Conjugate points on a limiting extremal

(Weierstrass integrals/Riemannian manifolds/nondegenerate extremals/indices of extremals)

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Contributed by Marston Morse, March 18, 1976

ABSTRACT *Theorem 1*, together with the author's extensions of the Sturmian Comparison Theorems, will suffice to establish the basic Theorem 27.3, p. 211 of *Global Variational Analysis: Weierstrass Integrals on a Riemannian Manifold. Mathematical Notes*, Princeton University Press, Princeton, N.J., 1976. Here and in the preceding reference there is given a Weierstrass Integral on a compact, connected, Riemannian manifold M_n , conditioned as in the above reference. Let γ be an arbitrary extremal of J joining a point P to a point Q , $\neq P$ on M_n . An extremal ζ of J joining A to B is termed *nondegenerate* if A and B are not conjugate on ζ . The *index* of ζ is by definition the "count" of conjugate points of A on ζ definitely preceding B .

THEOREM 1. *When the extremal γ is nondegenerate, extremals of J issuing from P with the same length as γ and sufficiently near γ in the sense of Fréchet, are nondegenerate and have the same index as γ .*

A manifold frame $((M))_\gamma^\nu$ based on γ . To prove *Theorem 1*, use will be made of Index Theorem 25.1, p. 187 of ref. 1. To that end we identify the extremal γ of *Theorem 1* with the extremal γ joining A_1 to A_2 in §25 of ref. 1. As on p. 184 of ref. 1, let

$$P, q_1, \dots, q_\nu, Q \quad [1]$$

be a sequence of $\nu + 2$ successive points of γ defining $\nu + 1$ successive "elementary extremal" subarcs of γ with J lengths $< m$, the constant of Definition 19.4, p. 133 of ref. 1. Set $m = n - 1$. As on p. 183 of ref. 1 let $((M))_\gamma^\nu$ be a "manifold frame" defined by the product

$$M^{(1)} \times \dots \times M^{(\nu)} \quad [2]$$

of disjoint differentiable m -manifolds which cut across γ at the respective points $q_1, \dots, q_\nu$ of [1] and satisfy Conditions (1) to (5) on p. 184 of ref. 1. Condition (1), for each i on the range 1, $\dots, \nu$, $M^{(i)}$ is defined by an injective C^∞ -diff

$$\alpha_i \rightarrow M^{(i)}(\alpha_i) : B_m \rightarrow M_n, \quad [3]$$

where $\alpha_i = (\alpha_i^1, \dots, \alpha_i^m) \in B_m$, with B_m an open origin-centered solid m -ball in R^m . Under Condition (2)

$$M^{(i)}(O) = q_i, \quad [4]$$

where O is the origin in R^m and $i = 1, 2, \dots, \nu$. Condition (5) on p. 184 of ref. 1, will be slightly strengthened by requiring that the elementary extremals of Condition (5) have lengths less than some fixed positive constant $< m$.

A diff Λ of $(B_m)^\nu$ onto $((M))_\gamma^\nu$. A surjective C^∞ -diff Λ of the ν -fold product $(B_m)^\nu$ onto $((M))_\gamma^\nu$ is implied by the ν diffs [3]. In the notation of [3] a point $v \in (B_m)^\nu$ is an $m\nu$ -tuple of the form,

$$v = (\alpha_1^1, \dots, \alpha_1^m; \dots; \alpha_\nu^1, \dots, \alpha_\nu^m) = (v_1, v_2, \dots, v_{m\nu}) \quad [5]$$

Under the diff Λ , a $m\nu$ -tuple $v \in (B_m)^\nu$ goes into a ν -tuple

$$\Lambda(v) = (p_1, \dots, p_\nu) = (M^{(1)}(\alpha_1), \dots, M^{(\nu)}(\alpha_\nu)) \in ((M))_\gamma^\nu \quad [6]$$

In particular $\Lambda(O) = (q_1, \dots, q_\nu)$ of [1], since [4] holds.

The broken extremal $E_\gamma(v)$. Given the $m\nu$ -tuple v of [5], $\Lambda(v)$ is the ν -tuple $(p_1, \dots, p_\nu)$ of [6]. The broken extremal defined by elementary extremals joining the successive points of the sequence

$$P, p_1, \dots, p_\nu, Q \quad [7]$$

will be denoted by $E_\gamma(v)$. Note that $E_\gamma(O) = \gamma$.

