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Joint density of eigenvalues in spiked multivariate models

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Abstract

The classical methods of multivariate analysis are based on the eigenvalues of one or two sample covariance matrices. In many applications of these methods, for example to high dimensional data, it is natural to consider alternative hypotheses which are a low rank departure from the null hypothesis. For rank one alternatives, this note provides a representation for the joint eigenvalue density in terms of a single contour integral. This will be of use for deriving approximate distributions for likelihood ratios and ‘linear’ statistics used in testing.

Keywords

principal components; covariance matrices; hypergeometric function; contour integral

1. Introduction

The eigenvalues of one or two sample covariance matrices play a central role in multivariate analysis. A long list of examples, including principal components analysis (PCA), canonical correlations analysis (CCA), multivariate analysis of variance (MANOVA) and multiple response linear regression are the main subject of many standard textbooks, such as Mardia et al. (1979); Anderson (2003).

Under the common assumption of Gaussian data, much is known about the joint and marginal distribution of the eigenvalues. For example, under the typical null hypotheses, the joint density of the eigenvalues has an explicit formula, derived in 1939 in the celebrated and independent work of Fisher, Girshick, Hsu, Mood and Roy. Under general alternatives, the joint density is given by an integral over a group of matrices. If the number of variables, and hence eigenvalues, is large, p say, as is common nowadays, this integral will be high dimensional, of dimension $O(p^2)$.

A remarkable classification of the joint density functions was given by James (1964), using hypergeometric functions of matrix argument. He showed how the classical multivariate methods could be organized into five cases, involving hypergeometric functions ${}_pF_q$ of different orders, specifically ${}_0F_0$, ${}_0F_1$, ${}_1F_0$, ${}_1F_1$, and ${}_2F_1$. Remarkable though this work is, and despite significant progress on the numerical computation of hypergeometric functions,

e.g. Koev & Edelman (2006), these expressions for the joint densities have proved challenging to work with in application.

In many high dimensional applications, however, it may be reasonable to consider alternative hypotheses which are low rank departures from the null. For some examples, see Johnstone & Nadler (2013). In this note we consider the simplest case, namely rank one deviations, and show that the joint eigenvalue density can then be reduced to a single (contour) integral.

We believe this integral representation to be of interest at least because it is amenable to approximation when dimension p is large, leading to simple approximations to at least certain aspects of these multivariate eigenvalue distributions.

We mention two examples of such applications.

- i. derivation of limiting Gaussian approximations for ‘linear statistics’ (including, for example, the likelihood ratio test, and ‘high-dimension-corrected’ likelihood ratio test, Onatski et al. (2013); Wang et al. (2013)). Particular cases (${}_0F_0$, ${}_0F_1$, ${}_1F_1$) have been given for complex data by Passemier et al. (2014a).
- ii. delineation of the region of contiguous alternatives to the null hypothesis, and description of the Gaussian limit for the log-likelihood ratio process inside the contiguity region. This leads to a comparative understanding of the power properties of various hypothesis tests, both traditional and new, in the contiguity region. This example has been studied in the case of PCA, corresponding to ${}_0F_0$, by Onatski et al. (2013), and work is in progress to apply the result of this note to the general ${}_pF_q$ cases.

We will adopt James’ systematization in order to give a unified derivation of our contour formulas. We give the rank one formula for ${}_pF_q$ in real and complex cases, Section 2. This can be converted directly into an expression for the joint density function for the eigenvalues in each of James’ five cases (for both $\mathbb{R}$ and $\mathbb{C}$). Section 3 illustrates this process in one case, testing equality of covariance matrices, for real data (i.e. ${}_1F_0$).

In the real case, the proof of Section 2 applies only to even dimension p . Section 4 gives a different proof valid for all integer p .

2. Contour integral representation for rank one

Let X, Y be $r \times r$ Hermitian matrices. The definitions of hypergeometric functions with one and two matrix arguments are given, for example, by James (1964), with separate expressions for real and complex cases.

