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EFFICIENT ALGORITHMS FOR THE OPTIMAL-RATIO REGION DETECTION PROBLEMS IN DISCRETE GEOMETRY WITH APPLICATIONS*

Xiaodong Wu

Department of Electrical and Computer Engineering, Department of Radiation Oncology, The University of Iowa, Iowa City, IA 52242, USA, xiaodong-wu@uiowa.edu

Abstract

In this paper, we study several interesting *optimal-ratio region detection (ORD)* problems in d -D ($d \geq 3$) discrete geometric spaces, which arise in high dimensional medical image segmentation. Given a d -D voxel grid of n cells, two classes of geometric regions that are enclosed by a single or two coupled smooth *heightfield surfaces* defined on the entire grid domain are considered. The objective functions are normalized by a function of the desired regions, which avoids a bias to produce an overly large or small region resulting from data noise. The normalization functions that we employ are used in real medical image segmentation. To our best knowledge, no previous results on these problems in high dimensions are known. We develop a unified algorithmic framework based on a careful characterization of the intrinsic geometric structures and a nontrivial graph transformation scheme, yielding efficient polynomial time algorithms for solving these ORD problems. Our main ideas include the following. We observe that the optimal solution to the ORD problems can be obtained via the construction of a convex hull for a set of $O(n)$ unknown 2-D points using the hand probing technique. The probing oracles are implemented by computing a minimum s - t cut in a weighted directed graph. The ORD problems are then solved by $O(n)$ calls to the minimum s - t cut algorithm. For the class of regions bounded by a single heightfield surface, our further investigation shows that the $O(n)$ calls to the minimum s - t cut algorithm are on a monotone parametric flow network, which enables to detect the optimal-ratio region in the complexity of computing a single maximum flow.

Keywords

Hand probing; parametric search; minimum closed set; optimal-ratio region detection; algorithms

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1. Introduction

In this paper, we study several *optimal-ratio region detection* problems in discrete geometry, which aims to find a “best” well-shaped region in a given d -dimensional voxel grid $\Gamma = [1..N]^d$ of $n = N^d$ cells. Many applications such as data mining,^{9,5,6} data visualization,¹ and computer vision,³ require that the target region be of a “good” shape. We develop in this paper efficient algorithms for computing two interesting classes of optimal-ratio geometric regions, the *coupled-surfaces-bounded regions* and the *smooth lower-half regions*, which model important applications in medical image analysis.^{21,2,17,24}

A central problem in medical image analysis is image segmentation, which aims to define accurate boundaries for the objects of interest captured by image data. Accurate three and higher dimensional image segmentation promises to revolutionize the current medical imaging practice. Many medical anatomies express layered structures. Detecting regions bounded by a single or coupled surfaces from image data is highly demanded. Figure 1 (a) shows a retinal layer in a slice of a 3-D optical coherence tomography (OCT) image; the retinal layer is bounded by two coupled heightfield (terrain-like) surfaces, the internal limiting membrane and the pigment epithelium. Figure 1(b) illustrates a schematic cross-sectional anatomy of a diseased artery. For non-heightfield structures, a common segmentation approach^{21,17} for those objects is to perform a resampling of the original image to produce another new image $\mathcal{J}$ in the geometric $\mathbf{xyz}$ -space. This so-called “unfolding” operation is done in such a way that the sought object surfaces in $\mathcal{J}$ are heightfield ones. For example, the unfolding of the tubular surface in Figure 2(a) to produce the heightfield surface in Figure 2(b) is by a resampling of the image based on the cylindrical coordinate transform. Thus, each of the sought surfaces in $\mathcal{J}$ for the object boundaries contains exactly one voxel in every column of $\mathcal{J}$ that is parallel to the $\mathbf{z}$ -axis. Then, the region bounded by heightfield surface(s) is segmented from $\mathcal{J}$. Some geometric constraints on these surfaces should be satisfied by the segmentation. First, the two sought surfaces must be non-crossing and within a given range of distance apart. Since many anatomical structures are smooth, the segmented surfaces must be sufficiently “smooth”. Generally speaking, the smoothness is related to the surface curvature and means that an object boundary cannot change abruptly. Motivated by this medical image segmentation and other applications, we formulate the following optimal-ratio region detection problems.

Let Γ be a given d -D ($d \geq 3$) voxel grid $[1..N]^d$ of $n = N^d$ cells. The domain $\mathcal{D}$ of Γ is the projection of Γ onto the first $d - 1$ dimensions (i.e., $\mathcal{D} = [1..N]^{d-1}$). For each $\mathbf{x} = (x_1, x_2, \dots, x_{d-1}) \in \mathcal{D}$, the voxel subset $\{(x_1, x_2, \dots, x_{d-1}, z) \mid 1 \leq z \leq N\}$ forms a column $Col(\mathbf{x})$, called the $\mathbf{x}$ -column of Γ . We denote the voxel $(x_1, x_2, \dots, x_{d-1}, z)$ of $Col(\mathbf{x})$ by $\mathbf{x}_z$. Two columns, $Col(\mathbf{x})$ and $Col(\mathbf{y})$ with $\mathbf{x} = (x_1, x_2, \dots, x_{d-1})$, $\mathbf{y} = (y_1, y_2, \dots, y_{d-1}) \in \mathcal{D}$, are *adjacent in the p -th dimension* if $\sum_{i=1}^{d-1} |x_i - y_i| = 1$ and $|x_p - y_p| = 1$ ($1 \leq p < d$). Each voxel $\mathbf{x}_z \in \Gamma$ is assigned an *on-surface cost* $b(\mathbf{x}_z)$ and an *in-region cost* $c(\mathbf{x}_z)$; both are an arbitrary real value. The on-surface cost of a voxel is closely related to the likelihood that it may appear on a desired surface, while the in-region cost measures the possibility of a given voxel preserving the expected regional property. In image segmentation, both on-surface and in-region costs can be determined using low-level image features.²¹

A *heightfield surface* in Γ is defined by a function $S: \mathcal{D} \rightarrow \{1, 2, \dots, N\}$ such that it satisfies the following *smoothness constraint*: Given $d-1$ smoothness parameters $\{\delta_i \mid i = 1, 2, \dots, d-1\}$ with each for one of the first $d-1$ dimensions, for any two voxels, $\mathbf{x}_z$ and $\mathbf{y}_{z'}$, on the surface (i.e., $z = S(\mathbf{x})$ and $z' = S(\mathbf{y})$), if their corresponding columns $Col(\mathbf{x})$ and $Col(\mathbf{y})$ are adjacent in the p -th dimension ($p = 1, 2, \dots, d-1$), then $|z - z'| \leq \delta_p$. Figure 3(a) illustrates the smoothness constraint in 3-D. A *coupled-surfaces-bounded region* is a region R in Γ enclosed by two interrelated heightfield surfaces S_1 and S_2 (i.e., $R = \{\mathbf{x}_z \mid \mathbf{x} \in \mathcal{D}, S_1(\mathbf{x}) \leq z \leq S_2(\mathbf{x})\}$, see Figure 3(b)), whose interrelation is specified by the *surface separation constraint*: Given two parameters $\delta^u, \delta^l \geq 0$, for each $\mathbf{x} \in \mathcal{D}$, $\delta^l \leq S_2(\mathbf{x}) - S_1(\mathbf{x}) \leq \delta^u$. The second class of regions we consider is called the *smooth lower-half region*, which is a region R bounded by a heightfield surface S (i.e., $R = \{\mathbf{x}_z \mid \mathbf{x} \in \mathcal{D}, 1 \leq z \leq S(\mathbf{x})\}$, see Figure 2(b)). The *net-cost* $\beta(R)$ of a region R is defined as the total on-surface cost of the heightfield surface(s) enclosing R plus the total in-region cost of the voxels in R , that is, for a coupled-surfaces-bounded region R enclosed by two coupled heightfield surfaces S_1 and S_2 ,