The Index Function I_γ based on $((M))_\gamma^\nu$. In Definition (25.8) of ref. 1, for each $v \in (B_m)^\nu$ one sets

$$I_\gamma(v) = J(E_\gamma(v)). \quad [8]$$

According to Index Theorem 25.1 on p. 187 of ref. 1 the nullity of τ of the critical $m\nu$ -tuple $v = O$ of I_γ is zero, since γ is nondegenerate by hypothesis. Moreover, by Index Theorem 25.1, at the place cited, the index κ of the critical $m\nu$ -tuple $v = O$ of I_γ is the count of conjugate points of P preceding Q on γ .

Unitary directions $r = (r^1, \dots, r^n)$ at P on M_n . Unitary directions r at P are defined in Definition 7.3 on p. 54 of ref. 1. In this note they are relative to a fixed presentation (φ, U) of an open neighborhood $\varphi(U)$ of P on M_n . Here U is an open neighborhood of the origin in the space U of points $u = (u^1, \dots, u^n) \in R^n$. The presentation (φ, U) has the form

$$u \rightarrow \varphi(u) : U \rightarrow M_n. \quad [8']$$

If u_0 is the origin in R^n , one supposes that $\varphi(u_0) = P$.

Extremals γ_r . Let γ_r denote the extremal of J issuing from P with a direction r which is unitary at P . Suppose that the length $L(\gamma_r) = L(\gamma)$. Let r_0 be the unitary direction of γ at P , so that $\gamma = \gamma_{r_0}$.

The endpoint Q_r of γ_r . Note that $Q_{r_0} = Q$. Let N be an open neighborhood of r_0 relative to the ellipsoid of unitary directions r at P introduced in Definition 7.3 on p. 54 of ref. 1. N will be taken so small that the following is true.

(a₁) The manifolds $M^{(i)}$ cut across γ_r in points uniquely determined by γ_r .

(a₂) Q_r is not conjugate to P on γ_r .

(a₃) The mapping $r \rightarrow Q_r : N \rightarrow M_n$ is an injective diff into M_n of class C^∞ .

(a₄) For $p_\nu \in M^{(\nu)}$, $0 < \text{distance}(p_\nu, Q_r) < m$. We shall prove *Theorem 1* by proving the following.

LEMMA 1. *If N_0 is a sufficiently small, open, connected neighborhood of r_0 , relative to N , extremals γ_r with $r \in N_0$, have the same index as γ .*

Proof of Theorem 1: We need information concerning the indices of extremals $\gamma_r \neq \gamma$. To that end a special index function Γ will replace the index function I_γ on p. 184 of ref. 1.

The broken extremal $E(v, r)$. Corresponding to an arbitrary $m\nu$ -tuple $v \in (B_m)^\nu$ let $(p_1, \dots, p_\nu)$ be the ν -tuple in $((M))_\gamma^\nu$ given by $\Lambda(v)$. For $r \in N$ the points

$$P, p_1, \dots, p_\nu, Q_r \quad [9]$$

are the vertices of a broken extremal, say $E(v, r)$, joining P to Q_r . Note that $E(v, r_0) = E_\gamma(v)$.

The Index Function Γ defined. We shall extend Definition [8]. For $(v, r) \in (B_m)^\nu \times N$ set

$$J(E(v, r)) = \Gamma(v, r). \quad [10]$$

Note that $\Gamma(v, r_0) = I_\gamma(v)$. Γ is of class C^∞ . In the following lemma the partial mapping

$$v \longrightarrow \Gamma(v, r) : (B_m)^\nu \longrightarrow R \quad [11]$$

is defined for each r fixed in N and will be denoted by $\Gamma(\cdot, r)$. Let

$$q_1(r), \dots, q_\nu(r) \quad [12]$$

be the ν points in which γ_r meets the respective manifolds $M^{(i)}$. Let v_r be the $m\nu$ -tuple in $(B_m)^\nu$ such that $\Delta(v_r) = (q_1(r), \dots, q_\nu(r))$. The $m\nu$ -tuple v_r is a critical point of $\Gamma(\cdot, r)$ as may be inferred from the proof of Lemma 25.1 on p. 186 of ref. 1. Theorem 1 is implied by Lemmas A and B which follow

LEMMA A. If the unitary direction r of γ_r at P is restricted to a sufficiently small, open, connected neighborhood N_0 of r_0 relative to N , the $m\nu$ -tuple v_r is a nondegenerate critical point of $\Gamma(\cdot, r)$ with an index κ_r which is the count of the conjugate points of P on γ_r preceding Q_r .

