Investigating the thermally induced acoustoelastic effect in isotropic media with Lamb waves

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Elastic wave velocities in metallic structures are affected by variations in environmental conditions such as changing temperature. This paper extends the theory of acoustoelasticity by allowing thermally induced strains in unconstrained isotropic media, and it experimentally examines the velocity variation of Lamb waves in aluminum plates (AL-6061) due to isothermal temperature deviations. This paper presents both thermally induced acoustoelastic constants and thermally varying effective Young's modulus and Poisson's ratio which include the third order elastic material constants. The experimental thermal sensitivity of the phase velocity ($\partial v_p / \partial \theta$) for both the symmetric and antisymmetric modes are bounded by two theories, the acoustoelastic Lamb wave theory with thermo-acoustoelastic tensors and the thermoelastic Lamb wave theory using an effective thermo-acoustoelastic moduli. This paper shows the theoretical thermally induced acoustoelastic Lamb wave thermal sensitivity ($\partial v_p / \partial \theta$) is an upper bound approximation of the experimental thermal changes, but the acoustoelastic Lamb wave theory is not valid for predicting the antisymmetric (A0) phase velocity at low frequency-thickness values, < 1.55 MHz mm for various temperatures. [<http://dx.doi.org/10.1121/1.4897310>]

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I. INTRODUCTION

Guided waves are able to propagate long distances in metals with little amplitude attenuation and are excellent to use for damage detection due to the full cross-section interrogation of the material.^{1,2} Environmental effects such as temperature,^{3,4} applied stress,^{5,6} and surface loading^{7,8} change the propagation velocity of guided waves and impede the implementation of traditional health monitoring methods in aerospace structures due to false positive readings. One of the biggest challenges in using guided waves for health monitoring is developing damage detection techniques that are sensitive to damage but not environmental changes.³ The structural health monitoring community needs to fully understand the effect of environmental factors on the system dynamics in order to identify dynamic features that are only sensitive to structural damage. We investigate the effect of thermally induced changes in the acoustoelastic tensors, their corresponding effect on Lamb waves in plates with stress-free boundary conditions, and develop the effective elastic moduli for the computation of Lamb wave thermal sensitivities with standard dispersion curves for isotropic media for the application of developing thermally insensitive damage metrics.

Toupin and Bernstein developed the theory of acoustoelasticity which describes the velocity change in waves with infinitesimal motion in a medium that is finitely predeformed or prestrained.⁹ An initially isotropic material becomes anisotropic under static uni/bi-axial loading which creates

directionally dependent elastic wave speeds in the material. While numerous articles have examined ultrasonic, bulk, and a variety of guided waves^{10–14} and thermal effects¹⁵ in acoustoelastic media, only the relevant developments of Lamb waves in acoustoelastic media are discussed here.

Several studies have documented experimental results that demonstrate environmental factors, such as isothermal temperature changes^{3,4} or applied stress,^{16,17} change Lamb wave velocity. Only a few studies have theoretical work with experimental validation, which addresses the change of dispersive Lamb wave velocities in metallic plates over a wide range of frequencies with respect to environmental factors; Gandhi *et al.* address the case of biaxial applied stress^{17,18} and Dodson and Inman investigate the case of isothermal temperature variation.¹⁹

Ledbetter and Reed elegantly state, “Many crystalline phenomena—thermal expansion, temperature and pressure derivatives of the second-order elastic constants,... are anharmonic effects related directly to the existence of third-order and higher order elastic constants.”²⁰ The previous investigation by Dodson and Inman used thermoelastic theory with the assumption of thermally varying second order elastic constants to investigate the analytical sensitivity of Lamb waves with respect to temperature.¹⁹ Garber and Granato have used lattice structures of cubic materials to show the temperature variation of material properties is related to the third and fourth order elastic constants.^{21,22} This work extends acoustoelasticity to isotropic tensors with temperature variations that are dependent upon the third order elastic constants and have not been previously presented in the literature. We also use the acoustoelastic foundation with thermal strain and isotropic elastic constants to

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develop thermally varying effective elastic moduli (Young's modulus and Poisson's ratio) which are composed of thermal variations and temperature independent second and third order elastic constants.

The present work utilizes acoustoelasticity with thermally induced strain to develop two alternate theoretical formulations to capture the change of Lamb wave velocity with temperature. Both new Lamb wave formulations assume thermally independent values for the elastic constants to illustrate the effect of the inclusion of the third order elastic constants in the dynamic thermo-elastic constitutive relations to better capture the change of Lamb wave velocity due to thermal variations. Both new formulations use the thermally induced acoustoelasticity to model the changes of Lamb wave velocity with temperature variations; the first uses anisotropic Lamb wave theory, while the second uses effective thermally varying elastic moduli and isotropic thermoelastic Lamb wave theory. We present the theoretical development of thermally induced acoustoelasticity and the two new formulations of Lamb wave velocity. We compare the results of the new Lamb wave formulations in the presence of isothermal variations to existing literature and experimental data over a large range of frequencies (0.4 to 1.6 MHz mm).

II. ELASTIC TENSORS OF THERMALLY INDUCED ACOUSTOELASTIC ISOTROPIC MATERIALS

In order to model the Lamb wave propagation we first develop the thermally induced acoustoelastic tensors. The infinitesimal wave motion in an acoustoelastic medium is derived in Appendix B and written as

$$A_{\alpha\beta\gamma\delta} \frac{\partial^2 u_\gamma}{\partial \chi_\beta \partial \chi_\delta} = \rho \frac{\partial^2 u_\alpha}{\partial t^2}, \quad (1)$$

where

$$A_{\alpha\beta\gamma\delta} = \left(C_{\beta\delta\epsilon\eta}^{(2)} e_{\epsilon\eta}^0 - \beta_{\beta\delta} \theta \right) \delta_{\alpha\gamma} + C_{\alpha\beta\gamma\delta}^{(2)} + C_{\alpha\beta\rho\delta}^{(2)} \frac{\partial u_\gamma}{\partial \chi_\rho} + C_{\rho\beta\gamma\delta}^{(2)} \frac{\partial u_\rho}{\partial \chi_\alpha} + C_{\alpha\beta\gamma\delta\epsilon\eta}^{(3)} e_{\epsilon\eta}^0, \quad (2)$$

where $C_{\alpha\beta\gamma\delta}^{(2)}$ is the second order elastic tensor, $C_{\alpha\beta\gamma\delta\epsilon\eta}^{(3)}$ is the third order elastic tensor, $\delta_{\alpha\beta}$ is the Kronecker delta function, u_α is the dynamic displacement, or wave motion in the material, u_α^0 is the initial finite displacement, and $e_{\gamma\delta}^0$ and $e_{\epsilon\eta}^0$ are the infinitesimal Cauchy strain tensors. Einstein's notation of summation over repeated indices is implied here and in later equations.

Equation (1) describes the infinitesimal wave motion in an acoustoelastic medium with large applied static stress and temperature variation and will be used in the development of the Lamb wave governing equations and dispersion curves. This section will examine the specific case of the initially isotropic elastic tensor, $A_{\alpha\beta\gamma\delta}$, subject to large temperature variations.