The definitions simplify in our special case in which X has rank one, with nonzero eigenvalue x . For $a \in \mathbb{C}$, let $(a)_k = a(a+1) \cdots (a+k-1)$, $(a)_0 = 1$ be the rising factorial, and for vectors of parameters $a = (a_l)_{l=1}^p, b = (b_l)_{l=1}^q$ with $a_l \in \mathbb{C}$ and $b_l \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, adopt the abbreviation

$$\rho_k = \rho_k(a, b) = \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k}.$$

If X has rank one as described, define

$${}_pF_q^\alpha(a, b; X, Y) = \sum_{k=0}^{\infty} \rho_k \frac{(1/\alpha)_k}{(r/\alpha)_k} \frac{x^k C_k^\alpha(Y)}{k!}. \quad (1)$$

Here $\alpha > 0$ indexes a one parameter family that includes the real ($\alpha = 2$) and complex ($\alpha = 1$) cases. Also, C_k^α are Jack polynomials (e.g. Macdonald (1995)): in the real case ($\alpha = 2$), they reduce to James' zonal polynomials (e.g. Muirhead (1982)), and in the complex case ($\alpha = 1$), to a normalization of the Schur functions (e.g. Dumitriu et al. (2007)). A contour formula for $C_k^\alpha(Y)$ is quoted below; for now we note that $C_k^\alpha(X) = x^k$, and (e.g. Wang (2012, eq. (245))) that

$$C_k^\alpha(I) = \prod_{j=0}^{k-1} \frac{r + \alpha j}{1 + \alpha j} = \frac{(r/\alpha)_k}{(1/\alpha)_k},$$

which explains the form of the two ratios in formula (1) as $C_k^\alpha(X)C_k^\alpha(Y)/C_k^\alpha(I)$.

The series (1) converges for all x, Y if $p = q$; for $x\|Y\| < 1$ if $p = q + 1$ (and $\|Y\|$ denotes the maximum eigenvalue in absolute value of Y) and finally diverges unless it terminates if $p > q + 1$ (e.g. Mathai et al. (1995)).

With this notation, the scalar generalized hypergeometric function, which does not depend on α , is

$${}_pF_q(a, b; x) = \sum_{k=0}^{\infty} \rho_k(a, b) \frac{x^k}{k!}.$$

The main result of this note can now be stated.

Proposition 1

Suppose that $p = q + 1$, X is rank 1 with positive eigenvalue x and that Y is positive definite with eigenvalues $(y_j)_{j=1}^r$.

- i. Suppose that r/α is a positive integer, say $r/\alpha = m + 1$, and that $a_l \notin \{1, \dots, m\}$ and $b_l \notin \{m, m - 1, m - 2, \dots\}$. Then,

$${}_pF_q^\alpha(a, b; X, Y) = \frac{\Gamma(m+1)}{x^m \rho'_m} \frac{1}{2\pi i} \int_{\Gamma} {}_pF_q(a-m, b-m; xs) \prod_{j=1}^r \frac{1}{(s-y_j)^{1/\alpha}} ds, \quad (2)$$

where the contour Γ starts at $-\infty$, encircles 0 and $\{y_j\}$ counterclockwise and returns to $-\infty$. Further, $a-m$ denotes the vector with entries $a_i - m$ and

$$\rho'_m = \rho_m(a-m, b-m).$$

Equality holds in the common domain of analyticity of both sides: $\mathbb{C}$ if $p = q$ and $\mathbb{C} \setminus (1, \infty)$ if $p = q + 1$.