$$\beta(R) = \sum_{i=1}^2 \sum_{\mathbf{x}_z \in S_i} b(\mathbf{x}_z) + \sum_{\mathbf{x}_z \in R} c(\mathbf{x}_z), \text{ while for a smooth lower-half region } R \text{ bounded by a heightfield surface } S, \beta(R) = \sum_{\mathbf{x}_z \in S} b(\mathbf{x}_z) + \sum_{\mathbf{x}_z \in R} c(\mathbf{x}_z).$$

The **optimal-ratio region detection (ORD)** problem seeks a desired-region R in Γ such that

the *ratio cost* $\alpha(R)$ induced by R , with $\alpha(R) = \frac{\beta(R)}{g(R)}$, is maximized, where $g(R)$ is a non-negative real-valued function of R . The objective function $\alpha(R)$ incorporates both boundary and regional information of the desired-region R . We consider both the coupled-surfaces-bounded regions and the smooth lower-half regions in Γ . The normalization over $g(R)$ is to avoid a bias to produce an overly large or small region resulting from data noise. The normalization techniques^{22,20} is commonly used for image segmentation. Let $|R|$ denote the number of voxels in the region R . In this paper, we consider the normalization functions which are *concave* with respect to $|R|$. The motivation for considering concave functions $g(R)$ is the observation that two frequently used ones in medical image analysis are concave.

One is $g(R) = \sqrt{|R|(n - |R|)}$, which is closely related to the *interclass variance* in discriminant analysis.¹⁴ To avoid the degeneracy of the solution (i.e., $|R| = n$), one may transform the costs of a voxel $\mathbf{x}_z$ in a way such that the net-cost $\beta(R)$ of the region R including the whole voxel grid Γ is 0 (e.g., we can set the new on-surface cost $b'(\mathbf{x}_z)$ for each voxel $\mathbf{x}_z \in \Gamma$ to be $[b(\mathbf{x}_z) - (b(\mathbf{x}_1) + b(\mathbf{x}_N))/2 - \sum_{\mathbf{y}_{z'} \in \Gamma} c(\mathbf{y}_{z'})/n]$). The second normalization function frequently used in medical image segmentation is $g(R) = |R|$, which is equivalent to the *volume* of R in the discrete space.

At one's first sight, the parametric search technique may be applicable for solving the ORD problems. An essential procedure in parametric search is, for a given parameter θ , to compute a desired-region R while maximizing $\beta(R) - \theta \cdot g(R)$. However, for a general concave function $g(R)$, that is not an easier problem at all. Trying every possible desired-region R in Γ is obviously not a desirable method. The difficulty is how to "distribute" the value of $\theta \cdot g(R)$ to the desired-region while it is not known before hand. Although when $g(R) = |R|$ our algorithm in Section 2.2 can be used to solve that problem, simply applying the parametric search technique may just lead to a *weakly* polynomial time algorithm (i.e.,

the running time is polynomial with respect to both the size of the input voxel grid Γ and the values of the on-surface and in-region costs of the voxels in Γ). Megiddo's approach¹⁸ may result in an improved *strongly* polynomial time algorithm (i.e., the running time is polynomial in the size of the input voxel grid and is independent on the values of the voxel costs). However, it requires the design of an efficient parallel algorithm for the maximum flow problem, which is hard to achieve.¹⁶

In this paper, we develop an interesting algorithmic framework, which leads to *strongly* polynomial-time algorithms for solving the ORD problems. We observe that each of the ORD problems is closely related to the construction of a convex hull for a set of $O(n)$ unknown 2-D points using the hand probing technique.^{8,7} The optimal solution actually defines a vertex of the constructed convex hull. The implementation of the probing oracle, which recognizes either one hull vertex or one edge at each call, is essentially the main challenge here. Inspired by our previous work for solving the layered net surface problem,²⁴ we judiciously characterizing the intrinsic self-closure structure of the problems, which enables us to implement such a probing oracle by computing a minimum s - t cut in a weighted directed graph. The ORD problems are then solvable by $O(n)$ calls to the minimum s - t cut algorithm. Interestingly, we are able to do much better for the detection of an optimal-ratio smooth lower-half region in Γ . We notice that the $O(n)$ calls to the minimum s - t cut algorithm are on a sequence of weighted directed graphs, which forms a *monotone parametric flow network*.^{13,10} Hence, the optimal-ratio smooth lower-half region in Γ can be detected in the complexity of computing a single maximum flow using Gusfield and Martel's algorithm.¹³ Furthermore, for the case of using the normalization function $g(R) = |R|$, we establish a connection between our convex hull model for the problem and the traditional Newton based approach for the fractional programming problem (see, e.g., Gondran and Minoux¹²). This connection enables us to apply Gallo *et al.*'s simple parametric minimum s - t cut algorithm,¹⁰ yielding an $O(n^2 \log n)$ time algorithm for computing an optimal-ratio smooth lower-half region.

Previous work on the optimal region detection problems mainly focused on computing optimal well-shaped regions in low dimensions. Fukuda *et al.*⁹ considered computing optimal rectangular, connected x -monotone, and rectilinear convex regions in a 2-D pixel grid Γ of n cells, and gave $O(n^{1.5})$ -, $O(n)$ -, and $O(n^{1.5})$ - time algorithms, respectively. In studying the optimal pyramid problem,^{5,6} Chun *et al.* gave a linear time algorithm for computing an optimal region called *point-stabbed union of rectangles* in 2-D. The optimal region detection problems in higher dimensions are so under-explored that few known methods actually address them. Chen *et al.*⁴ extended Chun *et al.*'s 2-D approach to searching for a minimum-weight stabbed union of orthogonal regions in d -D ($d \geq 3$). The most closely related work is Asano *et al.*'s in Ref.² and our previous work on optimal layered net surface problems²⁴. Asano *et al.*² considered to detect a connected x -monotone region in a 2-D image while maximizing the interclass variance. Their algorithm employed the hand-probing technique in computational geometry and dynamic programming schemes. We observe though that the hand-probing technique is applicable for detecting target objects of an arbitrary shape in high dimensional spaces. Wu *et al.*²⁴ studied the layered net surface problems which seek to optimally partition a given voxel grid into multiple disjoint regions

using heightfield surfaces. The basic idea is to formulate the layered net surface problems as computing a maximum-cost closed set in a vertex-weighted directed graph by exploiting the self-closure structure of the problem. In our ORD algorithms, we also characterize the self-closure structure for a key step towards solving our ORD problems, but the construction of the directed graph has some technical differences from that in Ref.²⁴ Our main contribution is to develop a unified algorithmic framework for solving the various ORD problems by extending our previous techniques for the optimal layered net surface problems using the hand probing method.