To be consistent about the order of magnitude we assume the initial stress, $S_{\alpha\beta}^0$, is small enough to be written as¹⁰

$$S_{\alpha\beta}^0 = C_{\alpha\beta\gamma\delta}^{(2)} e_{\gamma\delta}^0 - \beta_{\alpha\beta} \theta. \quad (3)$$

Here we assume our initial stress is *thermoelastic* as opposed to the traditionally used elastic formulation to capture the thermally induced strains in the material, which can be significant with sizable temperature variations. In the general case of fixed boundary conditions, the thermally generated strain will induce corresponding stresses in the system. For an isotropic body with thermally induced axial stress σ_j^θ in the χ_j directions, $j = 1, 2, 3$, the initial second Piola-Kirchhoff stress tensor can be written as

$$S_{\alpha\beta}^0 = \sigma_{\alpha\beta}^\theta \delta_{\alpha\beta}. \quad (4)$$

Setting Eqs. (3) and (4) equal to each other and substituting in the isotropic elastic constants $C_{\alpha\beta\gamma\delta}^{(2)}$ and $\beta_{\alpha\beta}$ given in Eqs. (C3) and (C1) then solving for the initial Cauchy strains, e_i^0 yields

$$e_{\alpha\beta}^0 = \frac{1}{2\mu} \sigma_{\alpha\beta}^\theta - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \sigma_{\gamma\gamma}^\theta \delta_{\alpha\beta} + \alpha \theta \delta_{\alpha\beta} \quad (5)$$

where λ and μ are Lamé constants, α is the isotropic thermal expansion coefficient and δ_{ij} is the Kronecker delta function. In the case of pure axial loads $\partial u_i^0 / \partial \chi_j = e_{ij}^0$; and the relation between the incremental stress and incremental strain in Eq. (B7) can now be written as

$$S_{\alpha\beta} = B_{\alpha\beta\gamma\delta} \frac{\partial u_\gamma}{\partial \chi_\delta}, \quad (6)$$

where $B_{\alpha\beta\gamma\delta}$ is an elastic tensor defined as

$$B_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta}^{(2)} + C_{\alpha\beta\rho\delta}^{(2)} e_{\gamma\rho}^0 + C_{\alpha\beta\gamma\delta\epsilon\eta}^{(3)} e_{\epsilon\eta}^0. \quad (7)$$

When the structure has free-free boundary conditions and an isothermal temperature variation occurs there is no thermally generated stress (i.e., $\sigma_{\alpha\beta}^\theta = 0$), and the initial stress on the system is zero and the non-zero strain is due to thermal expansion. From Eqs. (3) and (5),

$$S_{\alpha\beta}^0|_{\sigma_\theta=0} = C_{\alpha\beta\gamma\delta}^{(2)} e_{\gamma\delta}^0 - \beta_{\alpha\beta} \theta = 0, \quad e_{\alpha\beta}^0 = \alpha \theta \delta_{\alpha\beta}. \quad (8)$$

The dynamic thermo-acoustoelastic tensors A_{ijkl} and B_{ijkl} can be written in new isotropic-like forms when the boundary conditions are stress free since the nonzero initial strain $e_{\gamma\delta}^0$ is only due to isotropic thermal expansion,

$$\begin{aligned} A_{ijkl}^{(\sigma_\theta=0)} &= (\lambda + \alpha\theta(6l - 2m + n + 2\lambda)) \delta_{ij} \delta_{kl} \\ &\quad + \left(\mu + \alpha\theta \left(3m - \frac{1}{2}n + 2\mu \right) \right) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \\ B_{ijkl}^{(\sigma_\theta=0)} &= (\lambda + \alpha\theta(6l - 2m + n + \lambda)) \delta_{ij} \delta_{kl} \\ &\quad + \left(\mu + \alpha\theta \left(3m - \frac{1}{2}n + \mu \right) \right) (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \end{aligned} \quad (9)$$

where l , m , and n are the Murnaghan constants. The elastic properties maintain the isotropic-like form due to the thermal

expansion acting along each material axis. The applied thermal strain is a uniform hydrostatic tension, therefore the material maintains its isotropic symmetry. The fourth order elastic tensors A_{ijkl} and B_{ijkl} in Eq. (9) will be used to calculate the Lamb wave velocity and thermal sensitivities in the thermally induced acoustoelastic media formulation. We can also use the elastic tensors A_{ijkl} and B_{ijkl} to create effective isotropic elastic moduli.

In the case of stress-free boundary conditions, we can approximate an effective dynamic thermo-acoustoelastic (TAE) Young's modulus, E_{TAE} , and Poisson's ratio, ν_{TAE} , using the values of B_{ijkl} presented in Eq. (9). Effective dynamic thermo-acoustoelastic elastic constants can be written as

$$E_{\text{TAE}} = \left[\mu + \alpha \theta \left(3m - \frac{1}{2}n + \mu \right) \right] \times \left(\frac{2\mu + 3\lambda + \alpha \theta (2\mu + 3\lambda + 18l + 2n)}{\mu + \lambda + \alpha \theta (\mu + \lambda + 6l + m + \frac{1}{2}n)} \right), \quad (10a)$$

$$\nu_{\text{TAE}} = \frac{\lambda + \alpha \theta (6l - 2m + n + \lambda)}{2(\lambda + \mu + \alpha \theta (6l + m + \frac{1}{2}n + \lambda + \mu))}, \quad (10b)$$

using the definition of Young's modulus and Poisson's ratio in Appendix C. These effective dynamic thermo-acoustoelastic moduli can be used with the traditional isotropic elastic theory to calculate semi-theoretical Lamb wave velocities for various temperatures and thermal derivatives.

We have presented two different formulations for incorporating the thermal expansion into the acoustoelastic theory; the first gave us thermally varying fourth order elastic tensors A_{ijkl} and B_{ijkl} , and the second yielded effective dynamic thermo-acoustoelastic (TAE) Young's modulus, E_{TAE} , and Poisson's ratio, ν_{TAE} . These formulations allow us to model the thermal variations in the elastic constants using linear thermal expansion and allow the structure of the higher order elastic constants to guide the thermal variations, giving insight to the structure of how the elastic properties change with temperature. Next the thermally varying elastic tensors developed here are used in two different formulations of Lamb wave dispersion equations.

III. LAMB WAVES IN ACOUSTOELASTIC MEDIA

Lamb waves are guided waves that exist in infinite plates with stress-free surfaces. We present two different theoretical formations to model the Lamb wave velocity and changes in velocity due to thermal variations in the elastic tensors developed in Sec. II; a general anisotropic formulation utilizes the fourth order elastic tensors A_{ijkl} and B_{ijkl} and an isotropic thermoelastic formulation uses the effective Young's modulus, E_{TAE} , and Poisson's ratio, ν_{TAE} .

A. Lamb waves and thermal sensitivity in an anisotropic media

This formulation of the Lamb wave dispersion curves follows the framework developed by Ghahdi *et al.*^{17,18} for

finding the Lamb wave dispersion curves in an initially isotropic acoustoelastic medium with homogeneous applied biaxial load. The three displacements u_1 , u_2 , and u_3 act in the Cartesian axis x_1 , x_2 , and x_3 . Using the partial wave method, we assume the displacement u_j takes a plane wave solution of the form

$$u_j = U_j e^{i\kappa(x_1 + \beta x_3 - v_p t)}, \quad (11)$$

where U_j is the amplitude, κ is the wave number along the x_1 direction, β is the ratio of the wave number in the x_3 direction to the wave number in the x_1 direction, and v_p is the phase velocity of the composite guided wave. This form captures the six different partial waves which travel in the x_1 direction and satisfy the applied boundary conditions. The combination of these partial waves create the various modes [symmetric, antisymmetric, and shear horizontal (SH)] which propagate in the x_1 direction at their own corresponding velocity. Substituting displacement solution Eq. (11) into the acoustoelastic equation of motion Eq. (B11) gives the modified Christoffel equations

$$K_{jk}(\beta)U_j = 0. \quad (12)$$

The isotropic form of A_{ijkl} is used to solve the characteristic equation $|K_{jk}| = 0$ for the fractional wave numbers, β . Using only the β_m values corresponding to the symmetric and antisymmetric modes ($m = 1, \dots, 4$), the K_{jk} values can be written as

$$\begin{aligned} K_{11} &= v_p^2 \rho - A_{1111} - \beta^2 A_{1313}, \\ K_{13} &= -\beta(A_{1133} + A_{1331}), \\ K_{31} &= -\beta(A_{3311} + A_{3113}), \\ K_{33} &= v_p^2 \rho - A_{3131} - \beta^2 A_{3333}. \end{aligned} \quad (13)$$

