Holomorphic Embedding of Complex Curves in Spaces of Constant Holomorphic Curvature

(Wirtinger's theorem/Kaehler manifold/Riemann surfaces)

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ABSTRACT A special case of Wirtinger's theorem asserts that a complex curve (two-dimensional) holomorphically embedded in a Kaehler manifold is a minimal surface. The converse is not necessarily true. Guided by considerations from the theory of moduli of Riemann surfaces, we discover (among other results) sufficient topological and differential-geometric conditions for a minimal (Riemannian) immersion of a 2-manifold in complex projective space with the Fubini-Study metric to be holomorphic.

INTRODUCTION AND MOTIVATION

It is known [1-5] how Riemann's conception of moduli for conformal mapping of homeomorphic, multiply connected Riemann surfaces on one another led to the conception of moduli, or, better, *infinitesimal moduli*, as obstacles or *obstructions* to the realization of such a map between two given surfaces.

The question arises quite naturally as to whether analogous phenomena occur in other, in a certain sense naturally related, mapping problems.

Certainly a related problem is that of attempting to map a Riemann surface (complex curve) complex analytically into a higher-dimensional complex manifold. That there are some sorts of obstructions arising in general is clear. The question that presents itself is whether the obstructions appear as infinitesimal moduli related to, or resembling those of, the moduli problem of the first paragraph.

We recall that the infinitesimal moduli there are precisely the everywhere-finite complex analytic *quadratic differentials* (real on any boundaries) on the Riemann surfaces. More precisely: in attempting to construct the desired map of surface *one* on surface *two*, we transform the problem into a *differential-geometric extremal problem* by putting a Riemannian metric, consistent with the complex structure, on surface *two*; we then construct a map of *one* on *two* that is extremal for some differential-geometric functional and find that the pull-back to *one* of the metric on *two* generates a quadratic differential as described. The extremal map represents the *attempt*; the *success* or *failure* is determined by whether the quadratic differential vanishes identically or not.

In references [2-4], the extremal problem was suggested by Douglas's solution of the Plateau problem, so that the notion of *minimal surface* is evoked.

The relevance of all this to our new mapping problem becomes clearer when we recall Wirtinger's theorem [6], a special case of which asserts that a holomorphically embedded complex curve in a Kaehler manifold is a minimal surface. How-

ever, the converse is not true in general: the complex curve embedded as a real minimal surface is not necessarily holomorphically embedded—there are *obstructions*. With the moduli problem in mind, this fact suggests that we view the embedding of our complex curve as a real minimal surface in a Kaehler manifold as the solution to the differential-geometric extremal problem for our present mapping problem and that we then seek, as our infinitesimal moduli, *differential-geometric invariants on the curve whose vanishing forms necessary and sufficient conditions for the minimal embedding to be holomorphic*. Going further, we observe that minimal surface evokes the notion of *second fundamental forms*; while the moduli problem, again, suggests quadratic differentials. Hence, as our differential-geometric invariants, we seek quadratic differentials on the curve generated by the second fundamental forms of the embedding. In another, simpler, context Heinz Hopf [7] has done this.

Adopting an experimental approach, we take the simplest examples for our Kaehler manifold, namely, those with constant holomorphic curvature. Anticipating the necessity for some topological restrictions, since no local differential geometric conditions could force some pieces of minimal surface to be holomorphically embedded, we consider as a principal example a compact curve. Also, in view of the minimal area character of the extremal problem, we then assume the image is not homologous to zero, and we choose as the most interesting case of the ambient manifold the complex projective space with the Fubini-Study metric.

Armed with these heuristic considerations, the reader is better prepared for our main theorem below. See also *Remark 2* at the end of that section.

MAIN THEOREM AND PROOF

THEOREM. Let $M^n(x)$ be an n -(complex) dimensional Kaehlerian manifold of constant holomorphic curvature x , M a connected orientable 2-(real) dimensional Riemannian manifold, and $f:M \rightarrow M^n(x)$ a minimal Riemannian immersion. If $x = 0$, then the second fundamental form of M in $M^n(x)$ is formally holomorphic (compare definition below) relative to the complex structure on M induced by its Riemannian metric. If $x \neq 0$ then the second fundamental form of M in $M^n(x)$ is formally holomorphic if and only if: either at every point p of M J [the complex structure of $M^n(x)$] takes the tangent space to $f(M)$ at $f(p)$ into its orthogonal complement in the tangent space to $M^n(x)$ at $f(p)$, or for every point p of M J leaves the tangent space to $f(M)$ at $f(p)$

invariant, i.e., $f(M)$ is a complex submanifold of $M^n(\kappa)$. In particular, if $M^n(\kappa)$ is complex projective space with its Fubini-Study metric, and M is compact, connected, orientable, $f: M \rightarrow CP_n$ minimal such that $f(M)$ is not homologous to zero, then the formal holomorphicity of the second fundamental form implies $f(M)$ is a complex submanifold of CP_n .

