On the Possibility of a Hydrodynamic Model of the Electron

(classical model of the electron/hydrodynamic balancing of Coulomb repulsion)

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ABSTRACT We explore the possibility that the mutual repulsive forces of a uniformly charged sphere could be kept in balance dynamically by a steady circulation of the material, which is assumed to be a nonconducting perfect fluid of uniform density. An exact solution is obtained of Maxwell's equations and of the hydrodynamic equations in the nonrelativistic approximation, which satisfies the boundary conditions on the surface of the sphere. In this solution all the components of the velocity and of the magnetic field are found to vanish on the surface, but not the electric field. The pressure can also be made to vanish on the surface, but in the interior it turns out to be *negative*, which makes the present solution unacceptable.

Interest in the classical model of the electron of Abraham (1) and Lorentz (2), consisting of a charged sphere of finite radius a , has recently been revived with the aim of testing the supposition (3) that the *muon* is an excited state of the electron. A point-model of the electron precludes an excited state. The difficulty with the classical model of the electron is the need to postulate the existence of forces of nonelectromagnetic origin which will compensate for the mutual repulsion of the charges, the so-called "Poincaré stresses" (4-6). Thus, in Dirac's "bubble model" of the electron (3), the charge is concentrated on the surface, and a surface tension is invoked in order to prevent the charges from flying apart.

We wish to explore the possibility that the repulsive forces between the charges are held in balance by a *dynamic pressure* generated by a steady circulation of the material inside the sphere, which is assumed to be a perfect liquid. The material of the liquid filling the sphere of radius a , assumed to be incompressible, is specified by a mass density μ and a charge density ρ . Both μ and ρ are assumed, in the first instance, to be uniform, so that

$$(4\pi/3)\rho a^3 = e, \quad (4\pi/3)\mu a^3 = m, \quad [1]$$

e and m being identified with the charge and mass of electron. The electric field $\mathbf{E}$, the magnetic field $\mathbf{H}$, and the velocity field $\mathbf{u}$ are assumed to be steady, and we solve Maxwell's equations as well as the hydrodynamic equations, subject to the boundary condition of the vanishing at the surface $r = a$ of the pressure p , the radial components of $\mathbf{u}$, and of the electric current $\mathbf{j}$, while the magnetic field is made to match a magnetic potential-field outside the sphere. The basic assumption is that the velocity vector $\mathbf{u}$ and the current vector $\mathbf{j}$ are parallel and that, indeed,

$$\mathbf{j} = \rho \mathbf{u}. \quad [2]$$

In this initial attack, the hydrodynamic equations are taken in the nonrelativistic approximation, which constitutes a serious limitation. Still, it is of interest that the Maxwell

equations, including the nonlinear term representing the Lorentz force, and the nonlinear nonrelativistic hydrodynamic equations admit of an exact solution that more than satisfies the boundary conditions. I say "more than satisfies" because the convective vortex inside the sphere turns out to be such as to yield the vanishing of *all three* of the components of velocity and of the magnetic field on the surface, in addition to the vanishing of the radial component of the current. On the surface of the sphere there is, therefore, no sign of the existence of convection or of a magnetic field in the interior, and externally the model manifests itself only by exhibiting a radial electric field equal to (e/r^2) .

The solution for the convective model yields an expression for the pressure that must be made to vanish at the surface $r = a$, since outside the sphere the vacuum cannot sustain any pressure. This surface condition we are able to meet, showing that no "Poincaré stresses" need be introduced at the surface. However, the resulting pressure at the center of the sphere turns out to be *negative*. We have, therefore, to postulate that the perfect liquid can sustain a negative pressure, which is a "Poincaré stress" under a new guise. We must admit that the solution obtained for the hydrodynamic model, in its present nonrelativistic formulation, fails. It would seem that at this stage it would be premature to share Abraham's faith in his electromagnetic *Weltanschauung* (7) and to embrace a hydromagnetic *Weltanschauung*.

THE HYDROMAGNETIC EQUATIONS

We wish to find a stationary solution of Maxwell's equations

$$\text{curl} \mathbf{E} = -(1/c)(\partial \mathbf{H} / \partial t), \quad [3]$$

$$\text{curl} \mathbf{H} = (1/c)(\partial \mathbf{E} / \partial t) + (4\pi/c)\mathbf{j}, \quad [4]$$

$$\text{div} \mathbf{E} = 4\pi\rho, \quad [5]$$

$$\text{div} \mathbf{H} = 0, \quad [6]$$

and of the hydrodynamic equation for a perfect fluid:

$$(\partial \mathbf{u} / \partial t) + \boldsymbol{\omega} \times \mathbf{u} = -\text{grad}[(p/\mu) + \frac{1}{2}\mathbf{u}^2] + \mathbf{F}, \quad [7]$$

