A tree swaying in a turbulent wind: a scaling analysis

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Abstract A tentative scaling theory is presented of a tree swaying in a turbulent wind. It is argued that the turbulence of the air within the crown is in the inertial regime. An eddy causes a dynamic bending response of the branches according to a time criterion. The resulting expression for the penetration depth of the wind yields an exponent which appears to be consistent with that pertaining to the morphology of the tree branches. An energy criterion shows that the dynamics of the branches is basically passive. The possibility of hydrodynamic screening by the leaves is discussed.

Keywords Trees · Branches · Elasticity · Wind · Turbulence · Drag reduction

1 Introduction

The time criterion introduced by Lumley [1–3] plays a central role in the theory of drag reduction in turbulent flow. Drag reduction implies a reduction in the friction factor in pipe flow experiments. The time scale of turbulence at some spatial scale is compared with the main reaction time of a polymer chain, for instance. When they are equal, this signals a change in the dynamical behavior of the solution because the chain becomes markedly deformed. A turbulent flow acting on a chain is a bit similar to the action of an oscillatory force on a slightly damped spring. Nevertheless, the two systems may not be viewed as identical: the turbulent time scale is merely a statistical correlation time τ_c . Beyond τ_c the fluid is effectively decorrelated thus the chain is assumed to be swept along with the flow in some random direction once every period τ_c . The ratio of the two scales also figures in more recent elaborate theoretical work on drag reduction [4–6].

A major difficulty in understanding drag reduction has been that it is a phenomenon whose relative magnitude is never more than order unity. It is therefore of interest to study systems of a complexity close in spirit to that of polymers but where the object-flow inter-

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action is more amenable to analysis by experiment. A tree swaying under the influence of a turbulent wind is a case in point. Here, I apply a time criterion to this problem inspired by de Gennes' discussion of a polymer chain deformed by bulk turbulence [4, 5].

The deformation of trees modeled as elastic structures under winds has been investigated extensively recently [7–14]. However, it is difficult to simulate wind turbulence at extremely high Reynolds numbers. Accordingly, it is useful to present a scaling analysis of a tree swaying in a turbulent wind to try to delineate the relevant dimensional parameters involved.

2 Time criterion

Let us consider a tree of height H swaying under the influence of a moderate wind of average speed U . Simple observations by the author in the field prove that its branches oscillate rather haphazardly and quite independently of each other. Thus, I indeed hypothesize that this is so till evidence might prove otherwise. The aerodynamical flow circulating within the crown may be argued to be turbulent. The Reynolds number $\text{Re}_0 = HU/\nu$ of a tree of height $H \simeq 10$ m, say, swaying in a wind of speed $U \simeq 10$ m/s is a formidable 10^7 (the kinematic viscosity ν of air is about $0.15 \text{ cm}^2/\text{s}$). The Kolmogorov dissipation scale $\lambda = H/\text{Re}^{3/4}$ [15] is then only a minute 0.1 mm. I assume the turbulent cascade is essentially unperturbed by the tree branches and leaves (see below). The tree is thus termed a passive object. Yet the Reynolds number $\text{Re}_d = dV_d/\nu$ associated with some branch of diameter d equal to about 1 cm, say, is still not too large for Karman vortices to separate from its surface ($\text{Re}_d \simeq 10^2$; see ref. [16]). The velocity of the wind is split up into a systematic component and a fluctuational component $\vec{V}$. There is a stick boundary condition at ground level but the systematic profile is not the usual logarithmic function of height if the tree were absent [17] (For a forest of trees, a phenomenological, so-called exponential profile is known to hold [17] because the canopy does exert some influence on the wind profile.) For winds blowing on a flat landscape, the fluctuational term $\vec{V}(\vec{r})$ is known to be isotropic on empirical grounds as first shown by G. I. Taylor [18]. There is no reason to suspect this would be otherwise if the average profile were different. The fluctuational speed V_d of the air near a branch is about $U(d/H)^{1/3}$.

