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Non-Iterative Reconstruction with a Prior for Undersampled Radial MRI Data

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Abstract

This paper develops an FBP-MAP (Filtered Backprojection, Maximum *a Posteriori*) algorithm to reconstruct MRI images from under-sampled data. An objective function is first set up for the MRI reconstruction problem with a data fidelity term and a Bayesian term. The Bayesian term is a constraint in the temporal dimension. This objective function is minimized using the calculus of variations. The proposed algorithm is non-iterative. Undersampled dynamic myocardial perfusion MRI data were used to test the feasibility of the proposed technique. It is shown that the non-iterative Fourier reconstruction method effectively incorporates the temporal constraint and significantly reduces the angular aliasing artifacts caused by undersampling. A significant advantage of the proposed non-iterative Fourier technique over the iterative techniques is its fast computation time.

Keywords

Constrained image reconstruction; analytical reconstruction algorithm; tomography; undersampled MRI; radial sampled MRI

I. Introduction

The Fourier transform based image reconstruction is standard practice in MRI. However, when constraints need to be incorporated in the image reconstruction, iterative algorithms are utilized. One obvious drawback of iterative reconstruction methods is the long computation time. This situation is similar to that in x-ray CT, where the FBP (filtered backprojection) algorithm is the work horse and iterative algorithms are called for if constrained reconstruction is demanded.

We must point out that the main focus of this paper is not on compressed sensing MRI, which considers undersampled k -space data and uses iterative total variation (TV) constrained techniques for image reconstruction (Lustig et al 2006, Lustig et al 2008, Ye et al 2007, Block et al 2007, Chang et al 2006, Wajer 2001). This paper considers a less ill-conditioned problem, where the k -space is undersampled at each time frame, but the k -space

data are available at other time frames. Data at other time frame can assist the image reconstruction at current time frame. This data assisting image reconstruction method has been intensively studied in our group and other groups (Chen et al 2008, Francois et al 2009, Fessler and Noll 2004, Ramani and Fessler 2011, Adluru et al 2009, Adluru 2010, Cao and Levin 1995). In our group, iterative spatiotemporal constrained reconstruction (STCR) methods use the L_1 norm for the special constraint and the L_1 or L_2 norm for the temporal constraint (Adluru et al 2009).

Recently we developed a non-iterative FBP-MAP (Maximum *a Posteriori*) algorithm for low-dose x-ray CT image reconstruction (Zeng 2012). The FBP-MAP algorithm in (Zeng 2012) emulates an iterative algorithm and can provide almost the same result that the iterative algorithm achieves at a given iteration number. The current paper uses a different (the calculus of variations) method to derive an FBP-MAP algorithm and applies the algorithm to the field of undersampled MRI, in which the k -space is radially sampled.

Since we do not know how to deal with the L_1 norm in an analytical algorithm, the L_1 norm (or TV norm) spatial constraint is not considered; the temporal constraint is considered using the L_2 norm. The temporal constraint encourages the reconstruction to look like a reference image, which can be the image of the dynamic images using the averaged adjacent data. The derivation of the algorithm based on the calculus of variations will be presented in the next section. A dynamic cardiac MRI study is then used to illustrate the feasibility of the proposed method. Finally, some conclusions are drawn at the end.

II. Methods

The radially sampled k -space MRI data are in the Fourier domain. The k -space data are first converted into the spatial domain by a one-dimensional (1D) Fourier transform. The spatial-domain data have a real part and an imaginary part. Our strategy is to reconstruct a real-part image, Img_R , using the real-part spatial-domain data and to reconstruct an imaginary-part image, Img_I , using the imaginary-part spatial-domain data, separately. The final image is the

norm of these two reconstructions, that is, $\sqrt{Img_R^2 + Img_I^2}$. The method of reconstruct Img_R and the method of reconstruct Img_I are identical. In the following sections, we will assume that the projection data are given in the spatial domain and are real, without specifying whether they are the real-part or the imaginary part. Therefore, the image reconstruction method is the same as that for the Radon transform, which is the line-integral of the two-dimensional (2D) object.