Using Eq. (12) and the Christoffel Eq. (13) the displacement ratio $W_m = U_{3m}/U_{1m}$ is defined as

$$W_m = - \left[\begin{matrix} K_{13}(\beta_m) \\ K_{33}(\beta_m) \end{matrix} \right]^\dagger \left\{ \begin{matrix} K_{11}(\beta_m) \\ K_{31}(\beta_m) \end{matrix} \right\}, \quad (14)$$

where $\dagger$ represents the generalized inverse ($m = 1, \dots, 4$). Using the stress-strain relation given in Eq. (6), the stresses are

$$\left\{ \begin{matrix} S_{33} \\ S_{13} \\ S_{23} \end{matrix} \right\} = \sum_{m=1}^6 i\kappa \left\{ \begin{matrix} D_{1m} \\ D_{2m} \\ D_{3m} \end{matrix} \right\} U_{1m} e^{i\kappa(x_1 + \beta_m x_3 - v_p t)}, \quad (15)$$

where

$$\begin{aligned} D_{1m} &= B_{3311} + \beta_m B_{3333} W_m, \\ D_{2m} &= \beta_m B_{1313} + B_{1331} W_m, \\ D_{36} &= -D_{35} = \beta_5 B_{2323}. \end{aligned} \quad (16)$$

The stress free boundary conditions of the top and bottom of the plate are

$$\begin{aligned} S_{33}(x_1, x_2, \pm d, t) &= 0, \\ S_{j3}(x_1, x_2, \pm d, t) &= 0, \quad j = \{1, 2\}, \end{aligned} \quad (17)$$

where d is the half-thickness of the plate. Substituting Eqs. (15) and (16) into Eq. (17) with $x_1 = 0$ gives a system of six equations relating the amplitudes $U_{11}, \dots, U_{16}$ to the boundary condition. The determinant of the coefficients must equal zero in order for the wave speed solution to be nontrivial. It can be shown that the determinant yields the dispersion curves of

$$\frac{D_{11}D_{23}}{D_{13}D_{21}} = \left[\frac{\tan(\varphi\beta_1)}{\tan(\varphi\beta_3)} \right]^{\pm 1}, \quad (18a)$$

$$\sin(2\varphi\beta_5) = 0, \quad (18b)$$

where $\varphi = (\kappa d)/2 = (\omega d)/(2v_p)$, κ is the wave number, and ω is the angular frequency. Solving the dispersion Eqs. (18a) and (18b) gives the phase velocity dispersion curve as a function of angular frequency, $v_p(\omega)$, for the symmetric (S) for $+1$, antisymmetric (A) for -1 , and SH modes, respectively. The dispersion equations for the various modes are transcendental and must be solved numerically for the phase velocity over a chosen range of frequencies. The phase velocity, v_p , is the velocity of the individual sinusoids and the group velocity, v_g , is the speed of the modulating wave packet and found by the relation

$$v_g \equiv \frac{\partial \omega}{\partial \kappa} = v_p + \kappa \frac{\partial v_p}{\partial \kappa}. \quad (19)$$

The group velocity relation allows us to look at the energy propagation of a given wave packet and is used to check the validation of a new signal processing method presented in this paper.

We use the fourth order elastic tensors A_{ijkl} and B_{ijkl} developed in Sec. II with the Lamb wave dispersion equations given in Eq. (18a) to derive the theoretical thermal sensitivity in the next section.

1. Theoretical sensitivity to isothermal variations

Since the thermo-acoustoelastic tensors A_{ijkl} and B_{ijkl} are both linear functions of θ the terms $\partial A_{ijkl}/\partial \theta$ and $\partial B_{ijkl}/\partial \theta$ drive the isothermal temperature changes in dispersion curves in Eq. (18a). We assume that v_p and ρ are functions of θ , while the frequency ω is independent of environmental change. This sensitivity calculation will show how the Lamb wave velocity changes with respect to θ , and is a function of frequency. First, the change of K_{kj} with respect to θ is calculated by taking the derivative of Eq. (13). The change in the displacement ratios are

$$\begin{aligned} \frac{\partial W_m}{\partial \theta} &= \left[\frac{K_{13}(\beta_m)}{K_{33}(\beta_m)} \right]^{\dagger} \\ &\times \left(\frac{\partial}{\partial \theta} \left\{ \frac{K_{11}(\beta_m)}{K_{31}(\beta_m)} \right\} - \frac{\partial}{\partial \theta} \left[\frac{K_{13}(\beta_m)}{K_{33}(\beta_m)} \right] W_m \right). \end{aligned} \quad (20)$$

Next the thermal change in D_{km} is derived and used to calculate the thermal sensitivities of the dispersion curves in Eq. (18a) which can be written as

$$\begin{aligned} &\left(\frac{\partial D_{11}}{\partial \theta} D_{23} + D_{11} \frac{\partial D_{23}}{\partial \theta} \right) \cot(\varphi\beta_1) \\ &- D_{11} D_{23} \csc^2(\varphi\beta_1) \left(\frac{\partial \varphi}{\partial \theta} \beta_1 + \varphi \frac{\partial \beta_1}{\partial \theta} \right) \\ &- \left(\frac{\partial D_{13}}{\partial \theta} D_{21} + D_{13} \frac{\partial D_{21}}{\partial \theta} \right) \cot(\varphi\beta_2) \\ &- D_{13} D_{21} \csc^2(\varphi\beta_2) \left(\frac{\partial \varphi}{\partial \theta} \beta_2 + \varphi \frac{\partial \beta_2}{\partial \theta} \right) = 0, \end{aligned} \quad (21a)$$

$$\begin{aligned} &\left(\frac{\partial D_{11}}{\partial \theta} D_{23} + D_{11} \frac{\partial D_{23}}{\partial \theta} \right) \tan(\varphi\beta_1) \\ &+ D_{11} D_{23} \sec^2(\varphi\beta_1) \left(\frac{\partial \varphi}{\partial \theta} \beta_1 + \varphi \frac{\partial \beta_1}{\partial \theta} \right) \\ &- \left(\frac{\partial D_{13}}{\partial \theta} D_{21} + D_{13} \frac{\partial D_{21}}{\partial \theta} \right) \tan(\varphi\beta_2) \\ &+ D_{13} D_{21} \sec^2(\varphi\beta_2) \left(\frac{\partial \varphi}{\partial \theta} \beta_2 + \varphi \frac{\partial \beta_2}{\partial \theta} \right) = 0, \end{aligned} \quad (21b)$$

where Eqs. (21a) and (21b) are the thermal sensitivities for the symmetric and antisymmetric modes, respectively. These transcendental equations are dependent upon the phase velocity and are solved numerically for the change in phase velocity for a chosen range of frequencies.

The thermal sensitivities in Eq. (21) incorporate the thermal expansion based changes propagated into the fourth order elastic tensors A_{ijkl} and B_{ijkl} . The next section uses the second set of elastic constants proposed, E_{TAE} , and Poisson's ratio, ν_{TAE} , to calculate the thermal sensitivity of Lamb waves.

