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Exchange-Induced Relaxation in the Presence of a Fictitious Field

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Abstract

In the present study we derive a solution for two site fast exchange-induced relaxation in the presence of a fictitious magnetic field as generated by amplitude and frequency modulated RF pulses. This solution provides a means to analyze data obtained from relaxation experiments with the method called RAFFn (Relaxation Along a Fictitious Field of rank n), in which a fictitious field is created in a coordinate frame undergoing multi-fold rotation about n axes (rank n). The RAFF2 technique is relevant to MRI relaxation methods that provide good contrast enhancement for tumor detection. The relaxation equations for $n = 2$ are derived for the fast exchange regime using density matrix formalism. The method of derivation can be further extended to obtain solutions for $n > 2$.

Keywords

fictitious fields; relaxations; exchange; high rotating frames

INTRODUCTION

In the early days of NMR, it was shown by Whitfield and Redfield that electrical component of a circularly polarized radiofrequency (RF) field produces a stationary component of magnetization $\mathbf{M}$ perpendicular to the plane in which the RF field rotates [1]. The spin precesses around the magnetic field component which is oriented out of the plane of rotation of the circularly polarized RF field. Subsequently, Bendall et. al and Garwood et. al have described a radiofrequency (RF) pulse with the *sin/cos* amplitude ($\omega_1(t)$) and frequency ($\omega(t)$) modulation functions which produce a fictitious RF field component perpendicular to the plane of rotation of the effective frequency during the course of the pulse [2, 3].

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Recently we introduced a novel method entitled RAFF (Relaxation Along a Fictitious Field (the 2nd rotating frame technique, here RAFF2 ($n = 2$)) which provides possibility to generate relaxation dispersion in MRI by altering the amplitude and orientation of $\mathbf{H}_1(t)$. Subsequently, we extended this method to higher rotating frames ($3 \leq n \leq 5$) by utilizing consecutive transformations to the rotating frame of rank n around $y^{(n-1)}$ and $x^{(n-1)}$ axes. The new method which is entitled RAFF n (with n indicating rank of the rotating frame), generates fictitious RF field components operating in the rotating frames of rank n . Here, the final effective field in the rotating frame n , $\mathbf{H}^n$, is the vector sum of $\mathbf{D}_{n-1}$ and $\mathbf{C}_{n-1}$ [4]. Note, in this work magnetic field vectors will be expressed in angular frequency units (i.e., $\omega_0 = \gamma B_0 z$). By convention, the RF field (B_1) is applied along the x axis (x') of the rotating frame and the RF field vector is $\omega_1 = \gamma B_1 x'$, and the frequency offset $-\omega_0(t)z$. The non-adiabatic sweep of the effective RF field $\mathbf{H}_0$ produces a fictitious field component ($\mathbf{C}_1 = \gamma^{-1} d\alpha_0 / dt y'$), and the effective fictitious field $\mathbf{H}_1$ in the second rotating frame is the vector sum of $\mathbf{C}_1$ and the frequency offset $\mathbf{D}_1$ [4]. Here, γ is the gyromagnetic ratio and α_0 is the time-dependent angle between $\mathbf{H}_0$ and the first rotating frame quantization axis z' . The amplitude of the fictitious field can be varied by changing the relationship between $\mathbf{C}_1$ and $\mathbf{D}_1$ components [5].

The RAFF2 method provided high sensitivity to rat glioma gene therapy [6], demonstrating that RAFF n offers a unique source of contrast in MRI [7]. Moreover, with RAFF n , the gray/white matter (GM/WM) contrast in the human and rat brain increased with the increase of n reaching maximum with $n = 4$ and 5 [7]. Our theoretical analysis suggested that the increased GM/WM contrast is due to ability of RAFF4 and 5 to probe slow motion in the brain (e.g., the exchange kinetics is slow on the chemical shifts time scale). Importantly, the method allows to significantly reduce the specific absorption rate (SAR). Motivated by the promising findings of our experimental study of RAFF n in the human and rat brain [4] which demonstrated great potential of RAFF n in characterization of tumors and indicated its exceptional sensitivity to myelin, here we provide a theoretical description of exchange-induced relaxation, which is critical for the understanding of the MRI contrast obtained with RAFF n *in vivo* [6]. In this work, the analytical solution for anisochronous two site exchange (2SX) (i.e., exchange between sites with different chemical shifts) in the fast exchange regime (FXR) during RAFF2 was derived. The validity of the analytical solution was verified by comparing with Bloch-McConnell and product operator formalisms [5, 8]. The relaxation rates were derived for the case of a fictitious field in the 2nd rotating frame (SRF). Although the presented analytical solutions are valid for the description of the exchange-induced relaxations during RAFF2 only, the method of derivation [9–11] can be further extended to higher rotating frames of rank $n \geq 3$.