where $\mathbf{F}$ represents the Lorentz force per unit mass:

$$\mathbf{F} = (\rho/\mu)\mathbf{E} + (1/c\mu)\mathbf{j} \times \mathbf{H}, \quad [8]$$

$$\boldsymbol{\omega} = \text{curl} \mathbf{u}, \quad [9]$$

and

$$\text{div} \mathbf{u} = 0. \quad [10]$$

With the right-hand side of [3] vanishing, we can put

$$\mathbf{E} = -\text{grad} \Phi, \quad [11]$$

and, since ρ in [5] was assumed to be constant, we have

$$\Phi = (e/2a)[3 - (r^2/a^2)], \quad 0 < r < a, \quad [12]$$

$$\Phi = e/r, \quad r > a, \quad [13]$$

as a complete solution for the electric field. It follows from [4] and [8] that

$$\text{curl} \mathbf{H} = (4\pi/c)\mathbf{j} = (4\pi\rho/c)\mathbf{u}, \quad [14]$$

$$\mathbf{F} = -(\rho/\mu)\text{grad}\Phi + (\rho/\mu c)\mathbf{u} \times \mathbf{H}, \quad [15]$$

whereby [7] can be put in the form

$$\mathbf{P} \equiv [\boldsymbol{\omega} + (\rho/\mu c)\mathbf{H}] \times \mathbf{u} = -\text{grad}Q, \quad [16]$$

with

$$Q = (p/\mu) + \frac{1}{2}\mathbf{u}^2 + (\rho/\mu)\Phi. \quad [17]$$

In a spherical system of coordinates (r, θ, ϕ) , the boundary conditions to be met at the surface of the sphere $r = a$ are

$$u_r(a) = 0, \quad [18]$$

$$j_r(a) = 0, \quad [19]$$

$$p(a) = 0, \quad [20]$$

and a condition requiring that the internal magnetic field should match an external potential field.

AXIALLY SYMMETRIC SOLUTIONS

On the assumption that all the variables are independent of longitude ϕ , the velocity field, satisfying [10], can be derived from a stream function Ψ :

$$u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \Psi}{\partial \theta}, \quad [21]$$

$$u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial r}. \quad [22]$$

Similarly, [6] can be satisfied by putting

$$H_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad [23]$$

$$H_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}. \quad [24]$$

Since, by assumption, $\partial Q/\partial \phi$ vanishes, the consequent condition of the vanishing of P_ϕ yields the relation

$$r \sin \theta u_\phi + (\rho/\mu c)\psi = f(\Psi), \quad [25]$$

$f(\Psi)$ being an arbitrary function of the convection stream function Ψ . Substituting for $\text{curl} \mathbf{H}$ in [14], we arrive at

$$H_\phi = (4\pi\rho/cr \sin \theta)\Psi, \quad [26]$$

$$\begin{aligned} \frac{\partial^2 \psi}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta} \right) \\ = -\frac{4\pi\rho}{c} r \sin \theta u_\phi = \lambda^2 \psi - \frac{4\pi\rho}{c} f(\Psi), \end{aligned} \quad [27]$$

where

$$\lambda^2 = (4\pi\rho^2/\mu c^2). \quad [28]$$

Similarly the P_r and P_θ components of Eq. [16] yield

$$Q = g(\Psi), \quad [29]$$

$g(\Psi)$ being an arbitrary function of Ψ , and

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \Psi}{\partial \theta} \right) - \lambda^2 \Psi + \dot{f} \left[-\frac{\rho}{\mu c} \psi + f \right] \\ = \dot{g}(\Psi) r^2 \sin^2 \theta, \end{aligned} \quad [30]$$

the dot signifying, here and elsewhere, differentiation with respect to the argument.

Eqs. [27] and [30] form a simultaneous system of differential equations for the magnetic and convective stream functions ψ and Ψ . The material constants enter in the parameter λ^2 and in the scaling factor $(\rho/\mu c)$, the former being invariant to the sign of the charge through the factor ρ^2 . The components of $\mathbf{u}$ are expressed in terms of these functions by [21], [22], and [25], while the components of the magnetic field are given by [23], [24], and [26].