- ii. If instead $r/\alpha = m + \varepsilon$ for $\varepsilon \in (0, 1)$ and non-negative integer m , then under the same conditions

$${}_pF_q^\alpha(a, b; X, Y) = \frac{(\varepsilon)_m}{x^m \rho'_m} \frac{1}{2\pi i} \int_{\Gamma} s^{\varepsilon-1} {}_{p+1}F_{q+1}(a-m, 1, b-m, \varepsilon; xs) \prod_{j=1}^r \frac{1}{(s-y_j)^{1/\alpha}} ds. \quad (3)$$

- iii. If $\alpha = 2$, then formula (2) holds for **any** integer r , still with $m + 1 = r/2$, if the symbol $(a)_m$ is interpreted as $\lceil (a+m)/\rceil (a)$ for non-integer m .

Thus, in the real ($\alpha = 2$) and complex ($\alpha = 1$) cases of most interest in applications, formula (2) holds for all positive integer r .

Particular cases of (2) are already known: ${}_0F_0$ for both real and complex cases (Mo, 2012; Onatski et al., 2013), for general α , Wang (2012); Forrester (2011), and for the complex case only, ${}_0F_1$ (Dharmawansa, 2013) and ${}_1F_1$ (Passemier et al., 2014a). Wang (2012) also gives formula (3) in the ${}_0F_0$ case. A generalization of (i) to the multi-spike case has been given for ${}_0F_0$ by Onatski (2014) and recently extended to ${}_pF_q$ by Passemier et al. (2014b).

Proof—Parts (i) and (ii) are shown here; part (iii) uses a different argument and is deferred to Section 4. We begin with a result from Wang (2012, eq. (248)), which states that

$$(1/\alpha)_k \frac{C_k^\alpha(Y)}{k!} = \frac{1}{2\pi i} \int_{\Gamma'} \prod_{j=1}^r \frac{1}{(1-zy_j)^{1/\alpha}} \frac{dz}{z^{k+1}}.$$

Here the contour Γ' encircles zero and is chosen small enough so that all y_j^{-1} lie outside.

Insert this into (1) and interchange summation and integration to obtain

$${}_pF_q^\alpha(a, b; X, Y) = \frac{1}{2\pi i} \int_{\Gamma'} \prod_{j=1}^r \frac{1}{(1-zy_j)^{1/\alpha}} G(z; x) dz \quad (4)$$

where the series

$$G(z;x) = \sum_{k=0}^{\infty} \frac{\rho_k}{(r/\alpha)_k} \frac{x^k}{z^{k+1}}$$

converges for all x, z if $p < q$ and for $|x/z| < 1$ if $p = q + 1$.

Now write $r/\alpha = m + 1$ and introduce the variable $l = k + m$, so that

$$G(z;x) = \sum_{l=m}^{\infty} \frac{\rho_{l-m}}{(m+1)_{l-m}} \frac{x^{l-m}}{z^{l-m+1}} = \frac{m!}{x^m} \frac{z^{m-1}}{\rho'_m} \sum_{l=m}^{\infty} \frac{\rho_l(a-m, b-m)}{l!} \left(\frac{x}{z}\right)^l, \quad (5)$$

where we have used $(m+1)_{l-m} = l!/m!$, and noted that $(c)_{l-m} = (c-m)_l / (c-m)_m$ so that

$$\rho_{l-m}(a, b) = \frac{\rho_l(a-m, b-m)}{\rho_m(a-m, b-m)}.$$

Let $G_0(z;x)$ denote the function obtained by extending the summation in (5) down to $l = 0$, so that

$$G_0(z;x) = \frac{m!}{x^m} \frac{z^{m-1}}{\rho'_m} {}_pF_q(a-m, b-m; x/z).$$

Since we are adding a polynomial to G and a term that is analytic within the contour in (5), the value of the integral is unchanged. Hence

$${}_pF_q^\alpha(a, b; X, Y) = \frac{m!}{x^m} \frac{1}{\rho'_m} \frac{1}{2\pi i} \int_{\Gamma'} \prod_{j=1}^r \frac{z^{m-1}}{(1-zy_j)^{1/\alpha}} {}_pF_q(a-m, b-m; x/z) dz.$$