THEORY

Exchange Induced Relaxation

Previous work from our laboratory focused on the derivation of exchange-induced relaxations during adiabatic pulses of the hyperbolic secant (HS n) family [3], where the theoretical treatment of exchange-induced relaxation was given in the effective field frame, $\mathbf{H}_1(t)$ [12, 13]. Following our nomenclature, $\mathbf{H}_1$ is the effective field in first rotating frame

(FRF), and thus upon convention the $\mathbf{H}_1$ (here we adapted terminology used in [4]) is the vector sum of $\omega_1(t) \hat{x}$ and $\omega(t) \hat{z}$. Here, we present derivations of the exchange induced relaxations obtained in the SRF.

We consider a general time-dependent Hamiltonian which is written in the laboratory frame as:

$$H_T(t) = H_Z + H_{RF}(t) + H_{ex}(t), \quad (1)$$

where $H_T(t)$ is the total Hamiltonian defined as the sum of the Zeeman component H_Z , a time-dependent radiofrequency term, $H_{RF}(t)$, and the Hamiltonian representing the exchange in the system $H_{ex}(t)$ [9, 12–15]. We then assume that the radiofrequency term is a rotating term in phase with the precession of the applied RF [16]:

$$H_T(t) = \omega_0 I_z + \omega_1(t) \exp(-i\omega_{RF} t I_z) \tilde{I}_x \exp(i\omega_{RF} t I_z) + H_{ex}(t) \quad (2)$$

Here ω_0 is the Larmor frequency of the spin, $\omega_1(t)$ is the time-dependent RF amplitude, $\omega_{RF}(t)$ is the frequency of the applied RF and I_z is the z component of the spin angular momentum [17]. By using standard methods [18] and applying the operator $\exp(i\omega_{RF} t I_z)$ to the total Hamiltonian, it results:

$$\tilde{H}_T(t) = \Delta\omega(t) \tilde{I}_z + \omega_1(t) \tilde{I}_x + \tilde{H}_{ex}(t), \quad (3a)$$

where

$$\Delta\omega(t) = \omega_0 - \omega_{RF}(t) \quad (3b)$$

is the frequency offset in rad/s, ω_0 is the Larmor frequency of the resonance of interest. The single tilde above the operator signifies an operator in the first rotating frame [19]. Next, the total Hamiltonian given by Eq. (3) should be transformed to the second rotating frame (SRF) with the coordinate system $x''y''z''$, by applying the operator:

$$\exp(i\alpha_1(t) \tilde{I}_y), \quad (4)$$

where:

$$\alpha_1(t) = \text{ArcTan} \left(\frac{\omega_1(t)}{\Delta\omega(t)} \right). \quad (5)$$

This results in the attainment of a Hamiltonian in the SRF which has the form:

$$\tilde{\tilde{H}}_T(t) = H_1(t) \tilde{\tilde{I}}_z - (d(\alpha_1(t)/dt) \tilde{\tilde{I}}_y + \tilde{\tilde{H}}_{ex}(t)) \quad (6)$$

with the definition

$$H_1(t) = \sqrt{\Delta\omega(t)^2 + \omega_1(t)^2}. \quad (7)$$

The double tilde in Eq. (6) indicates an operator in the SRF [20]. One can apply an indefinite number of frames to the analysis of the spin dynamics when higher rotating frames with rank $n > 2$ more closely describe the relevant motional regime [21]. For the sake of simplicity, in this paper we consider the transformation when the fictitious field which is generated by the motion of the coordinate system around the X'' axis of the SRF is stationary (i.e., $H_2 \parallel Z''$) [22, 23]. This situation is relevant during the application of a RAFF2 pulse.

