the apparent luminosities l at specified observed frequencies or averaged with respect to given distributions thereof, the apparent angular diameters θ suitably associated with the distribution of this luminosity, number counts of the objects in specified redshift, magnitude, or angular diameter ranges, and the like. Redshifts, magnitudes, and variously defined angular diameters have been extensively observed and published over the past several decades, and are available for large samples of galaxies and substantial ones of quasars.

The proper course for the validation of a cosmological theory would appear to consist in the application of the theory of testing of statistical hypotheses to the comparison of its predictions with the relations observed for samples of the type cited. If the theory is thereby accepted, it is further subject to qualitative considerations of: compatibility with less statistical types of extragalactic observation and general physical ideas; freedom from isolated anomalies; and relative scientific economy and predictive capacity.

2. Observable stochastic relations

The application of general statistical principles to astronomical observations is complicated by a variety of special features. To minimize possible biases it seems useful to adopt such precautions as: (i) employment of model-independent

samples already published in the literature; (ii) adherence to uniform statistical procedures rather than variable ones derived from largely conjectural (even if physically not unreasonable) opinions about specific samples. (i) is straightforward, but (ii) requires development.

Consider the class of cosmological theories that imply relations of the form

$$z = F_1(r) + u, \quad l = F_2(r)(1 + z)^{-\alpha} L, \\ \theta = F_3(r, z) A, \dots \quad [1]$$

where F_1 , F_2 , and F_3 are given (possibly parameter-dependent) functions, and u , L , and $A, \dots$ are random variables (e.g., the peculiar motion, intrinsic luminosity and diameter, ...) associated with a random object of redshift z and spectral index α . In the case of the chronometric theory the redshift–distance relation is:

$$z = \tan^2(r/2) + u, \quad [2]$$

r being measured in suitable units; the geometry of the sphere then implies the further relations

$$l = L(1 + z)^{-\alpha} \sin^{-2} r, \quad \theta = A/\sin r. \quad [3]$$

In the case of homogeneous general relativistic models, the functions F_i depend additionally on the two parameters q_0 and Λ , and are well known. The values $q_0 = 1$ and $\Lambda = 0$ are representative of those generally adopted in the expansion theory and will be used here for illustrative purposes.

Due to the presence of the nonobservable quantity r in these relations, they cannot be directly subjected to observational test. In principle, the general procedure for making them so consists in the postulation of a (possibly parameter-dependent) distribution for r (possibly jointly with u , L , etc.); integration over r then yields relations involving only directly observable quantities whose joint probability distribution is in principle observable. But it is unnecessary to elaborate this procedure here, in view of the observational evidence that σ_u is negligible in comparison with the material values of z . This makes possible a considerable analytic simplification, which has been generally adopted: on neglecting the dispersion in u , r becomes a function of z alone, and all of the cited relations become observable, except that between z and r .

The most familiar of these relations, and one which is quite representative of those resulting from the Eqs. [3], is the redshift–magnitude relation

$$m = f(z) + 2.5(1 - \alpha)\log(1 + z) + M, \quad [4]$$

where the usual Pogson scale is employed, m being the apparent and M an absolute magnitude, and α the spectral index. In the chronometric theory, $f(z) = 2.5 \log z - 2.5 \log(1 + z)$; for the Friedmann model with $q_0 = 1$, the Hubble law $f(z) = 5 \log z$ is obtained. The prototypical question, and the chief one to which the remaining theoretical portion of this note is addressed is: how to test the hypothetical relation [4], for a given function $f(z)$, on the basis of published samples of redshift and magnitudes, in a statistically uniform, efficient, and consistent manner. The general method developed will be applicable to the similar relations derivable from the Eqs. [3], as well as to their conjunction. Astronomical sampling and selection problems are well represented by their impact on the testing of relation [4], which is treated next. The term $(1 - \alpha) \log(1 + z)$ is small in most applications (in which either z is small or α about 1), and particularly in comparison with the difference between the chronometric and Hubble laws; consequently it will be neglected in the following.