2. Our Algorithms for the ORD Problems

This section presents our strongly polynomial-time algorithms for computing an optimal-ratio region in Γ using a given concave normalization function $g(R)$. We first establish a convex hull model for the ORD problems, which involves the construction of the convex hull of a set of 2-D points with unknown coordinates by the *hand probing* method^{8,7}. The probing oracle is then implemented via finding in Γ an optimal desired-region for each of the parameters output by the probing process, which is solvable by computing a minimum s - t cut in a weighted directed graph.

2.1. The convex hull model

Asano *et al.*² developed a convex hull model for the detection of a connected x -monotone region in a 2-D image. We observe that the model also works for detecting a target object of an arbitrary shape. The proofs are similar to those in Ref.² To make this paper self-contained, we present the convex hull model in this section.

2.1.1. Construction of the convex hull by hand probing—The objective function

that we want to maximize is the ratio cost $\alpha(R)$ of a desired-region R with $\alpha(R) = \frac{\beta(R)}{g(R)}$, where $\beta(R)$ is the net-cost of R and $g(R)$ is a non-negative concave function of $|R|$. Note that $|R|$ denotes the total number of voxels in R and ranges from 0 to n . Observe that for each k ($0 \leq k \leq n$), if we are able to compute an optimal desired-region R_k^* in Γ such that the size of R_k^*

is k and the net-cost $\beta(R_k^*) = \sum_{i=1}^2 \sum_{\mathbf{x}_z \in S_i} b(\mathbf{x}_z) + \sum_{\mathbf{x}_z \in R_k^*} c(\mathbf{x}_z)$ is maximized, then we solve the problem. Unfortunately, that is not an easier problem at all. However, the view of the problem in such a way lays down a base for further exploiting the intrinsic geometric structure of the problem.

For each $k = 0, 1, \dots, n$, the pair $(k, \beta(R_k^*))$ defines a point in the 2-D $(\mathbf{x}, \mathbf{y})$ -plane, thus forming a set P of points with $P = \{(k, \beta(R_k^*)) | k = 0, 1, \dots, n\}$. Note that for some k 's, the problem may not have a feasible solution; we then simply let the net-cost be $-\infty$. Actually, we may not need to compute all the points in P in order to find the optimal-ratio desired-region in Γ , as stated in Lemma 1.

Lemma 1. *The point $(|R^*|, \beta(R^*))$ defined by an optimal-ratio desired-region R^* in Γ , must be a vertex of the upper chain $UH(P)$ of the convex hull $CH(P)$ of P .*

Proof. Let $\alpha^* = \frac{\beta(R^*)}{g(R^*)}$. Consider the curve $\xi : y = \alpha^* g(x)$ in the 2-D $(\mathbf{x}, \mathbf{y})$ -plane that the points in P lie on. Since $\beta(R^*) = \alpha^* g(R^*)$, the point $(|R^*|, \beta(R^*))$ is on the curve ξ . Notice that

$\alpha^* = \max_{R \subseteq \Gamma} \frac{\beta(R)}{g(R)}$. Thus, for any feasible region R in Γ , we have $\beta(R) \leq \alpha^* g(R)$, that is, every point $(k, \beta(|R_k^*|)) \in P (k=0, 1, \dots, n)$ lies below or on the curve ξ (see Figure 4(a)). Furthermore, due to the concavity of $g(x)$, all points in P lie below or on the tangent line l to ξ at the point $(|R^*|, \beta(R^*))$. Hence, $(|R^*|, \beta(R^*))$ is a vertex of the upper chain $UH(P)$ of the convex hull $CH(P)$ of P .

Thus, we only need to compute a subset of points in P that lie on the upper chain $UH(P)$ of convex hull $CH(P)$. Note that the convex hull model here actually is used mainly to get rid of the denominator $g(R)$ of the objective function. However, directly computing the hull vertices of $UH(P)$ appears to be quite involved. Inspired by the *hand probing* method,^{8,7} which can be viewed as recognizing a convex polygon by “touching with lines”, we use the following *probing oracle* to construct the upper chain $UH(P)$ even the coordinates of the points in P is unknown.

Given a slope θ , report the tangent line with slope θ to $CH(P)$ and the tangent point as well.

With this probing oracle, the convex polygonal chain $UH(P)$ can be constructed, as follows. Start with slopes $+\infty$ and $-\infty$ to find the two endpoints (the leftmost and rightmost points) of $UH(P)$. Now suppose that we have computed two vertices u and v on the hull and there is no vertex of $UH(P)$ between u and v being computed so far. Let θ be the slope of the line through u and v . Then, perform a probing oracle with respect to θ (see Figure 4(b)). Consequently, we either find a new vertex on $UH(P)$ between u and v or know that $\overline{uv}$ is a hull edge of $UH(P)$ (i.e., no vertex of $UH(P)$ between u and v). Thus, performing a probing oracle results in either a new vertex or a new hull edge of $UH(P)$. Hence, we have Lemma 2.

Lemma 2. *The convex polygonal chain with m vertices can be computed with $O(m)$ probing oracles.⁸*

Based on Lemmas 1 and 2, if we are able to implement such a probing oracle, the optimal-ratio desired-region in Γ can be computed by performing $O(n)$ probing oracles since $|UH(P)| = O(n)$. We next address the main challenge of efficient implementation of the probing oracles.

2.1.2. Implementation of the probing oracle—Given a real-valued parameter θ , we define the *parametric net-cost* of a desired-region R in Γ as the net-cost of R minus $\theta|R|$ (i.e., $\beta(R) - \theta|R|$), denoted by $\beta_\theta(R)$. We show in this section that the probing oracle can be implemented via computing in Γ an optimal desired-region with a maximized parametric net-cost.

For a given parameter θ , let $R^*(\theta)$ be an optimal desired-region with a maximized parametric net-cost. Recall that R_k^* denotes the optimal desired-region with size of k such that

$\beta(R_k^*) = \max_{R \subseteq \Gamma, |R|=k} \beta(R)$. Lemma 3 follows immediately from the relevant definitions.

Lemma 3. $\max_{R(\theta)} \beta_\theta(R(\theta)) = \max_k [\beta(R_k^*) - k\theta]$

Lemma 4. *There exists a tangent line to $UH(P)$ at the point $(j, \beta(R_j^*))$ with a slope θ if and only if $|R^*(\theta)| = j$ and $\beta(R^*(\theta)) = \beta(R_j^*)$.*

Proof. “ $\Rightarrow$ ” Suppose that $l: y = \theta x + b$ is a tangent line to $UH(P)$ at the point $(j, \beta(R_j^*))$. This implies that $b = \beta(R_j^*) - j \cdot \theta = \beta(R_j^*) - |R_j^*| \theta$. Note that $UH(P)$ is the upper chain of the convex hull $CH(P)$. Thus, for any $k \geq j$, the point $(k, k \cdot \theta + b)$ on l is on or above $UH(P)$. Alternatively, $k \cdot \theta + b \geq \beta(R_k^*)$, that is, $\beta(R_j^*) - j \cdot \theta \geq \beta(R_k^*) - k \cdot \theta$ for any $k \geq j$ (see Figure 5). Hence, the region R_j^* achieves $\max_k [\beta(R_k^*) - k\theta]$. Based on Lemma 3, R_j^* is an optimal desired-region in Γ such that $|R_j^*| = j$ and $\beta_\theta(R_j^*) = \max_{R(\theta)} \beta_\theta(R(\theta))$. Thus, $\beta(R^*(\theta)) = \beta(R_j^*)$.