A. Setup of the objective function

We consider the (2D) Radon transform and its reconstruction. Let the image to be reconstructed be $f(x, y)$ and its Radon transform be $[Rf](s, \theta)$, defined as

$$[Rf](s, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - s) dx dy, \quad (1)$$

where δ is the Dirac delta function, θ is the detector rotation angle, and s is the line-integral location on the detector. The Radon transform $[Rf](s, \theta)$ is the line-integral of the object $f(x, y)$. The coordinate systems are shown in Fig. 1. Image reconstruction is to solve for the object $f(x, y)$ from its Radon transform $[Rf](s, \theta)$. A popular image reconstruction method is the FBP algorithm, which consists of two steps.

In the first step of the FBP algorithm, $[Rf](s, \theta)$ is convolved with a kernel function $h(s)$ with respect to variable s , obtaining $q(s, \theta)$. The Fourier transform of the kernel $h(s)$ is the ramp-filter $|\omega|$, where ω is the Fourier variable with respect to s . The second step performs backprojection, which maps the filtered projection $q(s, \theta)$ into the image domain as

$$f(x, y) = \int_0^\pi q(s, \theta) \big|_{s=x\cos\theta+y\sin\theta} d\theta. \quad (2)$$

A common concern of this FBP reconstruction is its poor noise performance and angular aliasing artifacts. To reduce the noise and artifacts, this paper enforces a prior term in the objective function, and this approach is commonly used in iterative algorithms.

Let the noisy line-integral measurements be $p(s, \theta)$. The objective function v depends on the function $f(x, y)$ as follows.

$$v(f) = \| [Rf](s, \theta) - p(s, \theta) \|^2 + \beta \| f - g \|^2, \quad (3)$$

where the first term enforces data fidelity, the second term imposes prior information about the image f , the parameter $\beta > 0$ controls the level of influence of the prior information to the image f , and g is a reference image. Using integrals, v in (3) can be written as

$$v(f) = \int_{\theta=0}^\pi \int_{s=-\infty}^\infty \left[\int_{x=-\infty}^\infty \int_{y=-\infty}^\infty f(x, y) \delta(x\cos\theta + y\sin\theta - s) dy dx - p(s, \theta) \right]^2 ds d\theta + \beta \int_{x=-\infty}^\infty \int_{y=-\infty}^\infty [f(x, y) - g(x, y)]^2 dy dx. \quad (4)$$

B. Optimization

This section uses the calculus of variations to find the optimal function $f(x, y)$ that minimizes the objective function $v(f)$ in (4). The initial step is to replace the function $f(x, y)$ in (4) by the sum of two functions: $f(x, y) + \varepsilon \eta(x, y)$ (van Brunt 2004). The next step dv is to evaluate

$\frac{dv}{d\varepsilon} \big|_{\varepsilon=0}$ and set it to zero (van Brunt 2004). That is,

$$\begin{aligned} 0 = & 2 \int_{\theta=0}^\pi \int_{s=-\infty}^\infty \left\{ \left[\int_{x=-\infty}^\infty \int_{y=-\infty}^\infty f(x, y) \delta(x\cos\theta + y\sin\theta - s) dy dx - p(s, \theta) \right] \right. \\ & \times \left[\int_{x=-\infty}^\infty \int_{y=-\infty}^\infty \eta(x, y) \delta(x\cos\theta + y\sin\theta - s) dy dx \right] \Big\} ds d\theta \\ & + 2\beta \int_{x=-\infty}^\infty \int_{y=-\infty}^\infty [f(x, y) - g(x, y)] \eta(x, y) dy dx. \end{aligned} \quad (5)$$

In practice, the function $f(x, y)$ is compact, bounded, and continuous almost everywhere. After changing the order of integrals, we have

$$0 = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \eta(x, y) \times \left\{ \int_{\theta=0}^{\pi} \int_{s=-\infty}^{\infty} \left[\int_{\hat{x}=-\infty}^{\infty} \int_{\hat{y}=-\infty}^{\infty} f(\hat{x}, \hat{y}) \delta(\hat{x} \cos \theta + \hat{y} \sin \theta - s) d\hat{y} d\hat{x} - p(s, \theta) \right] \delta(x \cos \theta + y \sin \theta - s) ds d\theta \right. \\ \left. + \beta [f(x, y) - g(x, y)] \right\} dy dx. \quad (6)$$