Our proof is suggested by S. S. Chern's proof [8, 9] of a theorem of F. J. Almgren [10] and E. Calabi [11] that states that a minimal immersion of the 2-sphere into the standard 3-sphere of constant curvature 1 is totally geodesic. In particular, we follow Chern's notation and leave it to the reader to fill in the necessary details by reference to [8]. Our indices A, B, C, D will range from 1 to $2n$; j, k, l from 1 to 2; α, β, γ from 3 to $2n$; and we always sum repeated indices unless otherwise indicated. We now turn to the formulas needed for subsequent calculation.

Let $\{e_A\}$ denote a local orthonormal frame field on $M^n(\kappa)$ with dual basis of local 1-forms $\{\omega_A\}$. The complex structure on $M^n(\kappa)$ will be given locally by $Je_A = J_{AB}e_B$, where $J^2 = -1$, and J is orthogonal (in particular, J is skew-symmetric). The Levi-Civita connection of $M^n(\kappa)$ is given locally by a unique skew-symmetric matrix of local 1-forms (ω_{AB}) which satisfy the equations

$$d\omega_A = \omega_B \wedge \omega_{BA}. \quad (1)$$

The curvature matrix $\Omega_{AB} = (-1/2)R_{ABCD}\omega_C \wedge \omega_D$ (where $R_{ABCD} + R_{ABDC} = 0$) is skew-symmetric and satisfies the equations

$$d\omega_{AB} = \omega_{AC} \wedge \omega_{CB} + \Omega_{AB}, \quad (2)$$

and the metric, being of constant holomorphic curvature κ , is characterized by

$$R_{ABCD} = (\kappa/4)(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}). \quad (3)$$

Now let $f: M \rightarrow M^n(\kappa)$ be an immersion. We then use adapted frames $e_1, \dots, e_{2n}$ along $f(M)$, i.e., along $f(M)$ we assume e_1, e_2 are tangent to $f(M)$, and $e_3, \dots, e_{2n}$ are normal to $f(M)$. Then on M (we do not indicate f^* since there is no confusion) we have $0 = \omega_3 = \dots = \omega_{2n} = 0$, which implies by (1)

$$\begin{aligned} d\omega_k &= \omega_j \wedge \omega_{jk} \\ 0 &= \omega_j \wedge \omega_{j\alpha}. \end{aligned} \quad (4)$$

Therefore, by E. Cartan's lemma, $\omega_{j\alpha}$ can be written as

$$\begin{aligned} \omega_{j\alpha} &= \omega_k h_{kj\alpha} \\ h_{kj\alpha} &= h_{jk\alpha} \end{aligned}$$

and $\omega_k \omega_j h_{kj\alpha} e_\alpha$ is the second fundamental form of M in $M^n(\kappa)$. On M we also have by (3)

$$\Omega_{j\alpha} = (-\kappa/8)(J_{jk}J_{\alpha l} - J_{jl}J_{\alpha k} + 2J_{j\alpha}J_{kl})\omega_k \wedge \omega_l \quad (5)$$

and the covariant differential of the second fundamental form is given by

$$Dh_{jk\alpha} = dh_{jk\alpha} - \omega_j h_{ljk\alpha} - \omega_k h_{jlk\alpha} + h_{jk\beta} \omega_{\beta\alpha}. \quad (6)$$

The metric on M induces a complex structure on M such that $\omega = \omega_1 + i\omega_2$ is a $(1,0)$ -form, and relative to this complex

structure, we have

$$\omega_k \omega_j h_{kl\alpha} e_\alpha = (1/2)\{Re \omega^2(h_{11\alpha} - h_{22\alpha} - 2ih_{12\alpha}) + \omega\bar{\omega}(h_{11\alpha} + h_{22\alpha})\}e_\alpha.$$

We henceforth assume that the immersion is minimal, i.e., $h_{11\alpha} + h_{22\alpha} = 0$; therefore,

$$\omega_k \omega_j h_{kl\alpha} = \{Re \omega^2(h_{11\alpha} - ih_{12\alpha})\}e_\alpha,$$

i.e., the second fundamental form is the real part of a vector-valued quadratic differential. We define the second fundamental form to be formally holomorphic if

$$0 = \omega \wedge (Dh_{11\alpha} - iDh_{12\alpha}), \quad \alpha = 3, \dots, 2n$$

on M . One easily checks that the formula is invariant under change of adapted frame. We remark that if we know *a priori* that M is holomorphically embedded, then formally holomorphic is holomorphic.