B. Lamb waves and thermal sensitivity in an isotropic thermoelastic media

The dispersion equations for Lamb waves in an isotropic thermoelastic plate with stress-free boundary conditions are given by¹⁹

$$\frac{(\kappa^2 + s^2)^2}{4k^2sq} = \left[\frac{\tanh(qd)}{\tanh(sd)} \right]^{\pm 1}, \quad (22)$$

where the $+1$ is for the symmetric modes, -1 is for the antisymmetric modes, and $q^2 = \kappa^2 - (\omega/v_L(\theta))^2$, and $s^2 = \kappa^2 - (\omega/v_T(\theta))^2$, where $v_L(\theta) = \sqrt{(\lambda(\theta) + 2\mu(\theta))/\rho(\theta)}$ is the longitudinal velocity and $v_T(\theta) = \sqrt{\mu(\theta)/\rho(\theta)}$ is the transverse velocity in an unbounded isotropic medium, θ is the temperature change, $\rho(\theta)$, $\lambda(\theta)$ and $\mu(\theta)$ are the temperature dependent mass density, and Lamé's elastic moduli, respectively. In order to capture the geometric effects of thermal sensitivity the half-thickness of the plate d and the material density ρ are also temperature dependent.

Once again we can solve the transcendental dispersion equations for the phase velocity at each frequency of interest using the relation $\kappa = \omega/v_p$, where κ is the wave number, ω is the frequency, and v_p is the phase velocity.²³

To incorporate the thermal change of elastic constants, the isotropic thermoelastic formulation can use either the effective Young's modulus, E_{TAE} , and Poisson's ratio, ν_{TAE} presented in Sec. II or empirical data capturing the thermal

behavior of E and ν . The theoretical thermal sensitivity of the isotropic thermoelastic formation is derived in the next section.

1. Theoretical sensitivity to isothermal variations

The thermal sensitivities of the symmetric and antisymmetric modes are defined as the thermal change in the dispersion curves, Eqs. (22), with respect to temperature which are

$$\begin{aligned} & 2(\kappa^2 + s^2) \left(2\kappa \frac{\partial \kappa}{\partial \theta} + 2s \frac{\partial s}{\partial \theta} \right) \tanh(sd) \\ & + (\kappa^2 + s^2) \text{sech}^2(sd) \left(\frac{\partial s}{\partial \theta} d + \frac{\partial}{\partial \theta} s \right) \\ & - 8\kappa \frac{\partial \kappa}{\partial \theta} \tanh(qd) - 4\kappa^2 \frac{\partial s}{\partial \theta} q \tanh(qd) \quad \text{Symmetric} \\ & - 4\kappa^2 s \frac{\partial q}{\partial \theta} \tanh(qd) \\ & - 4\kappa^2 s q \text{sech}^2(qd) \left(\frac{\partial q}{\partial \theta} d + \frac{\partial d}{\partial \theta} q \right) = 0, \end{aligned} \quad (23a)$$

$$\begin{aligned} & 2(\kappa^2 + s^2) \left(2\kappa \frac{\partial \kappa}{\partial \theta} + 2s \frac{\partial s}{\partial \theta} \right) \tanh(qd) \\ & + (\kappa^2 + s^2) \text{sech}^2(qd) \left(\frac{\partial q}{\partial \theta} d + \frac{\partial d}{\partial \theta} q \right) \\ & - 8\kappa \frac{\partial \kappa}{\partial \theta} s q \tanh(sd) - 4\kappa^2 \frac{\partial s}{\partial \theta} q \tanh(sd) \quad \text{Antisymmetric} \\ & - 4\kappa^2 s \frac{\partial q}{\partial \theta} \tanh(sd) \\ & - 4\kappa^2 s q \text{sech}^2(sd) \left(\frac{\partial s}{\partial \theta} d + \frac{\partial d}{\partial \theta} s \right) = 0. \end{aligned} \quad (23b)$$

Now that the two different theoretical formations of Lamb wave thermal sensitivities have been presented the different theoretical sensitivities can be compared. The next section will compare the theoretical Lamb wave velocities and the thermal sensitivity of the general anisotropic formulation with the fourth order elastic tensors A_{ijkl} and B_{ijkl} , an isotropic thermoelastic formulation with the effective Young's modulus, E_{TAE} , and Poisson's ratio, ν_{TAE} , and the isotropic thermoelastic formulation using empirical data for the thermally varying Young's modulus and Poisson's ratio (previously presented by Dodson and Inman¹⁹).

IV. NUMERICAL IMPLEMENTATION

We use the elastic values in Table I for the calculation of the theoretical velocities and thermal sensitivity curves. Asay and Guenther experimentally determined the elastic constants for 6061-T6 aluminum using an ultrasonic pulse echo technique with approximately 3% error in the estimates.²⁴ We propagate the 3% error into our calculations to compute the upper and lower bounds of the theoretical curves to capture the experimental error. Note that elastic values presented are *dynamic* elastic moduli which can vary up to 8% from static elastic moduli in 6061-T6 aluminum.²⁵

The Lamb wave velocities and thermal sensitivities are examined and compared for three cases; the first case is

TABLE I. Material parameters for 6061-T6 aluminum determined by Asay and Guenther (Ref. 24).

Parameter	Value	Error	Units
$E(\Theta)$	$73.5 - 0.0416\Theta$	-	GPa
$\nu(\Theta)$	$0.336 - 0.00011\Theta$	-	-
ρ	2704	-	kg/m ³
λ	56.5	-	GPa
μ	27.5	-	GPa
l	-281.5	± 7.5	GPa
m	-339.0	± 5.91	GPa
n	-416.0	± 12.48	GPa

presented by Dodson and Inman¹⁹ and uses thermally varying elastic moduli E and ν with the thermoelastic Lamb wave theory shown in Sec. III B (thermoelastic_{TE}). The second case uses the thermo-acoustoelastic effective elastic moduli (E_{TAE} and ν_{TAE}) developed in Sec. II with thermoelastic Lamb wave theory shown in Sec. III B (thermoelastic_{TAE}). The third theoretical case uses the thermo-acoustoelastic tensors $A_{\alpha\beta\gamma\delta}$ and $B_{\alpha\beta\gamma\delta}$ developed in Sec. II with the acoustoelastic Lamb wave theory shown in Sec. III A.

The thermoelastic_{TE} and thermoelastic_{TAE} moduli and corresponding wave speeds in infinite media are shown in Figs. 1(a) and 1(b). Interestingly, with increasing temperature the Poisson's ratio (ν_{TAE}) increases with increasing temperature while ν_{TE} slightly decreases over the same temperature range and is illustrated in Fig. 1(a). The elastic

FIG. 1. (Color online) Thermally varying (a) Elastic constants (E and ν) and (b) longitudinal, v_L , and transverse, v_T , wave speeds in infinite media used as inputs to the thermoelastic_{TE} and thermoelastic_{TAE} Lamb wave theory.

wave speeds v_L and v_T for the thermoelastic_{TE} theory have a slower rate of change than the thermoelastic_{TAE} wave speeds can be seen in Fig. 1(b).

The phase and group velocities and associated thermal sensitivity curves were calculated for the A_0 and S_0 modes using the phase velocity dispersion relations [Eqs. (18a) and (22)] and the thermal sensitivity [Eqs. (21) and (23)] and can be seen in Figs. 2 and 3, respectively. All three theories have the same phase and group velocity values at the initial temperature of 20 °C. The phase and group velocities for both the S_0 and A_0 modes for all three theories are shown for 20 °C, 40 °C, and 60 °C in Fig. 2. The S_0 phase velocity decreases with increasing temperature for all cases. The order of greatest to least velocity amplitude at 60 °C is thermoelastic_{TE}, acoustoelastic, and thermoelastic_{TAE} for the range of 0.5 to 1.5 MHz mm and can be seen in Fig. 2(a). Alternatively for the A_0 phase velocities the three cases do not have constant order due to the thermal changes for all frequency ranges. For the higher frequencies, >1.55 MHz mm, the A_0 phase velocities predicted by the three theories follow the same order as the S_0 mode. For lower frequencies, <1.55 MHz mm, the acoustoelastic theory predicts an increase in A_0 phase velocity, while the thermoelastic formations both predict decreasing velocity with increasing temperature as illustrated in Fig. 3(b).

