Unification Theorem for the Two Basic Dualities of Homothetic Demand Theory

PAUL A. SAMUELSON

Department of Economics, Massachusetts Institute of Technology, Cambridge, Mass. 02139

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ABSTRACT When income elasticities of demand are all unity, every dollar being spent in the same proportions at all levels of income, a homogeneous-first-degree, concave utility function exists to serve as an unequivocal measure of real output. Dual to it, and with identical concavity and homogeneity properties, is the minimized-cost-per-unit-of-output, a function of prices. Distinct from this production dual is the indirect-utility dual, representing, except for algebraic sign, the maximized level of utility attainable as a function of prices relative to income. These basic alternative dualities are shown to be related by a unifying theorem: The logarithm of either of the pair of production-dual functions has for its indirect-utility dual the logarithm of the other function. What is shown to be the same thing, the indirect-utility dual of the output function is, except for sign, the reciprocal of the output's production-dual.

Optimality Demand Relations

If income elasticities are unity, the case of *homothetic* preferences, it is known that one cardinal indicator of utility can be written as a homogeneous-first-degree function, unique except for scale, namely

$$f[u(Q)] = q(Q) \equiv m^{-1}q(mQ), f'[] > 0 < m \quad (1)$$

where throughout lower-case symbols stand for scalars and capital letters stand for row vectors, as with quantities of goods $Q = (q_1, \dots, q_n) = (q_j)$, prices $P = (p_j)$, normalized prices $Y = P/e$, where $e = p_1q_1 + \dots + p_nq_n = P \cdot Q = Q \cdot P$.

In the regular cases considered here any cardinal indicator of utility, $u(Q)$, will be quasi-concave with positive partial derivatives and no satiation. This requires that the homogeneous-first-degree function $q(Q)$ be concave with positive partial derivatives.

The observed demand functions, $Q = D[P/e] = D[Y]$, $Y = D^{-1}[Q]$, are generated as the utility maximizing solution subject to the budget constraint

$$\begin{aligned} \text{Max}_Q q(Q) \text{ subject to } Y \cdot Q &= 1 \\ &= q(D[Y]) = -q^*(Y) \end{aligned} \quad (2)$$

where $q^*(Y)$ will be soon defined as the indirect-utility dual to $q(Q)$. This constrained maximum implies

$$\begin{aligned} \frac{\partial q(Q)/\partial q_1}{p_1/e} &= \dots = \frac{\partial q(Q)/\partial q_n}{p_n/e} \\ &= \left[\sum_1^n q_j \partial q(Q)/\partial q_j \right] / \sum_1^n q_j p_j/e = q(Q)/1 \end{aligned} \quad (3)$$

Hence, as in Samuelson [1],

$$\begin{aligned} Y &= \text{gradient } \log q(Q) = [y_j] \\ &= [\partial \log q(Q)/\partial q_j] = D^{-1}[Q] \quad (4) \\ [\partial y_i/\partial q_j] &= [\partial^2 \log q(Q)/\partial q_i \partial q_j] = [\partial^2 \log q(Q)/\partial q_j \partial q_i] \\ &= [\partial y_j/\partial q_i] \end{aligned}$$

Since $q(Q)$ is concave, and $\log q$ is concave, so will be $\log q(Q)$, and indeed strongly concave in the regular case, with its Hessian matrix of second partial derivatives negative definite rather than semi-negative-definite as with homogeneous $q(Q)$. This Hessian matrix of $\log q(Q)$ serves as the Jacobian matrix of the demand functions of $D^{-1}[Q]$; being symmetric negative definite, it will have an inverse matrix with the same properties. Hence by a global theorem of Gale and Nikaido [2] on uniqueness of inversion, we can derive from $D^{-1}[Q]$ our desired demand functions $Q = D[Y]$. Actually, as is well known and easily shown, just as $D^{-1}[Q]$ can be written as the gradient of an existent function $\log q(Q)$, so this implies that $D[Y]$ can be written as the gradient of a defined existent function of prices. The next sections will relate this existent function to the indirect-utility dual of $\log q(Q)$ and to $\log p(P)$, and where $p(P)$ will be defined as the production dual to $q(Q)$ itself.

PRODUCTION DUALITY

As in Shephard [3] and Samuelson [4], we define $q(Q)$'s production dual $p(P)$ by

$$p(P) = \text{Min}_Q P \cdot Q / q(Q) \leq P \cdot Q / q(Q) \quad (5)$$

with the equality holding only for (Q, P) and (Q, Y) pairings that can be shown by explicit differentiation of (5) to satisfy the optimality equations of (3). From (5) directly, we verify that $p(P)$ is homogeneous-first-degree, $p(P) \equiv m^{-1}p(mP)$ because $q(Q)$ has this property. It is not hard to verify that concavity of $q(Q)$ implies concavity of $p(P)$, and that the contour, $p(p_1/p_q, p_2/p_q, \dots) = 1$, provides the important "factor-price frontier." From the symmetry relations

$$p(P)q(Q) \geq P \cdot Q = Q \cdot P = e \quad (6)$$

$$p(Y)q(Q) \geq 1 = p(Y)q(D[Y]) = p(D^{-1}[Q])q(Q)$$

we see that just as $p(\cdot)$ is dual to $q(\cdot)$, so is $q(\cdot)$ dual to $p(\cdot)$. Thus, the dual of the dual to a function is reflexively itself.