“ $\Leftarrow$ ” The fact that $|R^*(\theta)| = j$ indicates that, for any feasible desired-region $R(\theta)$ in Γ , if $|R(\theta)| \geq j$, then $\beta(R(\theta)) - |R(\theta)| \cdot \theta \leq \beta_\theta(R^*(\theta))$. Thus, for any $k \geq j$, $\beta(R_k^*) - k \cdot \theta \leq \beta_\theta(R^*(\theta))$. Based on the assumption that $\beta(R^*(\theta)) = \beta(R_j^*)$ and $|R^*(\theta)| = j$, we have $\beta(R_k^*) - k \cdot \theta \leq \beta(R_j^*) - j \cdot \theta$ for any $k \geq j$. Consider a line $l: y = \theta x + b$ with $b = \beta(R_j^*) - j \cdot \theta$. Obviously, the point $(j, \beta(R_j^*))$ is on the line l and the point $(k, \beta(R_k^*))$ is on or below the line l for any $k \geq j$ (see Figure 5). Hence, line l is a tangent line to $UH(P)$ at the point $(j, \beta(R_j^*))$ with a slope θ .

Consequently, for a given slope θ , we need to compute an optimal desired-region $R^*(\theta)$ in Γ . If the size of $R^*(\theta)$ is j , based on Lemma 4, the line $l: y = \theta x + (\beta(R^*(\theta)) - j \cdot \theta)$ is a tangent line to $UH(P)$ at the point $(j, \beta(R^*(\theta)))$ with slope θ . We thus let $R_j^* = R^*(\theta)$. Based on Lemma 1, it suffices to compute the upper chain $UH(P)$ of the convex hull $CH(P)$ to detect an optimal-ratio desired-region in Γ , where $P = \{(k, \beta(R_k^*)) | k = 0, 1, \dots, n\}$. As stated in Lemma 2, all vertices on $UH(P)$ can be obtained by performing $O(n)$ probing oracles. Assume that for a given slope θ , it takes $O(T(n))$ time for computing in Γ an optimal desired-region $R^*(\theta)$ with a maximized parametric net cost. Then, each probing oracle can be implemented in $O(T(n))$ time from Lemma 4. We hence have the following lemma.

Lemma 5. *For a given non-negative concave normalization function $g(R)$, an optimal-ratio desired-region in Γ can be computed in $O(nT(n))$ time.*

Next, we develop in Sections 2.2 and 2.3, respectively, efficient algorithms for computing in Γ an optimal coupled-surfaces-bounded region and an optimal smooth lower-half region maximizing their parametric net-costs.

2.2. Computing an optimal coupled-surfaces-bounded region

Our algorithm for computing an optimal coupled-surfaces-bounded region $R^*(\theta)$ with a maximized parametric net-cost for a given parameter θ (namely, the MPNC problem) is inspired by our previous work for solving the layered net surface problem.²⁴ Instead of seeking an optimal “partition” of a given voxel grid into multiple disjoint regions using

heightfield surfaces as in Ref.,²⁴ here we “select” an optimal region bounded by two coupled heightfield surfaces in Γ . We characterize the self-closure structures of the MPNC problem, and then model it as a maximum-cost closed set problem based on a nontrivial graph transformation scheme. The graph construction has some technical differences from that in Ref.²⁴

A *closed set* $\mathcal{C}$ in a directed graph with arbitrary vertex costs $w(\cdot)$ is a subset of vertices such that all successors of any vertex in $\mathcal{C}$ are also contained in $\mathcal{C}$.^{19,15} The *cost* of a closed set $\mathcal{C}$, denoted by $w(\mathcal{C})$, is the total cost of all vertices in $\mathcal{C}$. The maximum-cost closed set problem seeks a closed set in the graph whose cost is maximized.

2.2.1. The self-closure structures of the MPNC problem—Recall that a coupled-surfaces-bounded region $R(\theta)$ in Γ is a region enclosed by two coupled heightfield surfaces, S_1 and S_2 , satisfying the surface separation constraint. WOLG, we assume that S_2 is “on top” of S_1 (i.e., for any $\mathbf{x} \in \mathcal{D}$, $S_2(\mathbf{x}) > S_1(\mathbf{x})$).

Given a set of $d - 1$ smoothness parameters $\{\delta_i \geq 0 | i = 1, 2, \dots, d-1\}$ with each for one of the first $d - 1$ dimensions, consider every voxel $\mathbf{x}_z \in \Gamma$ (i.e., $\mathbf{x} \in \mathcal{D}$ and $1 \leq z \leq N$) and each column $Col(\mathbf{y})$ adjacent to $Col(\mathbf{x})$ in the p -th dimension for every $p = 1, 2, \dots, d - 1$. The *lowest neighbor* of $\mathbf{x}_z$ on $Col(\mathbf{y})$ is the voxel $\mathbf{y}_{z'}$ with $z' = \max\{1, z - \delta_p\}$ (i.e., the voxel on $Col(\mathbf{y})$ with the smallest d -th coordinate that can possibly appear together with $\mathbf{x}_z$ on a same feasible heightfield surface in Γ). In Figure 6(a), supposing the smoothness parameter $\delta_1 = 1$, the lowest neighbor of $\mathbf{x}_3$ (resp., $\mathbf{y}_3$) on $Col(\mathbf{y})$ (resp., $Col(\mathbf{x})$) is $\mathbf{y}_2$ (resp., $\mathbf{x}_2$). To help exploit the spatial relations between two bounding surfaces S_1 and S_2 of a feasible coupled-surfaces-bounded region, we define below the *upstream* and *downstream* voxels of any voxel $\mathbf{x}_z \in \Gamma$ for the given surface separation constraint specified by two parameters δ^l and δ^u : the *upstream* (resp., *downstream*) of $\mathbf{x}_z$ is $\mathbf{x}_{z+\delta^l}$ (resp., $\mathbf{x}_{\max\{1, z-\delta^u\}}$) if $z + \delta^l \leq N$ (resp., $z - \delta^u \geq 1$). Intuitively, if $\mathbf{x}_z \in S_1$ (resp., $\mathbf{x}_z \in S_2$), then the upstream (resp., downstream) voxel of $\mathbf{x}_z$ is the voxel on $Col(\mathbf{x})$ with the smallest d -th coordinate that can be on S_2 (resp., S_1). We assume that in Figure 6(a) $\delta^l = 1$ and $\delta^u = 2$, then the upstream voxel of $\mathbf{x}_3$ is $\mathbf{x}_4$ and the downstream voxel of $\mathbf{x}_3$ is $\mathbf{x}_1$.