Let $\hat{B}_m$ denote the concentric subball of B_m with half the radius. Let N_0 be an open connected neighborhood of r_0 relative to N , so small that for $r \in N_0$ each $m\nu$ -tuple v_r is in $(\hat{B}_m)^\nu$. For $r \in N_0$ let T_r be the translation of $R^{m\nu}$ that takes the origin in $R^{m\nu}$ into v_r . For $r \in N_0$ and $v \in (\hat{B}_m)^\nu$, the values $\Gamma(T_r(v), r)$ are well defined and the following statement is clearly true:

(a) For r -fixed in N_0 the index κ_r and nullity of the critical $m\nu$ -tuple v_r of $\Gamma(\cdot, r)$ equal the index and nullity, respectively, of the critical $m\nu$ -tuple $v = O$ of $\Gamma(T_r(\cdot), r)$.

The proof of Lemma A completed. We shall evaluate the index and nullity of the critical $m\nu$ -tuple $v = O$ of $\Gamma(T_r(\cdot), r)$ with the aid of Index Theorem 25.1 on p. 187 of ref. 1, taking γ therein as γ_r . To this end our notation will be extended as

follows. T_r defines a translation of each $m\nu$ -tuple v in $R^{m\nu}$. In particular, let T_r^i be the translation of coordinates $\alpha_1^1, \dots, \alpha_i^m$ induced by T_r . For $i = 1, 2, \dots, \nu$ and for $(\alpha_1^1, \dots, \alpha_i^m) \in \hat{B}_m$, set

$$\hat{M}^{(i)}(\alpha_1^1, \dots, \alpha_i^m) = M^{(i)}(T_r^i(\alpha_1^1, \dots, \alpha_i^m)). \quad [13]$$

The m -dimensional differentiable manifolds

$$M^{(1)}, \dots, M^{(\nu)} \quad [14]$$

thereby defined cut across γ_r at the respective points [12]. The points [12] are the points $\hat{M}^{(i)}(O), \dots, \hat{M}^{(\nu)}(O)$. The product of the manifolds [14] is thus a "manifold frame" $((M))_{\gamma_r}^\nu$ based on the extremal γ_r . The index function I_{γ_r} based on $((M))_{\gamma_r}^\nu$ equals $\Gamma(T_r(\cdot), r)$, as one readily sees.

According to the Index Theorem 25.1, p. 187 of ref. 1, the critical $m\nu$ -tuple $v = O$ of I_{γ_r} (equivalently $\Gamma(T_r(\cdot), r)$) has a vanishing nullity, since γ_r is nondegenerate; it has an index, say κ_r , which is the count of conjugate points of P on γ_r preceding Q_r . Lemma A follows from this statement and from (a).

LEMMA B. The index κ_r of the critical $m\nu$ -tuple v_r of $\Gamma(\cdot, r)$ is independent of the choice of r in N_0 .

We evaluate κ_r for each $r \in N_0$. For i, j on the range $1, 2, \dots, m\nu$, we set

$$a_{ij}^r = \frac{\partial^2}{\partial v_i \partial v_j} \Gamma(v, r) \Big|_{v = v_r}. \quad [15]$$

Cf. (25.11) on p. 187 of ref. 1. The index κ_r of the critical $m\nu$ -tuple v_r of $\Gamma(\cdot, r)$ is the number of negative characteristic roots of the $m\nu$ -square matrix $\|a_{ij}^r\|$. According to Lemma A the critical $m\nu$ -tuple v_r of $\Gamma(\cdot, r)$ is nondegenerate. Hence the determinant $|a_{ij}^r| \neq 0$ so that, as r varies continuously in N_0 , κ_r cannot vary. Lemma B follows.

Theorem 1 follows from Lemmas A and B.

1. Morse, M. (1976) *Global Variational Analysis: Weterstrass Integrals on a Riemannian Manifold*. Mathematical Notes (Princeton University Press, Princeton, N.J.).