3. Problems of astronomical sampling

The most evident of the problems arising in the treatment of samples of apparently extragalactic objects is the inherent limitation on apparent magnitude, which results in a truncation that varies with redshift. Truncation is a familiar statistical problem, and may be treated here by adaptation of known general procedures. E.g., least-squares estimation may be appropriately extended to truncated samples in the form of maximum-likelihood estimation, employing the probability density

$$\Pi_{i=1} p(m_i, z_i; \mu, \sigma) \times \left[\int_{-\infty}^{m^*} p(m, z_i; \mu, \sigma) dm \right]^{-1} Y(m^* - m_i),$$

where $p(m, z; \mu, \sigma) = \exp[-(m - f(z) - \mu)^2 / 2\sigma^2]$, m^* is the limiting apparent magnitude, and Y is the Heaviside function ($= 1$ or 0 according as the variable is ≥ 0 or < 0). The resulting estimates of μ and the parameters (if any) on which the function $f(z)$ depends coincide with least-squares estimates in the ideal case $m^* = \infty$, and with the aid of high-speed computers may be obtained for other values of m^* . It should be recognized that only the bright end of the luminosity function is involved in statistical tests, and this portion of the luminosity function may be estimated directly from the observations. The resulting distributions are naturally model-dependent, but their dispersions may be estimated in a model-independent fashion, when, as in the cases of interest, $f(z)$ changes so slowly with z that adequately large samples can be formed which have nearly constant redshift. These estimates of the absolute dispersions in the truncated magnitude distributions at a fixed redshift show in fact only a slight dependence of these dispersions on redshift, in the cases of major samples of galaxies and quasars, outside local regions.

The most important of the remaining problems in the testing of the redshift-magnitude relation is the model-dependence of the appropriate aperture of measurement, i.e., one encompassing a fixed physical diameter at the distance theoretically indicated by the redshift of the object. This model-dependence is sufficiently strong that if small apertures are employed (thus avoiding the otherwise significant problems of contamination by the night sky and other luminous objects), it is impossible to distinguish observationally between different redshift-distance power laws in the low-

to-moderate redshift range in which all but a few galaxy observations presently fall, as a theoretical computation (3) shows. In the case of bright cluster galaxies, which have evolved in the work of Hubble and his successors as their preferred test objects, the model-dependence of the magnitude may well be enhanced by the relatively large and apparently variable size of the galaxies employed, and by ambiguity concerning the level of the cluster background luminosity to be subtracted.

4. Statistical analysis of redshift-magnitude samples

Consider a sample (z_i, m_i) , $i = 1, \dots, n$ of redshift-magnitude measurements of objects of a given type. It is assumed that this type is objectively designated in a way that is inherently independent of z , or alternatively that the hypothesis of such independence has been separately validated, or has been conjoined with the basic theoretical hypothesis under test. It is assumed also that the sample is either a random selection from the objects in space in designated ranges (of redshift, magnitude, and field of sky), or is at least random at each fixed redshift, i.e., the sample is valid for the estimation of the conditional distribution (up to the observable limit) of m for given z . The latter assumption holds, e.g., for a sample which is complete in z and m in a given region of the sky out to a specific limiting apparent magnitude, or is a randomized subsample of one which is complete in apparent magnitude.

The problem of testing the hypothesis $m = f(z) + M$, where $f(z)$ is a given function, is commonly approached through the comparison of the dispersions in apparent and absolute magnitudes. This clear-cut elementary procedure has the limitation that it provides no quantitative probability of obtaining a reduction in dispersion (if any) of the size actually observed; nor does it take into account the fact that the expected reduction σ_f^2 in variance is model-dependent. E.g., for the Hubble law the expected reduction is four times that for the chronometric theory, at low redshifts.

A useful statistic, whose form and distribution are effectively independent of both the luminosity function and the hypothesis under test, can be developed by successive adaptations of the simple statistic representing the reduction in variance of the theory in question. Consider initially the case in which the distribution of M is independent of z , so that in particular z -dependent truncation effects are absent; let the mean of the distribution be denoted μ and its variance σ^2 . The change (putative reduction) in variance from the apparent to the absolute magnitudes, using the population rather than samples means, is $Y = N^{-1}[\Sigma(m_i - \bar{F} - \mu)^2 - \Sigma(M_i - \mu)^2]$, where $\bar{F} = N^{-1}\Sigma F_i$, $F_i = f(z_i)$, and all sums Σ are taken over the range $i = 1, \dots, n$.^{*} As noted, the expectation of Y is model-dependent, and it seems natural (especially for comparative purposes) to consider the modified statistic $Y' = Y - \langle Y \rangle$ which is found to have the value

$$Y' = N^{-1}[\Sigma(m_i - \bar{F} - \mu)^2 - \Sigma(M_i - \mu)^2 - \Sigma(F_i - \bar{F})^2] = 2N^{-1}\Sigma M_i(F_i - \bar{F})$$