LEMMA. For $f: M \rightarrow M^n(\kappa)$ minimal we have

$$(3\kappa/4)(J_{1\alpha} - iJ_{2\alpha})J_{12\omega_1} \wedge \omega_2 = \omega \wedge (Dh_{11\alpha} - iDh_{12\alpha}).$$

Proof: Set $\Omega_\alpha = \omega_{1\alpha} + i\omega_{2\alpha}$, $H_\alpha = h_{11\alpha} + ih_{12\alpha}$. Then by (4,2,5) we have

$$\begin{aligned} d\omega &= -i\omega_{12} \wedge \omega \\ d\Omega_\alpha &= -i\omega_{12} \wedge \Omega_\alpha + \Omega_\beta \wedge \omega_{\beta\alpha} - \\ &\quad (3\kappa/4)(J_{1\alpha} + iJ_{2\alpha})J_{12\omega_1} \wedge \omega_2 \end{aligned} \quad (7)$$

and the minimality of the immersion implies $\Omega_\alpha = \bar{\omega}H_\alpha$, which implies

$$d\Omega_\alpha = i\omega_{12} \wedge \Omega_\alpha - \bar{\omega} \wedge dH_\alpha. \quad (8)$$

By subtracting (8) from (7), we obtain

$$\begin{aligned} 0 &= \bar{\omega} \wedge dH_\alpha - 2i\omega_{12} \wedge \Omega_\alpha + \Omega_\beta \wedge \omega_{\beta\alpha} - \\ &\quad (3\kappa/4)(J_{1\alpha} + iJ_{2\alpha})J_{12\omega_1} \wedge \omega_2 \\ &= \bar{\omega} \wedge (dH_\alpha + 2i\omega_{12}H_\alpha + H_\beta \omega_{\beta\alpha}) - \\ &\quad (3\kappa/4)(J_{1\alpha} + iJ_{2\alpha})J_{12\omega_1} \wedge \omega_2. \end{aligned}$$

Hence, taking conjugates, we get

$$\begin{aligned} 0 &= \omega \wedge (d\bar{H}_\alpha - 2i\omega_{12}\bar{H}_\alpha + \bar{H}_\beta \omega_{\beta\alpha}) - \\ &\quad (3\kappa/4)(J_{1\alpha} - iJ_{2\alpha})J_{12\omega_1} \wedge \omega_2 \\ &= \omega \wedge (Dh_{11\alpha} - iDh_{12\alpha}) - (3\kappa/4)(J_{1\alpha} - iJ_{2\alpha})J_{12\omega_1} \wedge \omega_2 \end{aligned}$$

by (6), and the lemma is proven.

Proof of the Theorem: If $\kappa = 0$ then the theorem follows directly from the lemma. If $\kappa \neq 0$ then the second fundamental form of M in $M^n(\kappa)$ is formally holomorphic if and only if $(J_{1\alpha} - iJ_{2\alpha})J_{12} = 0$, $\alpha = 3, \dots, 2n$ on all of M . Now $J_{12} = 0$ is equivalent to saying that J takes the tangent space to $f(M)$ into its orthogonal complement, and $J_{1\alpha} - iJ_{2\alpha} = 0$, $\alpha = 3, \dots, 2n$ is equivalent to saying that J leaves the tangent space to $f(M)$ invariant. By the nonsingularity, and continuity of J , we have the second statement of the theorem. For the final statement, all we need remark is that $J_{12} = 0$ on all of $f(M)$ implies that the fundamental Kaehler form vanishes identically on $f(M)$ (indeed, the fundamental Kaehler form is given by $\frac{1}{2}\omega_A \wedge J\omega_A$).

Remark 1: We leave it to the reader to check that for arbitrary Riemannian manifolds M^n the holomorphicity of the second fundamental form is equivalent to the invariance of the tangent space to M under the action of the curvature tensor as a map of 2-vectors to 2-vectors.

Remark 2: Standard Riemannian function theory shows (pluricanonical embedding) that any compact Riemann surface can be embedded *one-one* and nonsingularly in complex projective space of sufficiently high dimension. The point of the note, however, is not the existence of an embedding but the attempt to "reverse" Wirtinger's theorem, and the point of the first section is to show the heuristic arguments that led to the main theorem.

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