We can transform this Hamiltonian to a third rotating frame around an operator which has been defined as [24]:

$$\exp(i\alpha_2(t)\tilde{\tilde{I}}_x), \quad (8)$$

With the following Hamiltonian;

$$\tilde{\tilde{H}}_T(t) = H_2(t)\tilde{\tilde{I}}_z - d(\alpha_2(t)/dt)\tilde{\tilde{I}}_x + \tilde{\tilde{H}}_{ex}(t), \quad (9)$$

where

$$H_2(t) = \sqrt{H_1(t)^2 + (d\alpha_1(t)/dt)^2}. \quad (10)$$

We then define:

$$\tilde{\tilde{I}}_z = \tilde{\tilde{I}}_z \cos(\alpha_2(t)) + \tilde{\tilde{I}}_y \sin(\alpha_2(t)), \quad (11)$$

with

$$\alpha_2(t) = \text{ArcTan} \left(\frac{d\alpha_1(t)/dt}{H_1(t)} \right), \quad (12)$$

where the rotation occurs around the X'' axis of the SRF (or H_1 frame).

With the above background, we can now concentrate on the effect of the term $\tilde{\tilde{H}}_{ex}(t)$ on the relaxation of the spin in the H_2 frame. Following the seminal paper by Wennerstrom [9] and the development and modification of the theory by Deverell et al. [10, 14] and Davis et al. [11], we write the two-site exchange Hamiltonian as:

$$\tilde{\tilde{H}}_{ex}(t) = \delta(t)\tilde{\tilde{I}}_z, \quad (13)$$

where $\delta(t)$ is a stochastic operator that has a value of 1 when occupying either of the two sites A or B and zero otherwise. Next, an expression for $\tilde{\tilde{I}}_z$ is derived upon transformation from the laboratory frame to the second rotating frame. Following the formalism of Hubbard [25] we define the following transformations valid in the laboratory frame:

$$\exp(i\omega_{\text{RF}}tI_z)T_{10}\exp(i\omega_{\text{RF}}tI_z) = T_{10} \quad (14a)$$

$$\exp(i\alpha_1(t)I_y)T_{10}\exp(i\alpha_1(t)I_y)=\sum_{q_1=-1}^1 T_{1q_1}d_{0q_1}^{(1)}(\alpha_1(t)) \quad (14b)$$

$$\exp(i\alpha_2(t)I_x)T_{1q_1}\exp(-i\alpha_2(t)I_x)=\sum_{q_2=-1}^1 T_{1q_2}d_{q_1q_2}^{(1)}(\alpha_2(t))\exp(iq_2\pi/2) \quad (14c)$$

$$\exp(i\psi(t)I_z)T_{1q_2}\exp(i\psi(t)I_z)T_{1q_2}\exp(iq_2\psi(t)) \quad (14d)$$

Here T_{1q} are defined as first rank tensors of order q [26]. Combining these relations we obtain the following expression for the Hamiltonian for exchange in the H_2 frame (note, the angle α_2 is between Z' and Z''):

$$\delta(t)\tilde{\tilde{T}}_{10}=\sum_{q_2=-1}^1\sum_{q_1=-1}^1\exp(iq_2\psi(t))\exp(iq_2\pi/2)d_{q_1q_2}^{(1)}(\alpha_2(t))d_{0q_1}^{(1)}(\alpha_1(t))T_{1q_2}, \quad (15)$$

with the following definitions:

$$\psi(t)=\int_{t_0}^t H_1(\zeta)d\zeta \quad (16)$$

We define $d_{qq_1}^{(1)}(\alpha_1)$ as the rank 1 reduced Wigner matrix element with indices q and q_1 for an angular argument α_1 [17, 20, 27]. Here, we use the nomenclature for the angular momentum of the spin $1/2$ following the convention:

$$T_{10}=I_z \quad (17a)$$

$$T_{11}=-\frac{1}{\sqrt{2}}I_+ \quad (17b)$$

$$T_{1-1}=\frac{1}{\sqrt{2}}I_- \quad (17c)$$

To address the relaxation driven time-evolution governed by exchange we use standard second order perturbation theory as used with the semi-classical treatment of the interaction with the lattice and the RF field [18]. We write the following Master Equation for the rate of change of the density matrix in the TRF [28, 29]:

$$\frac{d\tilde{\tilde{\sigma}}(t)}{dt}=-\int_0^\infty\left\{[\tilde{\tilde{H}}_{ex}(t),[\tilde{\tilde{H}}_{ex}^\pm(t-\tau),\tilde{\tilde{\sigma}}(t)]]\right\}d\tau, \quad (18)$$

where the ensemble averaging is used [18, 29–32]. We note that the symbol $\pm$ indicates the taking of the hermitian adjoint, and in particular we adopt the convention:

$$T_{1q}^{\pm} = (-1)^q T_{1-q} \quad (19)$$

Using Eqs. (15–19) and using the secular approximation [27], we find:

$$\frac{d\tilde{\tilde{\sigma}}(t)}{dt} = - \int_0^{\infty} \{\delta(t)\delta(t-\tau)\} \sum_{q_2=-1}^1 \sum_{q_1=-1}^1 \exp(i\tau q_2 H_2(t)) (d_{q_1 q_2}^{(1)}(\alpha_2(t)))^2 (d_{0q_1}^{(1)}(\alpha_1(t)))^2 (-1)^{q_2} [T_{1q_2}, [T_{1-q_2}, \tilde{\tilde{\sigma}}(t)]] d\tau. \quad (20)$$

We assume an exponential dependence for the correlation function:

$$\{\delta(t)\delta(t-\tau)\} \propto \exp(-\tau/\tau_{ex}), \quad (21)$$

where τ_{ex} is the lifetime of the spin in either of the two exchanging sites [13, 14]. Then, by carrying out the integration, we derive the spectral density function:

$$J(q_2 H_2(t)) = k_{ex} \frac{\tau_{ex}}{(1 + ((q_2 H_2(t))\tau_{ex})^2)}, \quad (22)$$

where $k_{ex} = p_a p_b (\delta\omega)^2$. Substituting Eq. (22) into Eq. (20) we find:

$$\frac{d\tilde{\tilde{\sigma}}(t)}{dt} = - \sum_{q_2=-1}^1 \sum_{q_1=-1}^1 J(q_2 H_2(t)) (d_{q_1 q_2}^{(1)}(\alpha_2(t)))^2 (d_{0q_1}^{(1)}(\alpha_1(t)))^2 (-1)^{q_2} [T_{1-q_2}, \tilde{\tilde{\sigma}}(t)] d\tau. \quad (23)$$

We next seek to use Eq. (23) to find the rate of change in time of an observable Q in the TRF for the exchange mechanism. We use the following standard relations:

$$\frac{d\langle \tilde{\tilde{Q}}(t) \rangle}{dt} = Tr \left[\frac{d\tilde{\tilde{\sigma}}(t)}{dt} Q \right], \quad (24)$$

$$Tr[[T_{1q_2}, [T_{1-q_2}, \tilde{\tilde{\sigma}}(t)]Q] = Tr[[T_{1-q_2}, [T_{1q_2}, Q]]\tilde{\tilde{\sigma}}(t)] = \langle [T_{1-q_2}, [T_{1q_2}, Q]] \rangle [t]. \quad (25)$$

Here Tr indicates the taking of the trace operation and $\langle \rangle$ means the expectation value [21, 24].

Next we evaluate the expectation value in Eq. (25). This means we must obtain the value of the double commutator for typical values of Q such as I_0 and I_{+1} . We adopt a general approach using the Clebsch Gordan coefficients [17, 26, 33–35]:

$$\langle [T_{1-q_2}, [T_{1q_2}, T_{10}]] \rangle = 2 \sum_{M=-1}^1 (111M|1q_2 10)(1110|1-q_2 1M) \langle T_{10} \rangle [t] \quad (26a)$$

$$\langle [T_{1-q_2}, [T_{1q_2}, T_{1\pm 1}]] \rangle = 2 \sum_{M=-1}^1 (111M|1q_2 1\pm 1)(111\pm 1|1-q_2 1M) \langle T_{1\pm 1} \rangle (t) \quad (26b)$$

Using these relations in Eq.(23) we finally obtain:

$$\begin{aligned} \frac{d\langle \tilde{T}_{10} \rangle(t)}{dt} = & -2 \sum_{q_2=-1}^1 \sum_{q_1=-1}^1 \sum_{M=-1}^1 \sum_{M_1=-1}^1 J[q_2 H_2(t)] (-1)^{q_2} (d_{q_1 q_2}^{(1)}[\alpha_2(t)])^2 (d_{0 q_1}^{(1)}[\alpha_1(t)])^2 (111M|1q_2 10)(111M_1|1(\\ & - q_2)1M) \langle \tilde{T}_{10} \rangle(t) \end{aligned} \quad (27a)$$

$$\begin{aligned} \frac{d\langle \tilde{T}_{11} \rangle(t)}{dt} = & -2 \sum_{q_2=-1}^1 \sum_{q_1=-1}^1 \sum_{M=-1}^1 \sum_{M_1=-1}^1 J[q_2 H_2(t)] (-1)^{q_2} (d_{q_1 q_2}^{(1)}[\alpha_2(t)])^2 (d_{0 q_1}^{(1)}[\alpha_1(t)])^2 (111M|1q_2 11)(111M_1|1(\\ & - q_2)1M) \langle \tilde{T}_{11} \rangle(t) \end{aligned} \quad (27b)$$