The variates Y' for different hypotheses, while comparable in zero point, are however not comparable in scale, due to

^{*} Throughout the present article, the redshifts z_i are considered as non-stochastic variables, in keeping with normal astronomical procedure. Alternatively, we are concerned here only with the conditional distribution of m for given z , and not with the distribution of z . Questions of the latter type will be treated in the sequel.

model-dependence of the variance of Y' . It seems useful therefore to introduce in turn the further modification $Y'' = Y' / \langle Y'^2 \rangle^{1/2}$, which will retain vanishing expectation and have the additional feature of unit variance. It is found that $\langle Y'^2 \rangle = 4N^{-1}\sigma^2\sigma_f^2$, where $\sigma_f^2 = N^{-1}\Sigma(F_i - \bar{F})^2$; thus

$$Y'' = N^{1/2}[\sigma_m^2 - \sigma_M^2 - \sigma_f^2]/(2\sigma_M\sigma_f),$$

where

$$\sigma_M^2 \equiv N^{-1}\Sigma(M_i - \mu)^2$$

and

$$\sigma_m^2 \equiv N^{-1}\Sigma(m_i - \mu - \bar{F})^2.$$

By the central limit theorem, the distribution of Y'' approaches a normal distribution, of mean zero and unit variance, as $N \rightarrow \infty$. On the other hand, the population mean and dispersion are in practice not known; therefore Y'' is not an observed quantity; however, μ and σ^2 may be consistently estimated from the sample by $\bar{M} = N^{-1}\Sigma M_i$ and $s_M^2 = N^{-1}\Sigma(M_i - \bar{M})^2$, where $M_i = m_i - F_i$. If these sample estimates are substituted for the exact population parameters, a new statistic Y''' is obtained which is entirely observable; and retains the feature of being asymptotically normally distributed with vanishing mean and unit variance. Specifically,

$$Y''' = N^{1/2}[s_m^2 - s_M^2 - s_f^2]/(2s_Ms_f)$$

where $s_m^2 = N^{-1}\Sigma(m_i - \bar{m})^2$, $\bar{m} = N^{-1}\Sigma m_i$; $s_M^2 = N^{-1}\Sigma(M_i - \bar{M})^2$, with $M_i = m_i - F_i$ and $\bar{M} = N^{-1}\Sigma M_i$; and $s_f = \sigma_f$, for notational uniformity. This statistic Y''' has all of the essential attributes indicated earlier; its negative, $X = -Y'''$, may appropriately be called the 'normalized residual variance'; it is a monotone increasing function of the crude residual variance s_M^2 , for fixed values of s_m and s_f such as generally occur, but unlike s_M^2 has a universal, and luminosity-function independent, asymptotic probability distribution.

If the mean $\mu(z)$ and dispersion $\sigma(z)$ of the objects observed at redshift z in this range are materially z -dependent the statistic Y' has nonvanishing expectation value, and must be re-centered as $\tilde{Y}' = Y' - \langle Y' \rangle = 2N^{-1}\Sigma(M_i - \mu_i)(F_i - \bar{F})$, where $\mu_i = \mu(z_i)$. Proceeding as earlier in order to normalize the scale $\tilde{Y}'$ yields the variate

$$\tilde{Y}'' = \tilde{Y}' / \langle \tilde{Y}'^2 \rangle^{1/2} \\ = \Sigma(M_i - \mu_i)(F_i - \bar{F}) / [\Sigma\sigma_i^2(F_i - \bar{F})^2]^{1/2},$$

where $\sigma_i = \sigma(z_i)$, now recognizable as analogous to a Pearson correlation coefficient. Taking the negative of $\tilde{Y}''$ and replacing the population parameters by suitable consistent sample estimates $\bar{M}_i$ and s_i^2 , a generalization $\tilde{X}$ of X is obtained:

$$\tilde{X} = -\Sigma(M_i - \bar{M}_i)(F_i - \bar{F}) / [\Sigma s_i^2(F_i - \bar{F})^2]^{1/2}$$

having the same essential features of X . If, e.g., the earlier-indicated truncated-least-squares estimation procedure is used, $\tilde{X}$ may properly be computed with $\mu(z)$ and $\sigma(z)$ taken as the mean and dispersion of the truncation at $M = m^* - f(z)$ of the normal law having the estimated parameters.

