On completeness of the Bergman metric and its subordinate metric

(Carathéodory differential metric/Weil-Petersson metric/Teichmüller space)

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ABSTRACT It is proved that on any bounded domain in the complex Euclidean space C^n the Bergman metric is always greater than or equal to the Carathéodory distance. This leads to a number of interesting consequences. Here two such consequences are given. (i) The Bergman metric is complete whenever the Carathéodory distance is complete on a bounded domain. (ii) The Weil-Petersson metric is not uniformly equivalent to the Bergman metric in the Teichmüller space $T(g)$ of any Riemann surface of genus $g \geq 2$.

Let D be any bounded domain in the complex Euclidean space C^n ($n \geq 1$). Then D admits both the Bergman metric s_D and the Carathéodory differential metric α_D . The main purpose of this note is to prove the following

THEOREM 1. *Let D be a bounded domain in C^n . For each $z \in D$ and each $\xi \in C^n$*

$$\alpha_D(z, \xi) \leq s_D(z, \xi).$$

Let ρ_D and d_D be the distance functions on D which are induced from α_D and s_D , respectively. Then the Carathéodory distance c_D satisfies:

$$c_D \leq \rho_D \leq d_D$$

From this observation, we have

THEOREM 2. *Let D be any bounded domain C^n . Then the Bergman metric s_D is complete in D if the Carathéodory distance c_D is complete.*

Recently, S. Wolpert in ref. 1 and T. Chu have independently proved that the Weil-Petersson metric is not complete in the Teichmüller space $T(g)$ of a compact Riemann surface of genus $g \geq 2$. On the other hand, it follows from Theorem 2 and the completeness of the Carathéodory distance in $T(g)$ (see ref. 2) that the Bergman metric is complete in $T(g)$. Therefore, we conclude the following.

THEOREM 3. *In the Teichmüller space $T(g)$ of any Riemann surface of genus $g \geq 2$, the Weil-Petersson metric is not uniformly equivalent to the Bergman metric.*

For the proof of Theorem 1 we need the following

LEMMA. *Let f be a holomorphic function on D satisfying $|f(z)| \leq M$, $M > 0$. Then for each $z \in D$ and each $\xi \in C^n$,*

$$\left| \sum_{\nu=1}^n \frac{\partial f}{\partial z_\nu} \xi_\nu \right|^2 \leq M^2 \sum_{\alpha, \beta=1}^n \frac{\partial^2 \log K_D(z, \bar{z})}{\partial z_\alpha \partial \bar{z}_\beta} \xi_\alpha \bar{\xi}_\beta$$

where K_D is the Bergman kernel function of D .

PROOF: Set $a(t) = f(t)K_z(t)$, $b(t) = \sum_{\nu=1}^n \xi_\nu (\partial/\partial \bar{z}_\nu) [K_z(t)/K_z(z)]$, and $(a, b)_D = \int_D a \bar{b} d\omega$, where $d\omega$ is the Lebesgue measure in D and $K_z(t) = K_D(t, \bar{z})$.

By the Schwarz inequality

$$|(a, b)_D|^2 \leq (a, a)_D (b, b)_D. \quad [1]$$

Using the reproducing property of K_D (ref. 3), we have

$$(a, a)_D = (f K_z, f K_z) = (f \bar{f} K_z, K_z) \leq M^2 K_D(z, \bar{z}), \quad [2]$$

$$\begin{aligned} (b, b)_D &= \sum_{\mu, \nu=1}^n \xi_\mu \bar{\xi}_\nu \left(\frac{\partial}{\partial \bar{z}^\nu} \left(\frac{K_z}{K_z(z)} \right), \frac{\partial}{\partial \bar{z}^\mu} \left(\frac{K_z}{K_z(z)} \right) \right) \\ &= \sum \xi_\mu \bar{\xi}_\nu K_z^{-3}(z) \left[K_z(z) \frac{\partial^2 K_z(z)}{\partial z_\mu \partial \bar{z}_\nu} - \frac{\partial K_z(z)}{\partial z_\mu} \frac{\partial K_z(z)}{\partial \bar{z}_\nu} \right] \\ &= \frac{1}{K_D(z, \bar{z})} \sum_{\mu, \nu=1}^n \frac{\partial^2 \log K_D(z, \bar{z})}{\partial z_\mu \partial \bar{z}_\nu} \xi_\mu \bar{\xi}_\nu \quad [3] \end{aligned}$$

and

$$\begin{aligned} (a, b)_D &= \sum_{\nu=1}^n \xi_\nu \left(f K_z, \frac{\partial}{\partial \bar{z}_\nu} \frac{K_z}{K_z(z)} \right) \\ &= \sum_{\nu=1}^n \xi_\nu \frac{\partial}{\partial \bar{z}_\nu} \left(\frac{f(z) K_D(z, \bar{z})}{K_D(z, \bar{z})} \right) = \sum_{\nu=1}^n \xi_\nu \frac{\partial f}{\partial \bar{z}_\nu}. \quad [4] \end{aligned}$$

From [1] together with [2], [3], and [4], Lemma follows.

Theorem 1 is now an immediate consequence of Lemma when we recall that the Carathéodory differential metric of D is given (see ref. 4) by

$$\alpha_D(z, \xi) = \sup \left\{ \left| \sum_{\nu=1}^n \frac{\partial f}{\partial z_\nu} \xi_\nu \right| : f \in H(D, \Delta) \right\}, \quad [5]$$

where $H(D, \Delta)$ denotes the class of holomorphic functions f on D with values in the unit disc Δ in C .

It should be noted that the method of the proof of the above lemma was essentially due to K. H. Look in ref. 5.

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