We say that a voxel $\mathbf{x}_z$ is *below* a heightfield surface S if $S(\mathbf{x}) > z$, and denote by $LO(S)$ the subset of all voxels of Γ that are on or below S . For every voxel $\mathbf{x}_z \in LO(S)$, consider its lowest neighbor $\mathbf{y}_{z'}$ on any adjacent column $Col(\mathbf{y})$ of $Col(\mathbf{x})$. Let $r = S(\mathbf{x})$ and $\mathbf{y}_{r'}$ be the lowest neighbor of $\mathbf{x}_r$ on $Col(\mathbf{y})$ ($\mathbf{x}_r \in Col(\mathbf{x})$ is on S). Then by the definition of lowest neighbors, $r' \leq S(\mathbf{y})$. Since $z \leq S(\mathbf{x})$, we have $z' \leq r'$ due to the smoothness constraint. Thus, $z' \leq S(\mathbf{y})$, and further, $\mathbf{y}_{z'} \in LO(S)$. Hence, we have the following *single surface self-closure* structure.

Observation 1. For any feasible heightfield surface S in Γ , if a voxel $\mathbf{x}_z$ is in $LO(S)$, then every lowest neighbor of $\mathbf{x}_z$ is also in $LO(S)$.

Observation 1 characterizes the self-closure property of the set $LO(S)$ of any feasible heightfield surface S in Γ . However, our task is more involved since two coupled heightfield surfaces S_1 and S_2 enclosing a coupled-surfaces-bounded region R are inter-related. We thus

need to further examine the *inter-surface self-closure* structure between $LO(S_1)$ and $LO(S_2)$. Consider any voxel $\mathbf{x}_z \in LO(S_1)$. Since $z \leq S_1(\mathbf{x})$, we have $z + \delta^l \leq S_1(\mathbf{x}) + \delta^l$, that is, the upstream voxel of the intersection voxel of S_1 and $Col(\mathbf{x})$ is “above” the upstream voxel of $\mathbf{x}_z$. In the meanwhile, $S_2(\mathbf{x}) \leq S_1(\mathbf{x}) + \delta^l$ (by the definition of the upstream voxel). Hence, the upstream voxel of $\mathbf{x}_z$ (i.e., $\mathbf{x}_{z+\delta^l}$) is in $LO(S_2)$. Using a similar argument, the downstream voxel of $\mathbf{x}_z$ (i.e., $\mathbf{x}_{\max\{1, z-\delta^u\}}$) is in $LO(S_1)$. Thus, Observation 2 follows.

Observation 2. For any feasible coupled-surfaces-bounded region R enclosed by S_1 and S_2 , the upstream (resp., downstream) voxel of each voxel in $LO(S_1)$ (resp., $LO(S_2)$) is in $LO(S_2)$ (resp., $LO(S_1)$).

The self-closure structures shown in Observations 1 and 2 are crucial to our MPNC algorithm and suggests a connection between our target problem and the maximum-cost closed set problem.^{19,15} In our MPNC approach, instead of directly searching for an optimal coupled-surfaces-bounded region $R^*(\theta)$ bounded by S_1 and S_2 , we look for optimal $LO(S_1)$ and $LO(S_2)$ in Γ , such that $LO(S_1)$ and $LO(S_2)$ uniquely define S_1 and S_2 , respectively. Similar self-closure structures were observed in Ref.²⁴ for the layered net surface problem.

2.2.2. Computing an optimal coupled-surfaces-bounded region—For a given θ , we construct a vertex-weighted directed graph $G(\theta) = (V, E)$ from Γ , such that the maximum-cost closed set in $G(\theta)$ specifies an optimal coupled-surfaces-bounded region $R^*(\theta)$ of a maximized parametric net-cost in Γ . The construction of $G(\theta)$ crucially relies on the self-closure structures shown in Section 2.2.1. $G(\theta)$ contains two vertex disjoint subgraphs $\{G_i = (V_i, E_i) | i = 1, 2\}$; each G_i is constructed in reflecting the single surface self-closure structure of the MPNC problem and is used for searching S_i of $R^*(\theta)$. The separation constraints between S_1 and S_2 are enforced in $G(\theta)$ by a set of edges E_s , connecting the corresponding subgraphs G_1 and G_2 , in such a way to reflect the inter-surface self-closure structure. Thus, $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup E_s$.

We first show the construction of each $G_i = (V_i, E_i)$ ($i = 1, 2$). Every voxel $\mathbf{x}_z \in \Gamma$ corresponds to exactly one vertex $v_i(\mathbf{x}_z) \in V_i$. If voxel $\mathbf{x}_z$ appears on S_i , then voxels of $Col(\mathbf{x})$ “below” $\mathbf{x}_z$ (i.e., those voxels of $Col(\mathbf{x})$ whose d -th coordinate is $\leq z$) are all in $LO(S_i)$. Thus, for each $\mathbf{x}$ in the domain $\mathcal{D}$ of Γ and $z = 2, 3, \dots, N$, vertex $v_i(\mathbf{x}_z)$ has a directed edge to the vertex $v_i(\mathbf{x}_{z-1})$, forming a *chain* $Ch_i(\mathbf{x})$: $v_i(\mathbf{x}_N) \rightarrow v_i(\mathbf{x}_{N-1}) \rightarrow \dots \rightarrow v_i(\mathbf{x}_1)$ in G_i for $Col(\mathbf{x})$. These chains help capture the closure structure of $LO(S_i)$. Two chains $Ch_i(\mathbf{x})$ and $Ch_i(\mathbf{y})$ in G_i are said to be *adjacent* if their corresponding columns $Col(\mathbf{x})$ and $Col(\mathbf{y})$ in Γ are adjacent. Next, we put directed edges between any two adjacent chains in G_i to enforce the surface smoothness constraint. Based on Observation 1, for every voxel $\mathbf{x}_z$ of each $Col(\mathbf{x})$ in Γ and its lowest neighbor $\mathbf{y}_{z'}$ on each adjacent $Col(\mathbf{y})$ of $Col(\mathbf{x})$, we put into E_i a directed edge from $v_i(\mathbf{x}_z) \in Ch_i(\mathbf{x})$ to $v_i(\mathbf{y}_{z'}) \in Ch_i(\mathbf{y})$. The role of this edge is to ensure that if $\mathbf{x}_z \in S_i$, then S_i must contain a voxel in $Col(\mathbf{y})$ that is no “lower” than $\mathbf{y}_{z'}$ (i.e., $S_i(\mathbf{y}) \geq z'$). Figure 7(a) illustrates part of the construction of G_i corresponding to two adjacent columns in Γ . This completes the construction of each G_i .

We then put directed edges into E_s between G_1 and G_2 , to incorporate the surface separation constraint (see Figure 7(b)). From Observation 2, if voxel $\mathbf{x}_z \in S_1$, then its upstream voxel

$\mathbf{x}_{z+\delta^l}$ must be on or below S_2 (i.e., $\mathbf{x}_{z+\delta^l} \in LO(S_2)$). Thus, for each vertex $v_1(\mathbf{x}_z)$ with $z = N - \delta^l$ on the chain $Ch_1(\mathbf{x})$ in G_1 , a directed edge is put in E_s from $v_1(\mathbf{x}_z)$ to $v_2(\mathbf{x}_{z+\delta^l})$ on $Ch_2(\mathbf{x})$ in G_2 . Intuitively, these edges ensure that S_2 must be at a distance of *at least* δ^l “above” S_1 (i.e., for each $\mathbf{x} \in \mathcal{D}$, $S_2(\mathbf{x}) - S_1(\mathbf{x}) \geq \delta^l$). On the other hand, each vertex $v_2(\mathbf{x}_z)$ with $z > \delta^l$ on $Ch_2(\mathbf{x})$ of G_2 has a directed edge in E_s to vertex $v_1(\mathbf{x}_{z'})$ with $z' = \max\{1, z - \delta^u\}$ (note that $\mathbf{x}_{z'}$ in Γ is the downstream voxel of $\mathbf{x}_z$), making sure that S_2 must be at a distance of *no larger* than δ^u “above” S_1 (i.e., for each $\mathbf{x} \in \mathcal{D}$, $S_2(\mathbf{x}) - S_1(\mathbf{x}) \leq \delta^u$). Notice that in this construction, each vertex $v_1(\mathbf{x}_z)$ with $z > N - \delta^l$ has no edge to any vertex on $Ch_2(\mathbf{x})$, and each vertex $v_2(\mathbf{x}_z)$ with $z \leq \delta^l$ has no edge to any vertex on $Ch_1(\mathbf{x})$. These vertices (called *deficient vertices*) in $G(\theta)$ correspond to voxels in Γ that cannot possibly appear in any feasible coupled-surfaces-bounded region. Thus, it is safe to remove all those vertices from $G(\theta)$.