5. The trend with redshift in expansion-theoretic absolute magnitudes

In practice, there may be material z -dependence of the mean and dispersion of the truncated samples, but the inher-

ent inaccessibility of extragalactic objects implies that there can be no *a priori* knowledge about this question. The operational question here is whether the absolute magnitudes $M = m - f(z)$ form a population whose mean and dispersion depend strongly or only marginally on z . This is an f -, and hence model-, dependent question; indeed, if z -independent for one theory, the distribution of M is predictably z -dependent for a different theory. Analytically, a sample which is representative according to *Theory 1*, with associated function f_1 , for which the absolute magnitudes $M_1 = m - f_1(z)$ have a distribution which has no significant dependence on z , should have a trend in the *Theory 2* absolute magnitudes $M_2 = m - f_2(z)$ given by the function $f_1(z) - f_2(z)$. Precisely these circumstances are observed in the major galaxy sample (4), with *Theories 1* and *2* taken as the chronometric and expansion theories, and independently of whether the entire redshift range of the sample is included, or only the relatively well-observed and local-anomaly-free region $500 \leq cz \leq 2000$; or only the first 50, 100, or 200 galaxies in order of redshift, these being minimally affected by the observational magnitude cutoff; or whether galaxies are selected on morphological type (ellipticals, spirals, lenticulars, or various cross- and sub-divisions); or limited to the northern or southern celestial or galactic hemispheres, etc.

More generally, consider any sample, which is complete, or representative of a complete sample, out to a limiting magnitude m^* and redshift z^* ; and suppose that for *Theory 1*, the degree of truncation involved is negligible, so that the distribution of the absolute magnitudes $M_1 = m - f_1(z)$ shows no significant z -dependence. This sample can appropriately be used to validate *Theory 1*, by consideration of the X -statistic and other means. However, it will not necessarily invalidate *Theory 2*, for the mean and/or variances of its absolute magnitudes $M_2 = m - f_2(z)$ may be highly z -dependent,—i.e., *on the hypothesis that Theory 2 holds*, the sample may appear quite biased. The X -statistic affords a proper *phenomenological* indication of the quality of the fit of the theory to the actual observed sample; but, on a *hypothesis-testing basis*, *Theory 2* is entitled to attempt to impugn the validity of the sample, and thereby avoid rejection. There results a somewhat paradoxical situation, in that the worse the apparent fit of a theory to the sample, as represented by a strong trend in $\mu(z)$, the more difficult it appears to be to reject the theory definitively; for the strength of the trend is in monotone relation to the basis for impugning the fairness of the sample. However, it is an extremely unusual luminosity function for which successive truncations show a strong trend in mean without a corresponding trend in variance. Thus, the burden of proof would appear to be on *Theory 2*, to show that a physically reasonable luminosity function exists which can exhibit this feature, as well as to explain the coincidence that the truncations involved in most published samples are precisely in the region of stationary truncated dispersion. Indeed, 'Theory 2' is statistically not properly defined until it can operationally specify its luminosity function.

Moreover, the trend is so strong as to persist into expansion-theoretically proper subsamples, and thereby to tend to invalidate the theory on its own terms. Consider specifically the subsample of objects in the original sample which are sufficiently bright, *on the Theory 2 hypothesis*, to be included in the sample if at the larger redshift z^* . Analytically, all (m, z) pairs are included for which $z \leq z^*$ and $m - f_2(z) \leq m^* - f_2(z^*)$; here z^* is arbitrary, but for statistical efficiency and specificity may be chosen to maximize the

FIG. 1. The quality of the fit of theories to observed redshift-magnitude relations of large or statistically precisely delineated samples. The quantity X is approximately normally distributed with zero mean and unit variance, for a fair sample, and a correct theory. The variances σ^2 are in the apparent magnitude, and in the residuals from the chronometric and Hubble law predictions in each case, the zero points of the predictions being adjusted to the mean luminosity of each sample. The tendency toward order-of-magnitude equality of the variances in apparent magnitude and in residuals from the Hubble law for galaxy samples (or others at $z \leq 0.5$) is predicted by the chronometric theory, as is the larger variance in these residuals for larger-redshift samples. The data are described in refs. 4, 5, and 10-14. X -values in excess of 6 are given numerically above the peak shown at $X = 6$.

number N of such objects. On the *Theory 2* hypothesis, the absolute magnitudes of this subsample should be z -independent; in particular the mean of the first (in order of redshift) $[N/2]$ objects should not differ significantly from that of the next $[N/2]$ objects. In actual fact, the difference is significant at the 0.0025 probability level, for the sample (4), at conservative limiting magnitudes m^* such as 12.75 or 12.5, with $f_2(z) = 5 \log z$, and with or without deletion of the local region $cz < 500$.[†]

[†] The trend is in close agreement with that predicted, for the mean magnitudes of expansion-theoretically proper subsamples, by the chronometric theory with a truncated normal luminosity function whose parameters are fitted by maximum-likelihood estimation, i.e., the function of z given by the mean of $M + f_1(z) - f_2(z)$ in the range $M < m^* - f_2(z^*) + f_2(z) - f_1(z)$.