THE LINEAR CASE

We shall attempt to obtain a solution of Eqs. [27] and [30] for the case when $f(\Psi)$ and $g(\Psi)$ are assumed to be linear functions of Ψ :

$$f(\Psi) = \beta\Psi, \quad [31]$$

$$g(\Psi) = \alpha + \gamma\Psi, \quad [32]$$

where α , β , and γ are constants. Under these assumptions, both Eqs. [27] and [30] become linear in Ψ and ψ . We note that the basic solution of the homogeneous equation

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \psi}{\partial \theta} \right) + \lambda^2 \psi = 0, \quad [33]$$

which is finite at the origin, is (8)

$$\psi_n = -K_n(\pi\lambda r/2)^{1/2} J_{n+1/2}(\lambda r) \sin \theta (\partial P_n / \partial \theta). \quad [34]$$

We shall further restrict the discussion to the case $n = 1$ when the left-hand side of [30] has a factor $\sin^2 \theta$ which matches this factor appearing on the right-hand side. Writing then

$$\Psi = S(r) \sin^2 \theta, \quad \psi = s(r) \sin^2 \theta, \quad [35]$$

we find that the functions $S(r)$ and $s(r)$ are governed by the equations

$$S^* + (\lambda^2 - \beta^2)S = -(\beta\rho/\mu c)s - \gamma r^2, \quad [36]$$

$$s^* + \lambda^2 s = (4\pi\rho\beta/c)S, \quad [37]$$

where

$$S^* = -(d^2 S/dr^2) + (2/r^2)S. \quad [38]$$

Eliminating s between [36] and [37], we get

$$S^{**} + (2\lambda^2 - \beta^2)S^* + \lambda^4 S = -\lambda^2 \gamma r^2, \quad [39]$$

whose solution is

$$S = Aj(k_1 r) + Bj(k_2 r) - (\gamma/\lambda^2)r^2, \quad [40]$$

where

$$j(x) = (\pi x/2)^{1/2} J_{3/2}(x) = (\sin x/x) - \cos x, \quad [41]$$

$$k_1^2 = \frac{1}{2}\beta^2 - \lambda^2 + (\frac{1}{4}\beta^4 - \beta^2\lambda^2)^{1/2}, \quad [42]$$

$$k_2^2 = \frac{1}{2}\beta^2 - \lambda^2 - (\frac{1}{4}\beta^4 - \beta^2\lambda^2)^{1/2}. \quad [43]$$

From [36] it now follows that

$$(\rho/\mu c)s = Ak_2 j(k_1 r) + Bk_1 j(k_2 r) - (\gamma\beta/\lambda^2)r^2. \quad [44]$$

TABLE 1. The roots x_n of $J_{1/2}(x_n) = 0$.

n	1	2	3	4	5
x_n	5.763459	9.095011	12.322941	15.514603	18.689036
$j(x_n)$	-0.954126	0.981722	-0.990077	0.993750	-0.995696

$$j(x) = [(\sin x/x) - \cos x].$$

With S and s known, the velocity and magnetic field components are given by

$$u_r = (2/r^2)S \cos \theta, \quad [45]$$

$$u_\theta = -(\dot{S}/r) \sin \theta, \quad [46]$$

$$u_\phi = (1/r)[\beta S - (\rho/\mu c)s] \sin \theta, \quad [47]$$

$$H_r = (2/r^2)s \cos \theta, \quad [48]$$

$$H_\theta = -(\dot{s}/r) \sin \theta, \quad [49]$$

$$H_\phi = (4\pi\rho/rc)S \sin \theta. \quad [50]$$

Putting

$$k_1 a = x, \quad k_2 a = y, \quad [51]$$

we find that

$$\beta a = x + y, \quad \lambda^2 a^2 = xy. \quad [52]$$

The constants A , B , x , and y must be determined so as to fit the boundary conditions. Both [18] and [19] are satisfied by putting

$$S(a) = 0, \quad [53]$$

while the condition of the matching of the magnetic field to an external potential field requires that (9)

$$a\dot{s}(a) + s(a) = 0. \quad [54]$$

Putting now $p(a) = 0$ in [17], we find, with $\Psi(a)$ vanishing by [53], that

$$\alpha = (\rho e/\mu a) + \frac{1}{2}(\sin^2 \theta/a^2)[\dot{S}^2(a) + (\rho/\mu c)^2 s^2(a)]. \quad [55]$$

The bracketed term in [55] must vanish, since

$$\sin^2 \theta = (2/3)[1 - P_2(\cos \theta)], \quad [56]$$

so that the coefficient multiplying $P_2(\cos \theta)$ in [55] must vanish. Hence, the condition of the vanishing of the pressure on the surface requires that

$$\dot{S}(a) = 0, \quad [57]$$

$$s(a) = 0, \quad [58]$$

which, by [54], also implies that

$$\dot{s}(a) = 0. \quad [59]$$

This means that all the components of $\mathbf{u}$ and of $\mathbf{H}$ given by Eqs. [45]–[50] must vanish on the surface.