Up to this point, the graph construction establishes the connection between a closed set in $G(\theta)$ and a feasible coupled-surfaces-bounded region in Γ . For any feasible coupled-surfaces-bounded region R , we construct a closed set $\mathcal{C}_i \subseteq \emptyset$ in G_i for each S_i ($i = 1, 2$) in the following way. Initially, let $\mathcal{C}_i = \emptyset$. For each $\mathbf{x} \in \mathcal{D}$, we add to $\mathcal{C}_i$ the subset $\mathcal{C}_i(\mathbf{x}) = \{v_i(\mathbf{x}_z) | z \in S_i(\mathbf{x})\}$ of vertices on $Ch_i(\mathbf{x})$ of G_i . Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Based on the construction of $G(\theta)$ and Observations 1 and 2, it is easy to prove that $\mathcal{C}$ is a closed set in $G(\theta)$.

Lemma 6. *Any feasible coupled-surfaces-bounded region in Γ defines a closed set $\mathcal{C} \subseteq \emptyset$ in $G(\theta)$.*

Given any closed set $\mathcal{C} \subseteq \emptyset$ in $G(\theta)$, we specify in Γ a region R bounded by two coupled heightfield surfaces S_1 and S_2 , as follows. Note that we search for each S_i in subgraph G_i ($i = 1, 2$). Let $\mathcal{C}_i = \mathcal{C} \cap V_i$. For each voxel $\mathbf{x} \in \mathcal{D}$, denote by $\mathcal{C}_i(\mathbf{x})$ the set of vertices of $\mathcal{C}_i$ on the chain $Ch_i(\mathbf{x})$ of G_i . Based on the construction of G_i , it is not hard to show that $\mathcal{C}_i(\mathbf{x}) \neq \emptyset$. Let $r_i(\mathbf{x})$ be the largest d -th coordinate of the vertices in $\mathcal{C}_i(\mathbf{x})$. Then, define the function S_i as $S_i(\mathbf{x}) = r_i(\mathbf{x})$ for every $\mathbf{x} \in \mathcal{D}$. By applying a similar argument as in Refs.,^{24,23} we can prove that each S_i is a heightfield surface in Γ , and both S_1 and S_2 satisfy the surface separation constraint. Hence, Lemma 7 follows.

Lemma 7. *Any closed set $\mathcal{C} \subseteq \emptyset$ in $G(\theta)$ specifies a feasible coupled-surfaces-bounded region in Γ .*

Our goal is to compute a non-empty maximum-cost closed set in $G(\theta)$, which can specify an optimal coupled-surfaces-bounded region in Γ . We thus need to further assign a cost $w(\cdot)$ to each vertex in $G(\theta)$.

Note that the parametric net-cost of a coupled-surfaces-bounded region R for a given θ is

$$\beta_\theta(R) = \beta(R) - \theta|R| = \sum_{i=1}^2 \sum_{\mathbf{x}_z \in S_i} b(\mathbf{x}_z) + \sum_{\mathbf{x}_z \in R} c(\mathbf{x}_z) - \theta|R| = \sum_{i=1}^2 \sum_{\mathbf{x}_z \in S_i} b(\mathbf{x}_z) + \sum_{\mathbf{x}_z \in R} [c(\mathbf{x}_z) - \theta]$$

, where S_1 and S_2 are two bounding surfaces of R . Further, note that we are looking for $LO(S_i)$ ($i = 1, 2$) instead of S_i directly, by using the subgraph G_i . Thus, we want to find a way to assign vertex costs to V_i , so that for any feasible S_i in Γ , the cost $\sum_{\mathbf{x}_z \in S_i} b(\mathbf{x}_z)$ can be “distributed” to those vertices of G_i corresponding to $LO(S_i)$ (i.e., $\sum_{\mathbf{x}_z \in LO(S_i)} w(v_i(\mathbf{x}_z)) = \sum_{\mathbf{x}_z \in S_i} b(\mathbf{x}_z)$). We may go one step further to require that for any voxel $\mathbf{x}_r \in Col(\mathbf{x})$ in Γ , the

total cost of the vertices $v_i(\mathbf{x}_z)$ on $Ch_i(\mathbf{x})$ of G_i with $z = r$ (i.e., $\sum_{z=r} w(v_i(\mathbf{x}_z))$) be equal to the cost of $\mathbf{x}_r$. In addition, note that $R = LO(S_2) - (LO(S_1) \setminus S_1)$, and consider the removal of the deficient vertices as well. We have the following vertex-cost assignment scheme: For every $\mathbf{x} \in \mathcal{D}$,

$$w(v_1(\mathbf{x}_z)) = \begin{cases} b(\mathbf{x}_z) & \text{if } z=1, \\ [b(\mathbf{x}_z) - b(\mathbf{x}_{z-1})] + [\theta - c(\mathbf{x}_{z-1})] & \text{for } z=2, 3, \dots, N - \delta^l. \end{cases} \quad (1)$$

$$w(v_2(\mathbf{x}_z)) = \begin{cases} b(\mathbf{x}_z) + \sum_{z'=1}^z c(\mathbf{x}_{z'}) & \text{if } z=\delta^l+1, \\ [b(\mathbf{x}_z) - b(\mathbf{x}_{z-1})] + [c(\mathbf{x}_z) - \theta] & \text{for } z=\delta^l+2, \dots, N. \end{cases} \quad (2)$$

We can show that the parametric net-cost $\beta_\theta(R)$ of the coupled-surfaces-bounded region R defined by a closed set $\mathcal{C} \subseteq \mathcal{D}$ equals to the total vertex cost $w(\mathcal{C})$ of $\mathcal{C}$, and vice verse.

Note that in G_i , if a vertex $v_i(\mathbf{x}_q) \in C_i = \mathcal{C} \cap V_i$, then all vertices in $\mathcal{C}_i(\mathbf{x}) = \{v_i(\mathbf{x}_z) \in Ch_i(\mathbf{x}) | z \leq q\}$ are also in $\mathcal{C}_i$.

(3)

Based on Lemmas 6 and 7, we have the following fact.