FIG. 2. (Color online) Lamb wave phase and group velocities calculated at 20 °C, 40 °C, and 60 °C using the three different Lamb wave theories. The zoom windows show (a) the S_0 phase velocities over 0.5 to 1.5 MHz mm and (b) the A_0 phase velocities for the range of 0.4 to 0.8 MHz mm.

FIG. 3. (Color online) The thermal sensitivities of phase velocity and group velocity. The acoustoelastic and thermoelastic_{TAE} have upper and lower bounds (lighter lines of same line type) calculated using the 3% errors in the third order elastic constants.

The phase velocity sensitivity for all three cases is shown in Fig. 3. For both modes of propagation (A_0 and S_0) and the entire frequency range shown, the change in velocity using the thermoelastic_{TAE} and acoustoelastic theories are very close in amplitude with the difference less than 0.1 m/s/°C or within 5%. Overall, all three cases have a very similar trend with the order of greatest to least change of velocity magnitude is acoustoelastic, thermoelastic_{TAE}, and thermoelastic_{TE}.

Uniaxial and biaxial stress induced acoustoelasticity has angle dependent velocity variations.¹⁷ The thermal sensitivity of Lamb waves with free boundaries presented in this paper is independent of direction of wave propagation in the material due to the isotropic loading of the thermal expansion induced strain. This can be validated by the same method Gandhi *et al.* uses to rotate the elastic tensors $A_{\alpha\beta\gamma\delta}$ and $B_{\alpha\beta\gamma\delta}$ in anisotropic media.

Overall the three cases predict very similar velocity and thermal sensitivity curves, but the acoustoelastic theory follows a different trend for low frequency, <1.55 MHz mm, A_0 phase velocity. At low frequencies, <1.55 MHz mm, the A_0 change of phase velocity of the acoustoelastic theory does not match with the velocity difference between 40 °C and 20 °C. This discrepancy indicates that velocity changes due to temperature predicted by the acoustoelastic theory in this frequency range cannot be predicted by only a first order derivative. We test this observation by comparing the predictions to experimental data in the following section.

V. EXPERIMENTAL INVESTIGATION

To compare the theoretical thermal sensitivities presented in the previous sections with real data we conducted an experiment to measure the Lamb wave velocities in the frequency-thickness range of 0.3 to 1.6 MHz mm of an aluminum plate with simulated free-free boundaries over a large temperature range (20 °C to 70 °C).

A. Experimental setup

The test article is a 600 mm long, 600 mm wide, and 4.064 mm thick 6061 aluminum alloy plate. Several piezoelectric lead zirconate titanate (PZT) transducers are placed in a pitch-catch configuration on the plate. The transducers are circular APC 850 patches (APC International, Ltd²⁶) with a diameter of 6.35 mm and a thickness of 0.254 mm and are bonded to the plate with EPO-TEK 353ND epoxy (Epoxy Technology²⁷). The epoxy EPO-TEK 353ND was chosen for this experiment based on Raghavan and Cesnik's experimental temperature epoxy study.²⁸ Four PZTs were bonded to the plate, one was used as an actuator, a second was placed at a horizontal distance of 200 mm (0°), the third was placed at a vertical distance of 180 mm (90°), and the last was placed at a 45° orientation 267 mm away from the actuator. The sensor and actuator layout can be seen in Fig. 4.

When actuators are mounted on only one side of the plate, such is the case in this experimental setup, they produce both the longitudinal S and flexural A Lamb waves. Because of actuation strain being applied to only one side of the plate and creating a near-antisymmetric motion, the anti-symmetric (A) wave is created almost immediately and the symmetric (S) wave is created after the stress wave reflects from the boundary through the thickness of the plate causing a frequency dependent time delay.

The signals are sent and received using a National Instruments²⁹ (NI) PXI 6115 Multifunction DAQ device with a PXI 8109 controller. The transducers received a $3\frac{1}{2}$ cycle Hanning-windowed sinusoid excitation signal with a range of center frequencies and a peak-to-peak amplitude of 20 V. The excitation and response signals are measured with a sampling rate of 10 MHz and 14 bits of resolution. Each pitch-catch measurement was conducted 20 times with 50 ms delay between measurements, allowing for full decay of each excited wave before the excitation of the next wave.

The temperature of the plate was measured at each wave measurement with two K-type thermocouples using a NI USB 9211 card. The two thermocouples were averaged to estimate the plate temperature and the variation between the

two thermocouples was never more than 0.3 °C. The isothermal measurements were conducted in a Tenny environmental chamber with a Versa Tenn III controller. The test article was placed on four foam blocks, one at each corner of the plate. Measurements were taken at a temperature range of 20 °C to 70 °C with increments of 2.5 °C. The temperature in the chamber is programmed to change approximately 1 °C every 5 min with constant temperature plateaus every 2.5 °C for 30 min to maintain close to isothermal conditions on the plate for each measurement.

B. Signal processing

The phase velocity was estimated from the experimental data using the phase shifts between the actuated and received signals.³⁰ The actuation $[S_{\text{actuation}}(t)]$ and received $[S_{\text{received}}(t)]$ signals are assumed to have a harmonic form and can be written as an infinite series of sinusoids,

$$S_{\text{actuation}}(t) = \sum_1^{\infty} M_i(\omega) \sin(\omega t + \psi_0(\omega)),$$

$$S_{\text{received}}(t) = \sum_1^{\infty} N_i(\omega) \sin(\omega t + \psi_0(\omega) + \Delta\psi(\theta, \omega)),$$

where $M_i(\omega)$ and $N_i(\omega)$ are the amplitude coefficients of the actuated and received signals, $\psi_0 = \omega t_0$ is the initial phase of the signal corresponding to some initial time t_0 , and $\Delta\psi$ is the phase shift between the signals. The phase shift can be written as

$$\Delta\psi(\theta, \omega) = \Delta\psi_v(\theta, \omega) + 2\pi n, \quad (24)$$

where $\Delta\psi_v$ is the frequency and temperature dependent change in phase due to the velocity of wave propagation between the transducers. Because of the periodic nature of sinusoids the arbitrary $2\pi n$ phase shifts can be present. Using the velocity relation of $v = \Delta x / \Delta t$, the experimental velocity and phase-induced velocities can be written as

$$v_{\text{exp}}(\theta, \omega) = \frac{\Delta x \omega}{\Delta\psi_v(\theta, \omega)} \text{ and } v_{2\pi n}(\theta, \omega) = \frac{\Delta x \omega}{2\pi n}. \quad (25)$$

To estimate the phase velocity, we need to estimate the change in phase $\Delta\psi_v(\omega)$ between the corresponding frequency components of the actuation and measured signals. The maximum amplitude of the signals is centered on the primary center frequency, f_{center} because the excitation signals are narrowband. The phase estimate is most accurate in the frequency range of maximum amplitude. The frequency with the maximum amplitude was calculated for each received signal, and a frequency range of $f_{\text{max}} \pm 0.04$ MHz mm was used for the phase estimate. Ideally $f_{\text{max}} = f_{\text{center}}$, but due to experimental factors, such as imperfect transducer coupling to the structure and thermal effects, there is a small difference (<0.02 MHz mm or between 1% and 3%) between the actuated and received maximum frequency.