Eq. (6) is in the form of $\iint \eta(x, y) c(x, y) dy dx = 0$. Since $\eta(x, y)$ can be any arbitrary function, according to the calculus of variations, one must have $c(x, y) = 0$, which is the Euler-Lagrange equation (van Brunt 2004). The Euler-Lagrange equation in our case is

$$\int_{\theta=0}^{\pi} \int_{s=-\infty}^{\infty} \left[\int_{\hat{x}=-\infty}^{\infty} \int_{\hat{y}=-\infty}^{\infty} f(\hat{x}, \hat{y}) \delta(\hat{x} \cos \theta + \hat{y} \sin \theta - s) d\hat{y} d\hat{x} - p(s, \theta) \right] \delta(x \cos \theta + y \sin \theta - s) ds d\theta + \beta [f(x, y) - g(x, y)] = 0. \quad (7)$$

Eq. (7) can be further rewritten as

$$\int_{\hat{x}=-\infty}^{\infty} \int_{\hat{y}=-\infty}^{\infty} f(\hat{x}, \hat{y}) \left[\int_{\theta=0}^{\pi} \int_{s=-\infty}^{\infty} \delta(\hat{x} \cos \theta + \hat{y} \sin \theta - s) \delta(x \cos \theta + y \sin \theta - s) ds d\theta \right] d\hat{y} d\hat{x} + \beta f(x, y) \\ = \int_{\theta=0}^{\pi} \int_{s=-\infty}^{\infty} p(s, \theta) \delta(x \cos \theta + y \sin \theta - s) ds d\theta + \beta g(x, y). \quad (8)$$

Notice that

$$\int_{\theta=0}^{\pi} \int_{s=-\infty}^{\infty} p(s, \theta) \delta(x \cos \theta + y \sin \theta - s) ds d\theta = \int_{\theta=0}^{\pi} p(s, \theta) \big|_{s=x \cos \theta + y \sin \theta} d\theta = b(x, y) \quad (9)$$

is the backprojection of the projection data $p(s, \theta)$, and the backprojection is denoted as $b(x, y)$. It must be pointed out that $b(x, y)$ is not the same as $f(x, y)$ even when $p(s, \theta)$ is noiseless, because no ramp filter has been applied to $p(s, \theta)$.

Also notice that

$$\int_{\theta=0}^{\pi} \int_{s=-\infty}^{\infty} \delta(\hat{x} \cos \theta + \hat{y} \sin \theta - s) \delta(x \cos \theta + y \sin \theta - s) ds d\theta = \int_{\theta=0}^{\pi} \delta((x - \hat{x}) \cos \theta + (y - \hat{y}) \sin \theta) d\theta = \frac{1}{\sqrt{(x - \hat{x})^2 + (y - \hat{y})^2}} \quad (10)$$

is the point response function of the projection/ backprojection operator at point (x, y) if the point source is at $(x, \hat{y})$ (Zeng 2009).

Using (9) and (10), (8) becomes

$$\int_{\hat{x}=-\infty}^{\infty} \int_{\hat{y}=-\infty}^{\infty} f(\hat{x}, \hat{y}) \left[\frac{1}{\sqrt{(x-\hat{x})^2 + (y-\hat{y})^2}} + \beta \cdot \delta(x-\hat{x}, y-\hat{y}) \right] d\hat{y} d\hat{x} = b(x, y) + \beta g(x, y). \quad (11)$$

The left-hand-side of (11) is a 2D convolution. Taking the 2D Fourier transform of (11) yields

$$F(\omega_x, \omega_y) \times \left[\frac{1}{\sqrt{\omega_x^2 + \omega_y^2}} + \beta \right] = B(\omega_x, \omega_y) + \beta G(\omega_x, \omega_y), \quad (12)$$

or

$$F(\omega_x, \omega_y) = [B(\omega_x, \omega_y) + \beta G(\omega_x, \omega_y)] / \left[\frac{1}{\sqrt{\omega_x^2 + \omega_y^2}} + \beta \right]. \quad (13)$$

Here the capital letters are used to represent the Fourier transform of their spatial domain counterparts, which are represented in lowercase letters; ω_x and ω_y are the frequency variables for x and y , respectively.