It follows by dual symmetry that we can write parallel to (3)

$$\frac{\partial p(Y)/\partial y_1}{q_1} = \dots = \frac{\partial p(Y)/\partial y_n}{q_n} = \frac{Y \cdot \text{grad } p(P)}{Y \cdot Q} = p(P) \quad (7)$$

the necessary conditions for

$$q(Q) = \min_P Q \cdot P / p(P) \quad (8)$$

Likewise we use dual symmetry to derive the parallel to (4)

$$Q = D[Y] = \text{grad} \log p(Y) \quad (9)$$

$$[\partial q_i / \partial y_j] = [\partial q_j / \partial y_i] = [\partial y_i / \partial q_j]^{-1}$$

In a forthcoming paper, Swamy and Samuelson [5] show that $p(P^1)/p(P^0)$ and $q(Q^1)/q(Q^0)$ provide exact index numbers of price change and quantity change between two situations (P^0, Q^0) and (P^1, Q^1) , thus resolving for the homothetic case the century-old search for ideal index numbers.

INDIRECT-UTILITY DUAL

On the basis of pioneering work by Hotelling, Roy, Houthakker, Gorman, and others, the full indirect-utility dual is introduced in Samuelson [1]. This must not be confused with the production dual of the last section.

By definition, the full indirect-utility dual to $u(Q)$ (with its proper reversal of algebraic sign) is given by

$$\begin{aligned} \max_Q u(Q) \text{ subject to } Y \cdot Q = 1 \\ Q \\ = u(D[Y]) = -u^*(Y) \end{aligned} \quad (10)$$

It should be noted that the full dual as defined here is of opposite sign to the classical indirect utility function, a convention with simplifying advantage.

Hence,

$$u(Q) + u^*(Y) \leq 0 \text{ for } Y \cdot Q = Q \cdot Y = 1 \quad (11)$$

and where the equality sign holds only for optimal pairings $Q = D[Y]$ and $Y = D^{-1}[Q]$. From the symmetry of (11) we see that the demand problem can be formulated as maximizing the left-hand side of (11) with respect to either set of variables Q or Y with the other set held constant. Hence, it is clear that both the direct and the indirect demand functions must have the same properties, and so must $u(\cdot)$ and $u^*(\cdot)$. Just as $u^*(\cdot)$ is dual to $u(\cdot)$, so is $u(\cdot)$ dual to $u^*(\cdot)$, with the result that the dual to the dual of any function is itself reflexively, $u(Q) \equiv u^{**}(Q)$. (In the above statement, the word indirect has been omitted for brevity.) In the general case where $u(Q)$ is quasi-concave in its arguments, with its indifference contours being convex from below, $u^*(Y)$ will thereby be quasi-concave in its arguments too. Indeed the form of the functions is indistinguishable.

UNIFYING THE TWO DUALITIES

We can now specialize $u(Q)$ to $\log q(Q)$ and relate the two alternative definitions of duality. For observed pairings, $Q = D[Y]$, $Y = D^{-1}[Q]$, we know

$$\begin{aligned} p(Y)q(Q) &= 1 \\ \log p(Y) + \log q(Q) &= 0 \\ u^*(Y) + u(Q) &= 0 \end{aligned} \quad (12)$$

By identifying the respective functions of P and of Q , we see that if we choose in the beginning for our $u(Q)$ the particular utility indicator $\log q(Q)$, then we shall get for its full indirect-utility dual $u^*(Y) = \log p(Y)$. This suggests our unifying theorem.

THEOREM. *For homothetic demand, the cardinal utility indicator, $q(Q) \equiv m^{-1}q(mQ)$, that is homogeneous-first-degree and concave has a production dual $p(Y)$ or $p(P) \equiv m^{-1}p(mP)$ with similar properties. Then (a) the full indirect-utility dual to $\log q(Q)$, which except for sign represents the maximum of this function attainable with the given budget of normalized prices Y , will be found to be $\log p(Y)$; (b) the full indirect-utility dual to $q(Q)$ itself will be, with its sign reversed, the reciprocal of the production dual, $-q^*(Y) \equiv p(Y)^{-1}$*

Since the proof of (a) has already been sketched, it remains only to prove (b). We note that, if $u(\cdot)$ and $u^*(\cdot)$ are indirect-utility duals, so are $f[u(\cdot)]$ and $-f[u^*(\cdot)]$ from the definition in (10). Hence, pair up

$$f[u(Q)] = \exp u(Q) = \exp \log q(Q) = q(Q) \quad (13)$$

$$\begin{aligned} -f[u^*(Y)] &= -\exp [-u^*(Y)] = -\exp [-\log p(Y)] \\ &= -p(Y)^{-1} = q^*(Y) \end{aligned}$$

$$-q^*(Y) \equiv p(Y)^{-1} \quad \text{Q.E.D.}$$

It is worth emphasizing that being able to summarize the information contained in n different demand functions of n variables in terms of the gradient of a single function of n variables represents a tremendous economy of description, a fact well known to students of Gibbsian thermodynamics.

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