The experimental data has arbitrary $2\pi n$ phase shifts due to very low excitation at low frequencies and unwrapping of

FIG. 4. (Color online) Diagram of the plate in the experimental setup testing the thermal sensitivity with free-free boundary conditions.

the phase. The phase shifts are shown in Fig. 5. Using the theoretical velocity at room temperature (20 °C) as a reference we minimize the distance between the experimental and analytical phase velocities by applying $2\pi n$ phase shifts to the experimental estimates. The phase velocity theoretical values and experimental estimates with and without phase shifts can be seen in Fig. 5(a). The group velocity estimates of both the shifted and unshifted phase are identical and are shown in Fig. 5(b). The primary values of interest are the thermal sensitivities ($\partial v/\partial \theta$), so for this method to be valid the change in velocity due to $2\pi n$ shifts must be greater than the change in velocity due to thermal effects. The requirement can also be written as

$$\min\{\Delta v_{2\pi}\} > \max\{\Delta v_{\text{exp}}(\theta)\} \quad (26)$$

and needs to be satisfied. For the frequency-thickness range of 0.02 to 2.03 MHz mm the minimum change in the A_0 velocity due to 2π phase shifts is 63 m/s, while the maximum experimental change using the phase velocities measured at 20 °C and 70 °C is only 28 m/s, which satisfies the proposed constraint given in Eq. (26).

C. Experimental results and discussion

The phase velocity was experimentally measured using the test setup and signal processing methods presented in the previous sections. The experimental data and the theoretical velocities for the aluminum plate at 20 °C, 40 °C, and 60 °C are shown in Fig. 6. In Fig. 6(a) the experimental quasi- S_0 phase velocity is much greater than the theoretical velocities up until ≈ 1.5 MHz mm. This is hypothesized to be due to

FIG. 5. (Color online) Antisymmetric (A_0) experimental and theoretical (a) phase velocities and (b) group velocities illustrating the effect of $2\pi n$ phase shifts.

FIG. 6. (Color online) Experimental and theoretical (a) S_0 with a zoom window showing a close-up comparison of experimental data and theory for the range of 1.4 to 1.6 MHz mm and (b) A_0 phase velocities with zoom windows illustrating the range 0.3 to 0.32 MHz mm and 0.8 to 0.9 MHz mm. These velocities are for a 4 mm thick 6061 aluminum plate at 20 °C, 40 °C, and 60 °C.

the frequency dependent time shift for the S_0 mode caused by the actuation of only a single side of the plate. The comparison focuses on the antisymmetric mode due to the large difference between the pure analytic symmetric (S_0) mode velocities and the experimental quasi-symmetric mode.

The experimental data for A_0 in Fig. 6(b) phase velocity agrees fairly well with the theoretical velocities at the reference temperature ($\Theta_0 = 20$ °C). The experimental data indicates a negative slope with respect to temperature. The A_0 phase velocity predicted by acoustoelastic theory has the wrong direction of variation (increases instead of decreasing) as seen in both zoom windows in Fig. 6(b) (0.3 to 0.32 MHz mm and 0.8 to 0.9 MHz mm) in comparison to the experimental data and therefore the acoustoelastic theory does not accurately model the phase velocities of the antisymmetric mode with isothermal temperature changes.

The experimental thermal sensitivity is the slope of the least-squares line fitted to the velocities with for each frequency over the range of temperatures examined (20–70 °C). The linear fit was a good approximation with the minimum r^2 value over the entire frequency range being 0.87 and most r^2 values being greater than 0.95. The experimental and theoretical (with upper and lower experimental bounds)

phase velocity thermal sensitivities are shown in Fig. 7. The 90% confidence interval of the slope estimate is plotted as error bars. The experimental change of phase velocity is independent of orientation of propagation and agrees with the earlier theoretical result. The antisymmetric mode [A_0 in Fig. 7(b)] has a greater range of change in thermal sensitivity than the symmetric mode [S_0 in Fig. 7(a)] in the frequency range experimentally examined [0.4–1.6 MHz mm]. The experimental change in phase velocity values for the symmetric mode (S_0) in Fig. 7(a) are mostly contained by the acoustoelastic (lower bound) and thermoelastic_{TE} (upper bound) analytical curves.

The experimental thermal sensitivity values for the antisymmetric (A_0) mode in Fig. 7 thermoelastic_{TAE} (upper bound) theories in the frequency range of 0.4 to 1.6 MHz mm. Although the acoustoelastic theoretical thermal sensitivity was the lower bound for both modes, the thermoelastic_{TE} case predicts changes in velocity which are very close to the acoustoelastic theory and is a good approximate. Using the thermoelastic_{TAE} and thermoelastic_{TE} cases, approximate upper and lower bounds of the experimental thermal variation can be calculated with only the easier-to-implement thermoelastic Lamb wave theory.

The comparison of the experimental data to theoretical cases suggests that neither the thermoelastic formulation nor the thermo-acoustoelastic formulation independently captures all the change of Lamb wave velocities with

temperature. This section shows that the experimental and analytical thermal sensitivity are comparable, with the thermoelastic_{TE} and thermoelastic_{TAE} theories bounding the change of velocity with respect to temperature.

VI. CONCLUSION

We have extended acoustoelasticity to include thermal strains as the prestrain on the structure, this in turn gave rise to two different formulations for incorporating the thermal expansion into the acoustoelastic theory; first, the new isotropic form of the thermally varying elastic tensors $A_{\alpha\beta\gamma\delta}$ and $B_{\alpha\beta\gamma\delta}$ [in Eq. (9)] and second the effective dynamic Young's modulus and Poisson's ratio, E_{TAE} and ν_{TAE} [in Eq. (10)]. Both sets of thermally varying elastic constants allow us to model the thermal variations in the elastic constants using linear thermal expansion and allow the structure of the higher order elastic constants to guide the thermal variations giving insight to the structure of how the elastic properties change with temperature.

We present two theories modeling the thermal variations of Lamb waves; the first uses the general anisotropic formulation with the fourth order elastic tensors A_{ijkl} and B_{ijkl} to develop the Lamb wave velocities and the thermal sensitivity [in Eqs. (18a) and (21)] while the second uses an isotropic thermoelastic formulation with the effective Young's modulus, E_{TAE} , and Poisson's ratio, ν_{TAE} to model the Lamb wave velocities and the thermal sensitivity [in Eqs. (22) and (23)].

While the thermally induced acoustoelastic Lamb wave formulation is a good maximum upper bound for the first order thermal sensitivities, the acoustoelastic formulation is not valid for predicting velocity in the presence of thermal changes for the antisymmetric (A_0) mode. The experimental thermal change in phase velocity in the frequency range of 0.4 to 1.6 MHz mm for both the symmetric S and antisymmetric A modes are bounded by two theories, the thermoelastic Lamb wave theory with thermally varying elastic constants previously presented¹⁹ (upper bound) and the acoustoelastic Lamb wave theory with the new thermo-acoustoelastic tensors presented (lower bound). Using the thermoelastic Lamb wave theory with the effective dynamic thermo-acoustoelastic Young's modulus and Poisson's ratio as an approximate lower bound, the bounds of the thermal variation can be calculated with only the easier-to-implement thermoelastic Lamb wave theory. The thermally induced acoustoelastic formulations provides a physics-based model and deeper understanding on the mechanics governing the change in the elastic constants, and therefore, elastic wave velocities change in isotropic materials with isothermal variations.

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FIG. 7. (Color online) The experimental and theoretical phase velocity thermal sensitivities of the (a) S_0 and (b) A_0 Lamb wave modes. The acoustoelastic and thermoelastic_{TAE} have upper and lower bounds (lighter lines of same line type) calculated using the 3% errors in the third order elastic constants.