Eqs. (27a,b) are the longitudinal and transverse rate equations for the magnetization in the TRF for the exchange mechanism. It is important to emphasize that when $\alpha_1(t)$ goes to zero upon transformation from the third to the second frame, the corresponding reduced Wigner matrix element goes to the Kronecker delta in $q_1 q_2$ and q_1 equals to q_2 . Here, it is to be noted that $H_1 \approx H_2$. If the tilt angle $\alpha_1(t)$ goes to zero upon transformation from the second to the first frame, q_1 becomes zero by the properties of the reduced Wigner matrix element mentioned above and the rate constants become time independent. For the convenience of the reader, the following are the equations for the transverse and longitudinal relaxation rate constants in H_2 (TRF) frame:

$$\begin{aligned} R_{2\rho, H_2}(t) = & -p_a p_b \delta \omega^2 \left(\cos(\alpha_2(t))^2 \cos(\alpha_1(t))^2 \tau_{ex} + \frac{1}{2} \sin(\alpha_2(t))^2 \sin(\alpha_1(t))^2 \tau_{ex} + \frac{1}{2} \left(\cos(\alpha_1(t))^2 \sin(\alpha_2(t))^2 \right. \right. \\ & + \frac{1}{2} \cos(\alpha_1(t))^2 \sin(\frac{\alpha_2(t)}{2})^4 \\ & \left. \left. + \frac{1}{2} \sin(\alpha_1(t))^2 \cos(\frac{\alpha_2(t)}{2})^4 \right) \frac{\tau_{ex}}{1 + (H_2(t) \tau_{ex})^2} \right) \end{aligned} \quad (28a)$$

$$R_{1\rho, H_2}(t) = -p_a p_b \delta \omega^2 \left(\sin(\alpha_1(t))^2 \cos(\frac{\alpha_2(t)}{2})^4 + \sin(\alpha_1(t))^2 \sin(\frac{\alpha_2(t)}{2})^4 + \cos(\alpha_1(t))^2 \sin(\alpha_2(t))^2 \right) \frac{\tau_{ex}}{1 + (H_2(t) \tau_{ex})^2} \quad (28b)$$

Exchange-induced $R_{1\rho, H_2}(t)$ and $R_{2\rho, H_2}(t)$ during RAFF2

The derived theoretical expressions can be used to characterize the longitudinal and transverse relaxations during FS pulses operating in non-adiabatic regime, including those used in the RAFF2 method [3]. With RAFF2, the fictitious field is generated during the FS pulses having amplitude and frequency modulation functions defined by [36]:

$$\omega_1(t) = \omega_1^{\max} \sin(\omega_1^{\max} \tan(\alpha_2(t)) t) \quad (29)$$

$$\Delta\omega(t) = \omega_1^{\max} \cos(\omega_1^{\max} \tan(\alpha_2(t)) t), \quad (30)$$

where $\omega_1^{\max}$ is the peak power of the FS pulses. It had been demonstrated that MR relaxation can be affected by varying the orientation of $\mathbf{H}_2$ relatively to z'' axis of a SRF [5]. When the angle λ which defines the initial orientation $\mathbf{M}$ relatively to z' axis of the FRF is set to zero, both $R_{1\rho, H_2}$ ($=1/T_{1\rho, H_2}$) and $R_{2\rho, H_2}$ ($=1/T_{2\rho, H_2}$) contribute to the relaxation processes during RAFF2. When $\lambda = \alpha_2$, the relaxation is governed solely by $R_{1\rho, H_2}$ mechanisms. When $\lambda = 90^\circ - \alpha_2$, $\mathbf{M}$ undergoes precession in the plane perpendicular to $\mathbf{H}_2$, and thus the relaxation is solely $R_{2\rho, H_2}$ (Figure 1).