Eq. (13) is in the form of “backprojection first, then 2D filter” reconstruction approach and the Fourier domain 2D filter is $\sqrt{\omega_x^2 + \omega_y^2} / (1 + \beta \cdot \sqrt{\omega_x^2 + \omega_y^2})$ (Zeng 2009). By using the central slice theorem (Zeng 2009), an FBP algorithm, which performs 1D filtering first, then backprojects, can be obtained, and the Fourier domain 1D filter is the central slice of the 2D filter $\sqrt{\omega_x^2 + \omega_y^2} / (1 + \beta \cdot \sqrt{\omega_x^2 + \omega_y^2})$, which gives

$$\text{Fourier domain filter for the FBP-MAP algorithm} = |\omega| / [1 + \beta \cdot |\omega|]. \quad (14)$$

When $\beta = 0$, (14) reduces to the ramp filter $|\omega|$ of the conventional FBP algorithm.

C. Implementation

In a typical FBP-type algorithm, the projection data are first filtered by the ramp filter $|\omega|$, and then are backprojected into the image domain. In our newly derived FBP-MAP algorithm, the conventional ramp filter $|\omega|$ is replaced by $|\omega| / [1 + \beta \cdot |\omega|]$. Our backprojector is the same as in the conventional FBP algorithm.

Notice the right-hand-side of (11) or (12). There are two components contributing to the input of the FBP-MAP algorithm. In (11), $b(x, y)$ is the backprojection of the primary measurements $p(s, \theta)$ and $g(x, y)$ is the reference image. The reference image $g(x, y)$ can be expressed as the conventional FBP reconstruction from a set of secondary measurements $p_g(s, \theta)$. After ramp-filtering, $p_g(s, \theta)$ becomes $q_g(s, \theta)$. The combined input data for the FBP-MAP algorithm are sum of $p(s, \theta) + \beta q_g(s, \theta)$.

The implementation procedure of the proposed FBP-MAP algorithm is as follows:

Step 1: Prepare two sets of projection data: the primary set $p(s, \theta)$ and the secondary set $p_g(s, \theta)$.

Step 2: Apply the conventional ramp filter $|\omega|$ to the secondary projection set $p_g(s, \theta)$, obtaining $q_g(s, \theta)$.

Step 3: Form the new projection data set: $p(s, \theta) + \beta q_g(s, \theta)$.

Step 4: Apply the newly derived FBP-MAP filter $|\omega|/[1+\beta|\omega|]$ to the combined data set.

Step 5: Perform conventional backprojection.

In our implementation, the primary data are the current time frame data, while the secondary data are the summation of the data from the current time frame and the data from 4 other time frames.

The above procedure is used to obtain the reconstruction Img_R from the real-part of the spatial-domain data and obtain the reconstruction Img_I from the imaginary-part of the spatial-domain data. The final image is the norm of these two reconstructions, that is,

$$\sqrt{Img_R^2 + Img_I^2}.$$

C. Perfusion MRI study

To illustrate the feasibility of the proposed FBP-MAP algorithm, a human cardiac perfusion study was used (Adluru 2009). The MRI data were acquired with Siemens 3T Trio scanner, using phased array of coils, of which one was chosen to demonstrate the method. The scanner parameters for the radial acquisition were TR = 2.6 msec, TE = 1.1 msec, flip angle = 12°, Gd dose = 0.03 mmol/kg, slice thickness = 6 mm. Reconstruction pixel size was $1.8 \times 1.8 \text{ mm}^2$. Each image was acquired in a 62 msec readout. The acquisition matrix size for an image frame was 256×24 , and 60 sequential images were obtained at 60 different times. At each time frame, the k -space is sampled with 24 uniformly spaced radial lines over an angular range of 180°; however, the 24-line sampling patterns of the adjacent time frames are offset by 180°/96.