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APPENDIX A: DERIVATION OF THERMO-ACOUSTOELASTIC CONSTITUTIVE EQUATIONS

In order to describe the nonlinear dynamics of a general thermo-acoustoelastic system, the appropriate internal energy must be derived. The internal energy density of an elastic system is $\hat{U}_0(E_{ij}, \Theta)$, where E_{ij} is a strain tensor and Θ is the absolute temperature. When the thermal effects are included in the system, a free energy equation is needed to describe the dynamics. The Helmholtz free energy, $\phi(E_{ij}, \Theta)$, and entropy s for an anisotropic thermoelastic material is defined as

$$\phi(E_{ij}, \Theta) = \hat{U}_0(E_{ij}, \Theta) - \Theta[s(E_{ij}, \Theta)], \quad (\text{A1a})$$

$$s(E_{ij}, \Theta) = \partial\phi(E_{ij}, \Theta)/\partial\Theta. \quad (\text{A1b})$$

The strain-energy density $W(E_{ij}, \Theta)$ is defined as

$$W(E_{ij}, \Theta) = \rho\phi(E_{ij}, \Theta), \quad (\text{A2})$$

where ρ is the density of the material in the natural or unstressed state. In a hyperelastic material under isothermal deformation, all work done on the body by external forces is stored and can be recovered when the forces are removed. A hyperelastic material satisfies

$$S_{ij} = \partial W(E_{ij}, \Theta)/\partial E_{ij}, \quad (\text{A3})$$

where S_{ij} is the second Piola-Kirchhoff stress tensor. In order to capture the nonlinear dynamics of thermal changes and acoustoelastic theory the Helmholtz free energy $\phi(E_{ij}, \Theta)$ is expanded as a multivariable Taylor series about the natural state ($E_{ij} = 0, \Theta = \Theta_N$),³¹ letting $\theta = \Theta - \Theta_N$, the constant quantity $\phi(0, \Theta_N)$, the Helmholtz free-energy of the natural state, to be zero, the entropy and stress at the natural state both are equal to zero, $s(0, \Theta_N) = T(0, \Theta_N) = 0$, then

$$\partial W(0, \Theta_N)/\partial E_{ij} = \partial\phi(0, \Theta_N)/\partial\Theta = 0. \quad (\text{A4})$$

Truncating the series at the third order terms the strain-energy density $W(E_{ij}, \Theta)$ and second Piola-Kirchhoff stress tensor can be written as

$$\begin{aligned} W(E_{ij}, \Theta) = & \frac{1}{2}C_{ijkl}^{(2)}E_{ij}E_{kl} - \beta_{ij}E_{ij}\theta + \frac{p\theta^2}{2} \\ & + \frac{1}{6}C_{ijklmn}^{(3)}E_{ij}E_{kl}E_{mn} - \frac{1}{2}\gamma_{ijkl}^{(2)}E_{ij}E_{kl}\theta \\ & - \frac{1}{2}\eta_{ij}E_{ij}\theta^2 + \frac{q\theta^3}{6} \end{aligned} \quad (\text{A5})$$

and

$$\begin{aligned} S_{ij}(E_{ij}, \Theta) = & C_{ijkl}^{(2)}E_{kl} - \beta_{ij}\theta + \frac{1}{2}C_{ijklmn}^{(3)}E_{kl}E_{mn} \\ & - \gamma_{ijkl}^{(2)}E_{kl}\theta + \frac{1}{2}\eta_{ij}\theta^2, \end{aligned} \quad (\text{A6})$$

where $C_{ijkl}^{(2)}$ is the second order elastic tensor, $C_{ijklmn}^{(3)}$ is the third order elastic tensor, β_{ij} is the thermal expansion tensor, $\gamma_{ijkl}^{(2)}$ is the linear change of the second order elastic tensor with temperature, η_{ij} is the linear change of the thermal stress tensor, and then p and q are the second and third order thermal energy terms, respectively. Assuming that the structure has a prescribed slowly varying temperature distribution ($\Theta(\vec{x}, t)$), the entropy equation s is not needed and will not be further discussed. Assuming that the elastic moduli does not change with temperature (i.e., $\gamma_{ijkl}^{(2)} = 0$ and $\eta_{ij} = 0$) then Eq. (A6) becomes

$$S_{ij} = C_{ijkl}^{(2)}E_{kl} + \frac{1}{2}C_{ijklmn}^{(3)}E_{kl}E_{mn} - \beta_{ij}\theta, \quad (\text{A7})$$

which is the governing constitutive stress-strain relation used in this paper. Starting with Eq. (A7), if we assume only infinitesimal deformations, $C_{ijklmn}^{(3)} = 0$, then the strain energy converges to the thermoelastic strain energy

$$W(E_{ij}, \theta) = \frac{1}{2}C_{ijkl}^{(2)}E_{ij}E_{kl} - \beta_{ij}E_{ij}\theta \quad (\text{A8})$$

and the constitutive thermoelastic stress is

$$S_{ij}(E_{ij}, \theta) = C_{ijkl}^{(2)}E_{kl} - \beta_{ij}\theta. \quad (\text{A9})$$

APPENDIX B: THERMOELASTIC ACOUSTOELASTICITY

The dynamic acoustoelastic equations of motion are presented in this section following the derivations of Pao *et al.*¹⁰ and are extended to include applied isothermal strain. We examine three different states of the material, the natural, initial, and final state, each with their own coordinates and position vectors $\vec{\chi}$, $\vec{X}$, and $\vec{x}$, respectively. The components of the vectors will be represented by Greek (χ_α), uppercase Roman (X_K), and lowercase Roman letters (x_k) for the natural, initial, and final states.

A body in a free unstressed state, with temperature Θ_N , is in the natural state. When the body undergoes a finite static deformation with temperature Θ_0 , it is then in the initial state. The final state occurs when an infinitesimal deformation is superimposed on the initial position and the body has a temperature of Θ_f . We can write the initial displacement $\vec{u}^0$, the final displacement $\vec{u}^f$, and the dynamic displacement, or wave motion in the material, $\vec{u}$ as

$$\begin{aligned} \vec{u}^0(\vec{\chi}) &= \vec{X}(\vec{\chi}) - \vec{\chi}, \\ \vec{u}^f(\vec{\chi}, t) &= \vec{x}(\vec{\chi}, t) - \vec{\chi}, \\ \vec{u}(\vec{\chi}, t) &= \vec{x}(\vec{\chi}, t) - \vec{X}(\vec{\chi}) = \vec{u}^f(\vec{\chi}, t) - \vec{u}^0(\vec{\chi}), \end{aligned} \quad (\text{B1})$$

where t is time. The finite Lagrangian strain tensors, $E_{\alpha\beta}$, are functions of the displacements in the initial and final states and are written in index notation in the natural coordinates,

$$E_{\alpha\beta}^{(\cdot)} = \frac{1}{2} \left(\frac{\partial u_\alpha^{(\cdot)}}{\partial \chi_\beta} + \frac{\partial u_\beta^{(\cdot)}}{\partial \chi_\alpha} + \frac{\partial u_\gamma^{(\cdot)}}{\partial \chi_\alpha} \frac{\partial u_\gamma^{(\cdot)}}{\partial \chi_\beta} \right), \quad (\text{B2})$$

where $(\cdot) = 0, f$, respectively. Assuming the superimposed dynamic wave motion is infinitesimally small the incremental dynamic strain, $E_{\alpha\beta}$, and second Piola-Kirchhoff stress tensor measured in the natural coordinates, $S_{\alpha\beta}$, from the initial state to the final state can be written as