With

$$\tau \equiv \gamma a^4, \quad [60]$$

conditions [53] and [57] require that

$$A j(x) + B j(y) = \tau/xy, \quad [61]$$

$$A x j(x) + B y j(y) = 2\tau/xy, \quad [62]$$

while conditions [58] and [59] lead to

$$A y j(x) + B x j(y) = \tau(x + y)/xy, \quad [63]$$

$$A \dot{j}(x) + B \dot{j}(y) = 2\tau(x + y)/x^2 y^2. \quad [64]$$

It can be verified that conditions [61]–[64] are all met if x and y are different roots of the equation

$$x \dot{j}(x) - 2j(x) = 0, \quad J_{1/2}(x) = 0. \quad [65]$$

Putting now

$$z = r/a, \quad [66]$$

we finally get

$$S(r) = \frac{\tau}{y-x} \left[\frac{j(xz)}{xj(x)} - \frac{j(yz)}{yj(y)} \right] - \frac{\tau}{xy} z^2, \quad [67]$$

$$(\rho a/\mu c)s(r) = \frac{\tau}{y-x} \left[\frac{yj(xz)}{xj(x)} - \frac{xj(yz)}{yj(y)} \right] - \frac{\tau(x+y)}{xy} z^2. \quad [68]$$

SOLUTION FOR THE PRESSURE

We note from [55] that, with the bracketed term vanishing,

$$\alpha = \rho e/\mu a = e^2/ma. \quad [69]$$

We now substitute from [12], [29], [32], [35], and [45]–[47] into [17] to get the pressure:

$$\begin{aligned} (e^2/ma) + (\tau/a^4)S(r) \sin^2 \theta &= (p/\mu) + (e^2/2am)(3 - z^2) \\ &+ (2/r^4)S^2 \cos^2 \theta + \frac{1}{2}(\dot{S}/r)^2 \sin^2 \theta + \frac{1}{2}[\beta(S/r) \\ &- (\rho/\mu c)(s/r)]^2 \sin^2 \theta. \end{aligned} \quad [70]$$

It can be shown that [70] yields a negative value for the pressure $p(0)$ at the center of the sphere. Since for small x , $j(x)$, given by [41], approaches $(x^2/3)$, it follows from [67] and [68] that for small r ,

$$S(r) \rightarrow Kr^2, \quad s(r) \rightarrow Lr^2. \quad [71]$$

Hence, near the center, the surviving terms in [70] are

$$(e^2/ma) \simeq [p(0)/\mu] + (3e^2/2am) + 2K^2 + O(r^2), \quad [72]$$

showing that

$$p(0) < 0. \quad [73]$$

It should be noted that in the Hicks vortex, also, the pressure at the center is negative if its value on the surface is zero. The electric repulsion further lowers the pressure at the center.

One can derive a solution of the hydromagnetic equations for the case when the charged sphere rotates with constant angular velocity Ω about an axis passing through the center. Although the rotation contributes a positive term of $\frac{1}{2}\Omega^2 r^2 \sin^2 \theta$ to (p/ρ) , still at the center the pressure remains negative.

DISCUSSION

The solution [67] and [68] contains an amplitude factor τ which remains arbitrary. One could try to relate τ to the observed spin and magnetic moment of the electron. However, the solution obtained yields zero values for both the spin and the magnetic moment. Presumably in a relativistic solution, τ will be related to a^2c , but that remains to be seen.

If we evaluate the electric energy E_{ei} from [11] to [13], we find

$$E_{ei} = 4\pi \int_0^\infty (E^2/8\pi)r^2dr = (3/5)(e^2/a), \quad [74]$$

of which 1/6 is contributed by the electric field inside the sphere. Now substituting from [28] into [52], we get

$$(e^2/a) = mc^2(xy/3), \quad [75]$$

which yields

$$E_{ei} = (xy/5)mc^2. \quad [76]$$

Taking $x = 5.763$, $y = 9.095$ from Table 1, we find

$$E_{ei} = 10.48mc^2. \quad [77]$$

The magnetic energy contains the factor τ^2 , which is arbitrary. Whether a relativistic theory will yield a solution with positive pressure throughout and whether the "mass" of the model will come out close to the observed mass of the muon of $207m$ remains to be seen.

ELECTRICALLY CONDUCTING MEDIUM

In our discussion, we have assumed that the electrical conductivity of the material vanishes. If the conductivity of the

material is κ , then Eq. [2] must be replaced by

$$\mathbf{j} = \rho\mathbf{u} + \kappa[\mathbf{E} + (1/c)\mathbf{u} \times \mathbf{H}]. \quad [78]$$

In the dynamo theory of the earth's magnetic field and in the theory of magnetic stars, the term $\rho\mathbf{u}$ in [78] is neglected in comparison with the conduction terms. Axisymmetric solutions of the magnetohydrodynamic equations for a conducting material inside a sphere have been given by Chandrasekhar (10, 11). It is interesting that Prendergast (12) has essentially rediscovered Hicks's spherical vortex (8), and this not for the first time. The work of Ferraro (12) raises the question whether an oblate spheroidal shape of the model would ameliorate the negativity of the pressure in the interior.

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