Theoretical calculations of $R_{1\rho, H_2}(t)$ and $R_{2\rho, H_2}(t)$ during RAFF2 pulses for $\alpha_2 = 45^\circ$ are shown in Figure 2. The calculations were performed employing Eqs. 28 and using the Mathematica 7 software package. It can be seen that $R_{1\rho, H_2}(t)$ and $R_{2\rho, H_2}(t)$ relaxation rate constants resembles the modulation functions of RAFF2 pulses [4]. When the RAFF2 pulse with $\alpha_2 = \pi/4$ acts on magnetization that is initially along the z' axis, R_{RAFF2} will be the average of $T_{1\rho, H_2}^{-1}(t)$ and $T_{2\rho, H_2}^{-1}(t)$ over the duration of the pulse as given by [4]:

$$R_{\text{RAFF2}} = \frac{1}{2T_p} \int_0^{T_p} (T_{1\rho, H_2}^{-1}(t) + T_{2\rho, H_2}^{-1}(t)) dt. \quad (31)$$

Here T_p is the duration of *sine/cosine* pulses used in RAFF2. Finally, in Figure 3 the calculations performed with Eqs. (28–31) are shown in comparison with the results obtained with the invariant trajectory approach [8] and with the Bloch-McConnell formalism (Figure 5 of ref. [5]). The calculations of the exchange induced rate constants R_{RAFF2} for different exchange correlation times obtained using Eqs. (28–31) are in a good agreement with those rate constants obtained in Liimatainen et. al [5] It should be noted that the analytical solution is in good agreement with Bloch-McConnell and invariant trajectory approaches in the fast exchange regime (Fig.3). The three methods provide slightly different results although when the system is approaching intermediate exchange kinetics which is already outside of Redfield limit. The analytical procedure for the determination of exchange induced relaxations given in the 2nd rotating frame can be extended to high rotating frames for the analysis of the relaxations during RAFFn technique [4]. It should be noted that the derivations of exchange-induced relaxation rate constants in the SRF frame (Eq. 28) were performed for the general case when the angle $\alpha_2(t)$ is time dependent. However, we emphasize that α_2 is constant for RAFF2 specialized for the case when the effective field H_2 is stationary in the SRF frame.

CONCLUSIONS

An expression for exchange-induced relaxation for two site fast exchange in the doubly rotating frame has been derived. The method of derivation is based on the density matrix formalism, and is valid in the fast exchange regime. The method of the derivation can be

used to obtain equations to describe exchange-induced relaxations in higher rotating frames ($n > 2$). Good agreement was obtained between analytical solution and the description of the exchange-induced relaxations provided by Bloch-McConnell and invariant trajectory approaches.

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Research Highlights

A theory of exchange-induced relaxation in the presence of a fictitious field was derived.

The method can be generalized for description of relaxation in the rotating frames of rank $n \geq 3$

The derived expressions provide a means to analyze data from MRI experiments with the RAFFn method

Figure 1. Relaxation during RAFF pulse. H_2 = effective field of the rotating frame of rank $n=2$, α_2 = angle between H_2 and z'' , λ = angle at which magnetization is tilted before application of the RAFF2 pulse relative to z' . A) When $\lambda=\varepsilon$, magnetization is aligned with H_2 in the second rotating frame, and it relaxes along the H_2 with $R_{1\rho, H_2}(t)$. B) When $\lambda= 90^\circ-\varepsilon$, magnetization rotates in a plane perpendicular to H_2 , and relaxes with $R_{2\rho, H_2}(t)$.

$$\alpha_2 = 45^\circ$$

Figure 2.

Calculated longitudinal $R_{1\rho, \text{H}_2}(t)$ (top) and transverse $R_{2\rho, \text{H}_2}(t)$ relaxation rate constants during RAFF2 pulse time t as a function of exchange correlation times τ_{ex} for one orientation of the H_2 relatively to z'' (defined by α_2). For calculation, $\omega_1^{\text{max}}/(2\pi) = 625$ Hz, $\delta\omega = 0.85$ ppm at 4T, and $P_A P_B = 0.25$.

Figure 3.

Theoretical simulation of exchange-induced relaxations with RAFF2 using Bloch-McConnell formalism (squares), analytical solution given by Eqs. (28,29) (triangle) and invariant trajectory method (circle), for the $\alpha_2=45^\circ$ for different exchange-correlation times. The results of the calculations using Bloch-McConnell formalism and invariant trajectory approach are from Liimatainen et. al [5]. Exchange parameters: difference in chemical shifts, $\delta\omega=0.85$ ppm; populations of the exchanging sites $P_A=P_B=0.5$, maximal amplitude of pulses $\omega_1^{\text{max}}/(2\pi)=625$ Hz.