Lemma 8. *For a given θ , the region $R^*(\theta)$ specified by a maximum-cost non-empty closed set $\mathcal{C}^*$ in $G(\theta)$ is an optimal coupled-surfaces-bounded region in Γ with a maximized parametric net-cost.*

A maximum non-empty closed set in $G(\theta)$ can be computed,^{19,10,23} as follows. Define a new directed graph $G_{st}(\theta) = (V \cup \{s, t\}, E \cup E_{st})$. The vertex set of $G_{st}(\theta)$ is the vertex set V of $G(\theta)$ plus a source s and a sink t . The edge set of $G_{st}(\theta)$ is the edge set E of $G(\theta)$ plus a new edge set E_{st} consisting of the following edges: The source s is connected to each vertex $v \in V$ by a directed edge of cost $w(v)$ and every vertex $v \in V$ is connected to the sink t by a directed edge of cost 0. The weight of each edge in E is assigned $+\infty$. Note that $G_{st}(\theta)$ has $O(n)$ vertices and $O(n)$ edges. By using Goldberg and Tarjan's minimum s - t cut algorithm,¹¹ we compute in $G(\theta)$ a maximum-cost closed set $\mathcal{C}^* \subseteq \mathcal{D}$ in $O(n^2 \log n)$ time.

Lemma 9. *For a given θ , the MPNC problem can be solved in $O(n^2 \log n)$ time. From Lemmas 5 and 9, we have the following theorem.*

Theorem 1. *Given a d -D ($d \geq 3$) voxel grid Γ of n cell and a non-negative concave normalization function $g(R)$, an optimal coupled-surfaces-bounded region R^* with a*

maximum ratio cost $\alpha(R^) = \max_R \frac{\beta(R)}{g(R)}$ can be computed in $O(n^3 \log n)$ time.*

Observing that $g(R) = \sqrt{|R|(n - |R|)}$ is a non-negative concave function and $g(R) = |R|$ is a degenerated concave function, Corollaries 1 and 2 immediately follow from Theorem 1.

Corollary 1. Given a d -D($d-3$) voxel grid Γ of n cell, an optimal coupled-surfaces-

bounded region R^* with a maximum ratio cost $\alpha(R^*) = \max_R \frac{\beta(R)}{\sqrt{|R|(n-|R|)}}$ can be computed in $O(n^3 \log n)$ time.

Corollary 2. Given a d -D($d-3$) voxel grid Γ of n cell, an optimal coupled-surfaces-

bounded region R^* with a maximum ratio cost $\alpha(R^*) = \max_R \frac{\beta(R)}{|R|}$ can be computed in $O(n^3 \log n)$ time.

2.3. Computing an optimal-ratio smooth lower-half region

This section presents our $O(n^2 \log n)$ time algorithm for computing an optimal-ratio smooth lower-half region in Γ using a non-negative concave normalization function $g(R)$.

From Lemma 5, a key subroutine is that, for a given parameter θ , computing an optimal smooth lower-half region $R^*(\theta)$ in Γ such that the parametric net-cost $\beta\theta(R^*(\theta))$ of $R^*(\theta)$ is maximized. Recall that a smooth lower-half region R in Γ is bounded by a heightfield surface S . We observe that the single surface self-closure structure as in Section 2.2.1 still holds for the smooth lower-half region. Thus, a similar graph transformation scheme as in Section 2.2.2 is used to construct a vertex-weighted directed graph $G(\theta) = (V, E)$, yet here $G(\theta)$ has only one subgraph used for the search of the only bounding heightfield surface S of the smooth lower-half region $R(\theta)$ for a given θ . Each voxel $\mathbf{x}_z \in \Gamma$ corresponds to exactly one vertex $v(\mathbf{x}_z)$ in $G(\theta)$. The cost of each vertex in $G(\theta)$ is assigned, as follows.

$$w(v(\mathbf{x}_z)) = \begin{cases} b(\mathbf{x}_z) + [c(\mathbf{x}_z) - \theta] & \text{if } z=1, \\ [b(\mathbf{x}_z) - b(\mathbf{x}_{z-1})] + [c(\mathbf{x}_z) - \theta] & \text{for } z=2, 3, \dots, N. \end{cases} \quad (4)$$

It is well-known that computing a maximum-cost closed set $\mathcal{C}$ in $G(\theta)$ is equivalent to compute a minimum s - t cut in $G_{st}(\theta)$.^{19,10} In $G_{st}(\theta)$, the source s has a directed edge to every vertex $v(\mathbf{x}_z)$ in $G(\theta)$ with a cost of $w(v(\mathbf{x}_z))$, the sink t has a directed edge with a cost of 0 from every vertex in $G(\theta)$, and the cost of all other edges is $+\infty$. Obviously, based on the vertex-cost assignment scheme (4), the cost of every edge from source s is a non-increasing function of θ and all other edges in $G_{st}(\theta)$ have a constant cost with respect to θ . Thus, $G_{st}(\theta)$ is a monotone parametric flow network.^{13,10}

To compute an optimal-ratio smooth lower-half region in Γ , as in Section 2.2, we need to compute an optimal smooth lower-half region with a maximized parametric net-cost for each of a sequence of parameters $\{\theta_1, \theta_2, \dots, \theta_m\}$ generated by the hand probing process (see Section 2.1), where $m = O(n)$. Due to the monotonicity of $G_{st}(\theta)$, we can apply Gusfield and Martel's parametric minimum cut algorithm¹³ to compute all those $O(n)$ optimal smooth lower-half regions in Γ in the complexity of solving a single maximum flow problem. Note that negative edge costs in the various related minimum cut problems can be made nonnegative using the transformation suggested in Ref.¹⁰

Theorem 2. Given a d -D ($d \geq 3$) voxel grid Γ of n cell, an optimal-ratio smooth lower-half region in Γ using a given non-negative concave normalization function $g(R)$ can be computed in $O(n^2 \log n)$ time.

Corollary 3. Given a d -D ($d \geq 3$) voxel grid Γ of n cell, an optimal smooth lower-half region

R^* with a maximum ratio cost $\alpha(R^*) = \max_R \frac{\beta(R)}{\sqrt{|R|(n - |R|)}}$ can be computed in $O(n^2 \log n)$ time.

Corollary 4. Given a d -D ($d \geq 3$) voxel grid Γ of n cell, an optimal smooth lower-half region

R^* with a maximum ratio cost $\alpha(R^*) = \max_R \frac{\beta(R)}{|R|}$ can be computed in $O(n^2 \log n)$ time.

3. Improved Algorithm for the Detection of an Optimal-ratio Smooth Lower-half Region Using $g(R) = |R|$

In Section 2.3, Gusfield and Martel's parametric minimum cut method¹³ is used to achieve an $O(n^2 \log n)$ time algorithm for the detection of an optimal-ratio smooth lower-half region using $g(R) = |R|$. However, their parametric minimum-cut algorithm is quite complex and may not be practically efficient. Interestingly, we are able to establish a connection between our convex hull model and the traditional Newton-based approach for the fractional programming problem (see, e.g., Gondran and Minous¹²). This connection enables us to find an order to compute the vertices on the convex hull, such that all those vertices can be computed by using Gallo *et al.*'s simple parametric minimum s - t cut algorithm¹⁰ in the complexity of solving a single maximum flow problem.