If one sums up the measurements from temporally adjacent 4 time frames, the summed k -space will have a 96-line sampling pattern, uniformly distributed over an angular range of 180°. In our image reconstruction method, each time frame requires current 24-line measurement P and associated time-averaged 96-line measurement $\hat{P}$, which uses the measurements from the current 24-line data, two “immediately after” 24-line measurements, and two “immediately before” 24-line measurements. A symbolic expression for $\hat{P}$ is given as:

$$\hat{P}(t) = P(t-2)/2 + P(t-1) + P(t) + P(t+1) + P(t+2)/2. \quad (15)$$

The fact that one image was acquired with 24 views (i.e., 24 radial lines in the k -space) makes the k -space undersampled.

Both the conventional FBP algorithm and the proposed non-iterative FBP-MAP algorithm were used to reconstruct the images. In our MRI data acquisition, each k -space radial readout had 256 samples. After zero-padding, the length N was chosen as 1024, and the frequency variable ω took discrete values of $2\pi n/N$, for integers n . Parameter β was chosen as 0.07, which was selected by experience and the noise level of a data set. Parameter β controls the influence of the reference image, and $\beta=0$ implies that the reference image is not used.

III. Results

Figure 2 shows the reconstruction results from the undersampled MRI data using both conventional FBP algorithm and proposed FBP-MAP algorithm. The images obtained by the convention method (see the 1st row of Fig. 2) suffer from severe angular aliasing artifacts and are noisy. The angular aliasing artifacts are significantly reduced and the noise is suppressed in the images obtained by the proposed FBP-MAP method (see the 2nd row of Fig. 2).

There are 60 time frames in the study, and results form a subset of 6 time frames are shown in Fig. 2, and the time spacing is 10 time frames. Due to the nature of dynamic imaging, the image intensity of a series of image changes over time. In order to able to view each individual reconstruction properly, each image in the 1st and 2nd rows is displayed from zero to its maximum value.

The 3rd row of Fig. 2 shows the difference between the 1st row and the 2nd row. The purpose of the difference image is to check whether the constrained reconstruction is able to track the dynamic changes and shapes overtime. The images in the 3rd row are obtained as follows. Both of the FBP images and FBP-MAP images are first normalized. If one treats the time as the third dimension, the normalization procedure is three-dimensional, with the first two dimensions as spatial and the third dimension as temporal. The normalized “3D” FBP image has a maximum value of unity and so does the normalized “3D” FBP-MAP image. The “3D” difference image is calculated as the normalized FBP image minus the normalized FBP-MAP image. The difference image for all time frames is displayed using a common gray-scale interval of $[-0.2, 0.2]$ in the 3rd row of Fig. 2. In the cardiac region, the difference image is almost zero, implying that the constrained reconstruction faithfully follows the intensity change and the cardiac motion as represented by the FBP reconstruction.

IV Conclusions

This paper introduces an FBP-MAP algorithm, based on minimizing an objective function that contains a data fidelity term and a constraint (Bayesian) term. The constraint encourages the image solution to look like a short-term time-average image in a dynamic data acquisition. Since no spatial constraint is used in the FBP-MAP algorithm, the FBP-MAP images do not look as sharp as those obtained by the iterative algorithm that also contains the TV constraint. It is our future research goal to include the spatial constraint into the FBP-MAP algorithm. This new algorithm is Fourier transform based and is non-iterative. The

most distinguished advantage of this new algorithm over the iterative algorithms is its fast computation time.

This paper also makes an important observation when the primary and secondary projection data sets are to be combined. These two data sets cannot be combined by a simple weighted summation. The secondary (that is, less important) data set needs to be filtered by a ramp filter before the weighted summation can be performed, in which the DC component is removed and the low frequency components are significantly suppressed.

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Figure 1.
Coordinate systems for 2D Radon transform.

Figure 2.

Comparison of conventional FBP reconstruction (1st row) and proposed FBP-MAP reconstruction (2nd row), using undersampled dynamic MRI data. The 3rd row shows the difference images between the 1st and 2nd rows. Six time frames are shown from left to right.