FIG. 2. The same as in Fig. 1, for samples that are smaller or may otherwise be less cogent. The data are described in refs. 5 and 15-20, Fig. 1 of ref. 7, and Fig. 1 of ref. 14.

6. Statistical summary

Figs. 1 and 2 show the basic statistics of the $m-z$ relations of most of the major published samples relevant to the validation of the chronometric hypothesis. The phenomenological comparison between the chronometric and expansion theories appears remarkably one-sided in favor of the former. The only exception to this is an older sample of brightest cluster galaxies; but, in addition to the factors cited at the end of Sec. 3, a not wholly successful attempt to distinguish between cD galaxies (supergiant ellipticals) to be deleted from the sample and normal giant ellipticals considered appropriate for inclusion would clearly bias the sample against the chronometric theory in the direction ($X < 0$) which is found; compare ref. 5.

It is relevant to note also that a model-independent and practical definition of "cluster" is not available, and that bright cluster galaxy samples have evolved historically as a means of validating the Hubble law. In his original paper (7), Hubble made no analysis of his redshift-magnitude data, and concluded solely on the basis of distance estimates which he described as for the most part "subject to considerable probable errors but . . . believed to be the most reasonable values at present available," that the results "establish a roughly linear relation between velocities and distances"; but compare Fig. 2.

The results for galaxy samples selected on region of sky refer to the coordinates defined by de Vaucouleurs in connection with his supergalaxy hypothesis (8). The regions in question were proposed by de Vaucouleurs as ones indicated by the hypothesis as relatively likely to conform to different

FIG. 3. Redshift-magnitude relations of bright extragalactic objects. References regarding data are given under Figs. 1 and 2; also ref. 21.

redshift-distance power laws. However, in the region $|B| < 30$ most favorable to the Hubble law, according both to the supergalaxy hypothesis and the present results, the chronometric law provides a distinctly better fit and has indeed a residual dispersion only 6% in excess of the minimal possible for a power law. If regions are selected without reference to the supergalaxy hypothesis, the results are quite strongly indicative of a uniform redshift-distance square law (more exactly, its observable implications) in all regions of the sky.

7. Nonstatistical discussion

Closely associated with the $m-z$ relation are several phenomena which appear quite anomalous from the expansion-theoretic standpoint, among them the following.

(i) **Energetics of Quasars.** On the expansion hypothesis, quasars are extraordinarily bright and fast-moving; thus 0Q172 is about 50,000 times as bright as the average nearby bright galaxy (specifically, average of 50 galaxies of lowest redshift from ref. 4), and has velocity $>0.9c$. On the chronometric hypothesis, the dispersion in intrinsic luminosity

among galaxies and quasars combined is comparable to that among household light bulbs, and they need not be moving at all. Fig. 3 gives an idea of the excellent overall fit to the $m-z$ relation of the general run of extragalactic objects provided by the chronometric theory.

(ii) **Superluminal Velocities.** The quasar 3C279 was estimated (8) to have an apparent velocity about $10c$, if at the distance predicted by the Friedmann model with $q_0 = 1$ and a Hubble parameter of $75 \text{ km s}^{-1} \text{ mpc}^{-1}$. On the chronometric hypothesis, with R inferred by using the same value of H at a distance of 10 mpc, the apparent velocity of this quasar is about $0.5c$. Other strikingly large apparent velocities of extragalactic objects are similarly reduced to distinctly subluminal levels.

(iii) **Brightness of ScI Galaxies at Different Redshifts.** Rubin *et al.* (9) have reported a relatively marginal difference in brightness between two groups of galaxies thought to be of fairly uniform intrinsic luminosity, although the redshifts of the groups are quite different. The formal deviation at a probability level of 10^{-6} (5σ) on the expansion theory is reduced to one of probability 0.05 (2σ) on the chronometric theory.

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