The difficulty to apply Gallo *et al.*'s simple parametric minimum s - t cut algorithm¹⁰ is that the dynamically generated sequence of slopes $\{\theta_1, \theta_2, \dots, \theta_m\}$ by the hand probing method may not be in sorted order. We observe that that problem can be overcome by the establishment of a connection between our convex hull model and the traditional Newton-based approach for the fractional programming problem (see, e.g., Gondran and Minous¹²). The following Newton-based method for computing an optimal-ratio smooth lower-half region in Γ is adapted from a general approach for the fractional programming problem.

PROCEDURE ItrORDV-SLH

Step 0: Set $i = 0$. Compute a smooth lower-half region $R^{(i)}$ in Γ and let $\theta_i = \frac{\beta(R^{(i)})}{|R^{(i)}|}$.

Step 1: Compute $R^{(i+1)}$ such that the parametric net-cost $\beta_{\theta_i}(R^{(i+1)}) = \max_R \{\beta_{\theta_i}(R) = \beta(R) - \theta_i |R|\}$, where R is any feasible smooth lower-half region in Γ .

Step 2: If $\beta_{\theta_i}(R^{(i+1)}) = 0$, then $R^* = R^{(i)}$; otherwise, let $\theta_{i+1} = \frac{\beta(R^{(i+1)})}{|R^{(i+1)}|}$, replace i by $i + 1$ and go to Step 1.

The output of this procedure is a sequence of slopes $\{\theta_1, \theta_2, \dots\}$ and their corresponding smooth lower-half regions $\{R^{(1)}, R^{(2)}, \dots\}$. It is well-known that the smooth lower-half region R^* whose parametric net-cost $\beta_\theta(R^*) = 0$ for a certain θ , is the optimal-ratio region in

Γ maximizing the ratio-cost, i.e., $\alpha(R^*) = \max_R \frac{\beta(R)}{|R|}$. The efficiency of ItrORDV-SLH depends on the number of times Step 1 has to be executed. We will bound the number of executions of Step 1 by establishing a connection between our convex hull model and the ItrORDV-SLH procedure. Lemma 10 states a good property of ItrORDV-SLH (note that the same property holds for the general fraction programming problem¹⁰).

Lemma 10. *The sequence of slopes $\{\theta_i\}$ generated by ItrORDV-SLH is strictly increasing.*

Let $k_i = |R^{(i)}|$. We can show that $(k_i, \beta(R^{(i)}))$ is a vertex on the upper chain $UH(P)$ of the convex hull $CH(P)$ as in Section 2.1.2. A key observation here is that all the points $\{(k_i, \beta(R^{(i)})) | i = 1, 2, \dots\}$ defined by $\{R^{(i)} | i = 1, 2, \dots\}$ are pairwise different vertices on $UH(P)$. Otherwise, assume that there exist $R^{(j)}$ and $R^{(j')}$ such that $j \neq j'$ and $(k_j, \beta(R^{(j)})) = (k_{j'}, \beta(R^{(j')}))$.

From Step 2 of Procedure ItrORDV-SLH, $\theta_j = \frac{\beta(R^{(j)})}{|R^{(j)}|}$ and $\theta_{j'} = \frac{\beta(R^{(j')})}{|R^{(j')}|}$. Thus, we have $\theta_j = \theta_{j'}$, which is a contradiction to Lemma 10. Note that the number of vertices on $UH(P)$ is bounded by $O(n)$. Hence, we have the following lemma.

Lemma 11. *Procedure ItrORDV-SLH terminates in $O(n)$ executions of Step 1.* Based on Lemmas 10 and 11, Procedure ItrORDV-SLH requires to compute in Γ an optimal smooth lower-half region with a maximized parametric net-cost for each of the sequence of $O(n)$ slopes $\{\theta_i\}$ that are in increasing order. Due to the monotonicity of the graph $G_{st}(\theta)$, which is used to compute such smooth lower-half regions in Γ , we can apply Gallo *et al.*'s simple parametric minimum s - t cut algorithm¹⁰ to compute all those optimal regions in the complexity of solving a single maximum flow problem. Hence, Theorem 3 follows.

Theorem 3. *Given a d -D($d \geq 3$) voxel grid Γ of n cell, an optimal-ratio smooth lower-half region in Γ using the normalization function $g(R) = |R|$ can be computed in $O(n^2 \log n)$ time.*

4. Conclusion

In this paper, an algorithmic framework, which leads to strongly polynomial-time algorithms for solving the optimal-ratio region detection problems, had been developed. We formulate the problems as computing a sequence of $O(n)$ minimum s - t cuts in the corresponding weighted directed graphs, by using the hand probing technique and exploiting the self-closure structures of the problems. Interestingly, for the class of regions bounded by a single heightfield surface, we have shown that the $O(n)$ calls to the minimum s - t cut algorithm are on a monotone parametric flow network, enabling the detection of the optimal-ratio region in the complexity of computing a single maximum flow. However, the algorithms for computing the optimal-ratio coupled-surfaces-bounded region might not be efficient enough for large medical image datasets in practice. How to improve the time complexity for the coupled-surfaces-bounded region detection is of our great interesting.

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Figure 1.

(a) A retinal layer in a slice of a 3-D optical coherence tomography (OCT) image. Each 3-D OCT image is composed of a number of 2-D radial scans, as schematically shown on the left. (b) A schematic cross-sectional anatomy of a diseased artery.

Figure 2.

(a) Illustrating a tubular object in a 3-D image and the “unfolding” operation by a cylindrical coordinate transform. (b) “Unfolding” the image in (a) to form a new 3-D image. The tubular object in (a) corresponds to a smooth lower-half region to be detected in (b) and the boundary of the tubular object in (a) corresponds to a heightfield surface in (b).

Figure 3.

(a) Illustrating the smoothness constraint in 3-D. Three adjacent columns shown with smoothness parameters $\Delta_1 = 2$ and $\Delta_2 = 1$. (b) A region R enclosed by two coupled heightfield surfaces S_1 and S_2 .

Figure 4.

(a) Illustrating the proof of Lemma 1. (b) The construction of a convex hull using the hand probing technique.

Figure 5.
Illustrating the proof of Lemma 4.

Figure 6.

Illustrating the self-closure structures. (a) An example 3-D voxel grid Γ . Only two adjacent columns of voxels in the first dimension are shown. The smoothness parameter $\delta^l = 1$ and the surface separation constraint $\delta^l = 1$ and $\delta^u = 2$. (b) Illustrating the single surface self-closure structure. The arrows point to the lowest neighbor of each voxel on the adjacent column. For a feasible heightfield surface S , $LO(S)$ consists of the voxels indicated by solid cycles. The lowest neighbor of any voxel in $LO(S)$ is in $LO(S)$. (c) Illustrating the inter-surface self-closure structure (only upstream voxels are shown). To improve readability, we duplicate the voxel grid Γ and two feasible heightfield surfaces, S_1 and S_2 , are shown in the separate voxel grids. The arrows point to the upstream voxel of each voxel on the same column. The solid cycles indicate the voxels in $LO(S_1)$ and $LO(S_2)$. The upstream voxel of any voxel in $LO(S_1)$ is in $LO(S_2)$.

Figure 7.

Illustrating the graph construction. (a) Constructing a subgraph G_i ($i = 1, 2$) from the voxel grid Γ in Figure 6(a). (b) Incorporating the surface separation constraint into G (with $\delta^l = 1$ and $\delta^u = 2$).