In view of the nonalignment of the branches of a tree, the total sum of Karman vortices arising from the interaction of the wind with the branches may be supposed to be a random variable. On the whole, one concludes that the air circulating throughout the crown is in the inertial regime. On dimensional grounds, the fluctuational velocity obeys Kolmogorov's relations like

$$V(r) \simeq (\varepsilon r)^{1/3}, \quad (1)$$

where the magnitude $V(H) = \mathcal{O}(U)$ and the rate of dissipation ε is of order U^3/H . In the inertial regime pertaining to scales much larger than the dissipation scale λ_1 it is assumed dissipation is absent [15]. Equation (1) follows from dimensional analysis.

I anticipate that the turbulent wind penetrates dynamically into an outer shell-like region of the tree crown of thickness ℓ_* (which will be termed the penetration depth). Thus, branches within this region are excited but those in the inner core are quiescent. I note that in practice the length $\ell(d)$ of a branch measured from the tip increases monotonically with d . The tip itself must have a nonzero diameter d_0 otherwise the branch could not bear a bud at its end. The bud forms new leaves, higher order branches and causes the main branch to be elongated each year. Accordingly, a Taylor expansion of $d(\ell)$ ought to be possible at small ℓ

$$d(\ell) = d_0 \left(1 + \alpha_1 \ell + \alpha_2 \ell^2 + \dots \right) \quad (2)$$

with empirical coefficients α_1 , α_2 , etc. By contrast, at large ℓ one expects a fractal structure [19–21]

$$d \sim \ell^\beta. \quad (3)$$

An eddy of the turbulent air encompassing a section of branch of length ℓ typically has a characteristic velocity $V(\ell) \simeq (\varepsilon \ell)^{1/3}$ (1). The time scale of the flow is

$$\tau_\ell \simeq \frac{\ell}{V(\ell)} \simeq \frac{\ell^{2/3}}{\varepsilon^{1/3}} \simeq \frac{H}{U} \left(\frac{\ell}{H} \right)^{2/3}. \quad (4)$$

In general, the size of the crown is of the order of the height of the tree [10].

For the moment, let us suppose that a section of branch within the annular region has a uniform diameter d . Then a time scale associated with the bending oscillations of the branch is given by

$$\tau_b \simeq \frac{\ell^2}{d} \left(\frac{\rho_w}{Y} \right)^{1/2} \simeq \frac{\ell^2}{dS}. \quad (5)$$

In effect the bending energy of the branch is $E\ell/2R_c^2$ [22] where E is the bending modulus and R_c is the radius of curvature of a bent branch. The deflection z of the branch from a straight form is $z \simeq \ell^2/R_c$ and Hooke's modulus equals $k \simeq E/\ell^3 \simeq d^4Y/\ell^3$ where Y is Young's modulus (here, for simplicity within a scaling analysis, we consider the wood to be an isotropic material with one characteristic modulus). The section of branch has a mass $m \simeq d^2\ell\rho_w$ with ρ_w the density of the wood and the characteristic time scale of a bending oscillation is $\tau_b = (m/k)^{1/2}$ yielding (5). The speed of sound in wood is $S = (Y/\rho_w)^{1/2}$.

Ultimately, at some penetration depth ℓ_* , the bending oscillation of the branch is in concert with turbulent eddies acting upon it. Accepting the time criterion $\tau_\ell \simeq \tau_d$, we have

$$d_* \simeq \frac{\ell_*^{4/3}}{H^{1/3}} \left(\frac{U}{S} \right). \quad (6)$$

This is interpreted as follows:

1. At large ℓ , we require $\beta > 4/3$ if the argumentation is to be self-consistent. Hence, the scaling analysis predicts a lower bound on the exponent β . Experimental estimates are, for example, $\beta = 1.37$ for pine trees and $\beta = 1.38$ for walnut trees [10]. This is not strictly true for the tree structures are regarded as discrete fractals whereas (3) is continuous. These experimental exponents are surprisingly close to the theoretical minimum value.
2. At low ℓ , we need to know the value of the coefficients α_1 and α_2 to proceed. Equation (2) and (6) would indicate that there may sometimes be two solutions which would conflict with the premise of the theory outlined here.
3. The typical aspect ratio of a branch predicted from (6) is of the order of U/S which is consistent with values in the field ($U = 10$ m/s; $S = 3000$ m/s).
4. Equation (6) is only correct if it is very slowly varying ($\alpha_1 d_0 \ll 1$). A detailed analysis bears this out (using the results of [23]).
5. There are of course smaller branches attached to a given branch and so forth but their mass is relatively small. Equation (6) remains valid to the leading order.
6. The mass of the leaves attached is also of the order of the mass of a branch [11, 12]. This implies (6) is correct to within $\mathcal{O}(1)$.

A cluster of leaves on a branch may be viewed as a porous medium so one expects the flow surrounding a test leaf to be screened somehow by the blocking of the aerodynamics by other leaves. At intermediate Reynolds numbers, a quadratic nonlinearity may be retained

in the Navier-Stokes equation to set up a convenient nonlinear version of the Brinkman screening theory [24]. Recently, however, a modification of the Oseen approximation [25] has been proposed to increase its range of validity well beyond $\text{Re} = \mathcal{O}(1)$ [26]. The trick is to use a higher renormalized viscosity (here denoted by ν). Thus, the velocity satisfies the modified Oseen equation, which will be used in Section 4.

In the next section I discuss the possible impact of the elastic tree energy on the aerodynamic turbulence.

3 Energy criterion

In the analysis above, the tree has been regarded as a passive entity. In order to assess whether this assumption is correct, I next consider an energy criterion as previously introduced by de Gennes in the case of drag reduction by polymer [4, 5]. A turbulent eddy of size ℓ and volume ℓ^3 is assumed to fluctuate coherently and isotropically on a scale ℓ (it also encompasses smaller eddies fluctuating at shorter time scales). Hence, the kinetic energy density (equal to the magnitude of the Reynolds stress) of such an eddy is $\rho_a V^2(\ell)$ where ρ_a is the density of air. Using (1), it is possible to write

$$U_{\text{kin}}(\ell) \simeq \rho_a U^2 \left(\frac{\ell}{H} \right)^{2/3}. \quad (7)$$

Now the bending energy of a branch is the average of Ez^2/ℓ^3 and since z is at most $\mathcal{O}(l)$, an upper bound on the bending energy density is

$$U_{b,\text{max}}(\ell) \simeq \frac{d^4 Y}{\ell^3}. \quad (8)$$

The section of the branch is enclosed by a blob of air of volume ℓ^3 . The ratio of the two densities is given by

$$R = \frac{U_{b,\text{max}}}{U_{\text{kin}}} = \frac{\rho_w}{\rho_a} \left(\frac{d}{\ell} \right)^2 \quad (9)$$

with the use of (6). This is about 10^{-1} so the branch is inferred to fluctuate passively. Nevertheless, a densely branched tree may conceivably have $R = \mathcal{O}(1)$ if there are enough branches within ℓ^3 .

4 Leaf aerodynamics

The Reynolds number of air flow near a leaf in the tree is typically about a hundred. Thus, we try to study the aerodynamics in the modified Oseen approximation [25]

$$\frac{\partial \vec{V}}{\partial t} + \vec{U} \cdot \vec{\nabla} \vec{V} = -\rho_a^{-1} \vec{\nabla} p + \nu \Delta \vec{V} \quad (10)$$

where $\vec{U}$ is the background velocity and p is the pressure. The air may be regarded as incompressible because U is much smaller than the velocity of sound in air. Let us set $\text{Re} \equiv 0$ momentarily and assume the cluster of leaves surrounding a branch gives rise to a hydrodynamic screening length ζ . A full analysis involving all particles is complicated [27] but a compact treatment introducing ζ at the beginning and a Schwinger variational principle yields identical results fast [28]. Here, a simple scaling analysis is given to see whether

hydrodynamic screening between the leaves may exist. In the stationary limit, we have upon enforcing screening via a Darcy term

$$\nu \Delta \vec{V} - \frac{\nu}{\zeta^2} \vec{V} = \rho_a^{-1} \vec{\nabla} p. \quad (11)$$