$$E_{\alpha\beta} = E_{\alpha\beta}^f - E_{\alpha\beta}^0, \quad (\text{B3a})$$

$$S_{\alpha\beta} = S_{\alpha\beta}^f - S_{\alpha\beta}^0. \quad (\text{B3b})$$

The equation of equilibrium from the natural state to the initial state is given in natural coordinates for the static finite deformation and equations of motion for the final state are³²

$$\frac{\partial}{\partial \chi_\beta} \left[S_{\beta\alpha}^0 + S_{\beta\gamma}^0 \frac{\partial u_\alpha^0}{\partial \chi_\gamma} \right] = 0, \quad (\text{B4a})$$

$$\frac{\partial}{\partial \chi_\beta} \left[S_{\beta\alpha}^f + S_{\beta\gamma}^f \frac{\partial u_\alpha^f}{\partial \chi_\gamma} \right] = \rho \frac{\partial^2 u_\alpha^f}{\partial t^2}, \quad (\text{B4b})$$

where ρ is the density in the natural state. The equations of motion for the infinitesimal dynamic motion can be approximated by subtracting Eq. (B4a) from Eq. (B4b), and disregarding higher order terms due to the infinitesimal displacement assumption the equation of motion as a function of the second Piola-Kirchhoff stress tensor is

$$\frac{\partial}{\partial \chi_\beta} \left[S_{\beta\alpha} + S_{\beta\gamma}^0 \frac{\partial u_\alpha}{\partial \chi_\gamma} + S_{\beta\gamma} \frac{\partial u_\alpha^0}{\partial \chi_\gamma} \right] = \rho \frac{\partial^2 u_\alpha}{\partial t^2}. \quad (\text{B5})$$

The thermo-acoustoelastic constitutive stress relation derived in Appendix A and shown in Eq. (A7) is used and the second Piola-Kirchhoff stress in the initial and final states can be written as

$$S_{\alpha\beta}^{(\cdot)} = C_{\alpha\beta\gamma\delta}^{(2)} E_{\gamma\delta}^{(\cdot)} + \frac{1}{2} C_{\alpha\beta\gamma\delta\epsilon\eta}^{(3)} E_{\gamma\delta}^{(\cdot)} E_{\epsilon\eta}^{(\cdot)} - \beta_{\alpha\beta} \theta^{(\cdot)}, \quad (\text{B6})$$

where $C_{\alpha\beta\gamma\delta}^{(2)}$ is the second order elastic tensor, $C_{\alpha\beta\gamma\delta\epsilon\eta}^{(3)}$ is the third order elastic tensor, $\beta_{\alpha\beta}$ is the thermoelastic tensor, θ^0 and θ^f are the temperature variations [$\theta^{(\cdot)} = \Theta_{(\cdot)} - \Theta_N$] between the current state, initial or final, and the natural state of the system where $(\cdot) = 0, f$. Assuming that the initial and final states are at the same temperature ($\theta^f = \theta^0 = \theta$) and the higher order terms have no influence, then the incremental second Piola-Kirchhoff stress tensor, $S_{\alpha\beta}$, in Eq. (B3b) can be written as

$$S_{\alpha\beta} = C_{\alpha\beta\gamma\delta}^{(2)} E_{\gamma\delta} + C_{\alpha\beta\gamma\delta\epsilon\eta}^{(3)} e_{\gamma\delta}^0 e_{\epsilon\eta}, \quad (\text{B7})$$

where $e_{\gamma\delta}^0$ and $e_{\epsilon\eta}$ are the infinitesimal Cauchy strain tensors defined as

$$e_{\gamma\delta} = \frac{1}{2} \left(\frac{\partial u_\alpha}{\partial \chi_\beta} + \frac{\partial u_\beta}{\partial \chi_\alpha} \right), \quad e_{\gamma\delta}^0 = \frac{1}{2} \left(\frac{\partial u_\alpha^0}{\partial \chi_\beta} + \frac{\partial u_\beta^0}{\partial \chi_\alpha} \right). \quad (\text{B8})$$

Employing Eqs. (B3a), (B5), (B7), and (B8), the symmetries of the elastic tensors in a homogeneous material and neglecting the third order displacement terms yields the equation of motion

$$\frac{\partial}{\partial \chi_\beta} \left[A_{\alpha\beta\gamma\delta} \frac{\partial u_\gamma}{\partial \chi_\delta} \right] = \rho \frac{\partial^2 u_\alpha}{\partial t^2}, \quad (\text{B9})$$

where

$$A_{\alpha\beta\gamma\delta} = \left(C_{\beta\delta\epsilon\eta}^{(2)} e_{\epsilon\eta}^0 - \beta_{\beta\delta} \theta \right) \delta_{\alpha\gamma} + C_{\alpha\beta\gamma\delta}^{(2)} + C_{\alpha\beta\rho\delta}^{(2)} \frac{\partial u_\gamma^0}{\partial \chi_\rho} + C_{\rho\beta\gamma\delta}^{(2)} \frac{\partial u_\rho^0}{\partial \chi_\alpha} + C_{\alpha\beta\gamma\delta\epsilon\eta}^{(3)} e_{\epsilon\eta}^0 \quad (\text{B10})$$

and $\delta_{\alpha\beta}$ is the Kronecker delta function. For applied biaxial stresses that only vary in the thickness (χ_3), such as bending stresses or thermal variations in χ_3 , or for homogeneously applied stresses the equation of motion in Eq. (B9) simplifies to

$$A_{\alpha\beta\gamma\delta} \frac{\partial^2 u_\gamma}{\partial \chi_\beta \partial \chi_\delta} = \rho \frac{\partial^2 u_\alpha}{\partial t^2}. \quad (\text{B11})$$

This equation describes the infinitesimal wave motion in an anisotropic medium with large applied static stress and temperature variation and are used in the development of the Lamb wave governing equations and dispersion curves.

APPENDIX C: ELASTIC CONSTANTS IN AN ISOTROPIC MEDIUM

The general anisotropic case of the second order elastic constant $C_{ijkl}^{(2)}$ has 21 different values for the 81 different components due to the symmetry of the tensor. For isotropic materials, the second order elastic tensor can be written with only two different values as

$$C_{ijkl}^{(2)} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (\text{C1})$$

where λ and μ are Lamé elastic constants and δ_{ij} is the Kronecker delta function. In the linear elastic case for uniaxial loading Young's modulus, E , is defined as the ratio of the uniaxial tensile stress and tensile strain and Poisson's ratio ν and is defined as the ratio of the axial strain and the lateral strain. E and ν can be written in terms of Lamé constants

$$E = \mu(3\lambda + 2\mu)/(\lambda + \mu), \quad (\text{C2a})$$

$$\nu = \lambda/(2(\lambda + \mu)). \quad (\text{C2b})$$

The thermal strain term β_{ij} for the isotropic case can be written as³³

$$\beta_{ij} = (3\lambda + 2\mu) \alpha \delta_{ij}, \quad (\text{C3})$$

where α is the isotropic coefficient of thermal expansion.

The third order elastic tensor needs 56 different values to describe the 729 components in the isothermal anisotropic case.¹⁰ The isotropic assumption simplifies this to three different values and can be written as³⁴

$$C_{ijklmn}^{(3)} = 2 \left(l - m + \frac{n}{2} \right) \delta_{ij} \delta_{kl} \delta_{mn} + 2 \left(m - \frac{n}{2} \right) \times (\delta_{ij} I_{lkmn} + \delta_{kl} I_{mnij} + \delta_{mn} I_{ijkl}) + \frac{n}{2} (\delta_{ik} I_{jlmn} + \delta_{il} I_{jlmn} + \delta_{jl} I_{ikmn}), \quad (\text{C4})$$

where

$$I_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2 \quad (C5)$$

and l , m , and n are the Murnaghan elastic constants.

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