The (pre-averaged) velocity perturbation by a point-like force f (a delta function) is then [27]

$$V \sim \frac{e^{-r/\zeta} f}{\nu \rho_a r}. \quad (12)$$

which is simply a screened form of the usual hydrodynamic decay. The friction coefficient of a leaf, ω is then an average in terms of the pair correlation function $g(\vec{r})$ [29]

$$\omega^{-1} \simeq \left\langle \frac{e^{-1/\zeta}}{\nu \rho_a r} \right\rangle. \quad (13)$$

For leaves viewed as platelets of surface area S_p , $g(\vec{r})$ scales as r^{-1} so that the friction coefficient becomes

$$\omega \simeq \frac{\nu \rho_a S_p}{\zeta}. \quad (14)$$

If the cluster of leaves is enclosed in a column of length l and cross-section A (volume $\Omega = lA$), the pressure difference Δp is expressed by (11) (Darcy's law)

$$A \Delta p = \Omega \nu \rho_a U / \zeta^2. \quad (15)$$

On the other hand, this force on N leaves is also given by (14)

$$A \Delta p = N \nu \rho_a S_p \zeta^{-1} U. \quad (16)$$

We finally obtain an expression for the hydrodynamic screening length

$$\zeta \simeq \frac{\Omega}{NS} = S^{1/2} \varphi^{-1}. \quad (17)$$

The variable φ is a “hydrodynamic” volume fraction $S_p^{3/2} N / \Omega$.

We next scale relevant terms in (10) and (11) by introducing an intermediate scale r with swarm size $\gg r \gg \zeta$. The impact of inertia is denoted by the dimensionless quantity J

$$J \equiv \frac{\text{inertial term}}{\text{screening term}} = \frac{U \zeta^2}{r \nu}, \quad (18)$$

though the Reynolds number pertaining to a single leaf in the flow field is

$$\text{Re}_\ell \equiv \frac{U S_p^{1/2}}{\nu}. \quad (19)$$

We therefore have

$$\frac{J}{\text{Re}_\ell} = \frac{S_p^{1/2}}{r \varphi^2}. \quad (20)$$

One expects $\varphi = \mathcal{O}(1)$ or possibly even larger. Hence, the ratio J may be smaller than unity at substantial scales r . Inertia could be neglected in that case so the computation of the screening length is self-consistent. The flow at scaler is laminarized by the strong screening collectively caused by the leaves.

5 Concluding remarks

The main expression derived here is (6) which gives the penetration depth ℓ_* in terms of the branch diameter d_* . The morphology of the tree imposes a second relation so that a unique ℓ_* is obtained provided the exponent β is not lower than $4/3$. This is in accord with recent measurements of β which are slightly above this number. Note that there are other theories providing another estimate for β . Reasoning based on hydraulic networks [19, 20] or elasticity theory [21] yield $\beta = 3/2$. On the other hand, the fractal approximation deduced here is bound to break down at small ℓ . As mentioned earlier, a Taylor expansion like (2) could lead to inconsistencies. However, we know that wood hardens because it loses moisture over time [30]. The ℓ dependence of the time scale given by (5) would lessen if this effect were taken into account. Multiple solutions could then be obviated.

The scaling picture presented here is only a zeroth-order theory. In reality, the inertial regime may be perturbed by dissipation arising from air flow about the leaves. Canopies have been investigated in some detail. It is found that Kolmogorov's laws do not always hold at large distances [31] with implications for the restricted validity of (6). Viscous damping [32] within the wood of tree branches has also been neglected. Moreover, at high wind speeds leaves may reconfigure and curl up [33].

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