The change of variables $z = 1/s$ yields

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma'} \frac{z^{m-1}}{\prod_{j=1}^r (1-zy_j)^{1/\alpha}} F(x/z) dz &= \frac{1}{2\pi i} \int_{\Gamma''} \frac{1}{s^{m+1}} \frac{F(xs)}{\prod_{j=1}^r (1-y_j/s)^{1/\alpha}} ds, \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{F(xs)}{\prod_{j=1}^r (s-y_j)^{1/\alpha}} ds, \end{aligned}$$

where the image Γ'' of Γ' is deformed to Γ as described in the Proposition statement in order to avoid the branch cut in the final formula. Here we use the analytic continuations of ${}_pF_q$: entire for $p < q$ and for $p = q + 1$ analytic off the positive real axis $(1, \infty)$. The result follows.

When $r/\alpha = m + \varepsilon$ we modify the argument. In (5), replace $(m+1)_{l-m}$ by $(m+\varepsilon)_{l-m} = (\varepsilon)_l / (\varepsilon)_m$ to obtain

$$G(z; x) = \frac{(\varepsilon)_m}{x^m} \frac{z^{m-1}}{\rho'_m} \sum_{l=m}^{\infty} \frac{\rho_l(a-m, b-m)(1)_l}{(\varepsilon)_l} \frac{1}{l!} \left(\frac{x}{z}\right)^l.$$

Proceeding as before, and extending the summation to $l = 0$, so that

$$G_0(z; x) = \frac{(\varepsilon)_m}{x^m} \frac{z^{m-1}}{\rho'_m} {}_{p+1}F_{q+1}(a-m, 1, b-m, \varepsilon; x/z),$$

we obtain formula (3).

3. Example

Consider the problem of testing equality of covariance matrices—the ${}_1F_0$ case in James (1964). Thus, suppose that $n_1, n_2 \geq p$ and that $p \times n_1$ and $p \times n_2$ real data matrices $X = [X_1 \cdots X_{n_1}]$ and $Y = [Y_1 \cdots Y_{n_2}]$ have columns X_i, Y_j with mean zero and covariance matrices Σ_1 and Σ_2 respectively. A signal detection application is described in Johnstone & Nadler (2013, Sec. 3).

Suppose that the observation vectors are independent Gaussian, so that $A_1 = XX'$ and $A_2 = YY'$ have Wishart distributions $W_p(n_1, \Sigma_1)$ and $W_p(n_2, \Sigma_2)$ respectively. Then James (1964, eq. (65)) gives an expression for the joint density of the eigenvalues (f_j) of $A_1 A_2^{-1}$. To state it, we introduce notation $|A| = \det(A)$, $F = \text{diag}(f_j)$ and $\Delta = \sum_1 \sum_2^{-1}$. We transform this expression, following Muirhead (1982, p. 313–4), to obtain for $n = n_1 + n_2$ and $f_1 > f_2 > \cdots > f_p$,

$$p(f; \Delta) = \frac{c_{p, n_1, n_2}}{|\Delta|^{n_1/2}} \frac{|F|^{(n_1-p-1)/2}}{|I+F|^{n/2}} {}_1F_0\left(\frac{n}{2}; I - \Delta^{-1}, F(I+F)^{-1}\right) \prod_{j < j'}^p (f_j - f_{j'}), \quad (6)$$

where in this real case, $\alpha = 2$, we have written ${}_1F_0$ for ${}_1F_0^2$. The normalization constant is given in terms of the multivariate gamma function (Muirhead, 1982, p. 61) by

$$c_{p, n_1, n_2} = \frac{\pi^{p^2/2} \Gamma_p(\frac{1}{2}n)}{\Gamma_p(\frac{1}{2}p) \Gamma_p(\frac{1}{2}n_1) \Gamma_p(\frac{1}{2}n_2)}.$$

In the spirit of application (ii) in the Introduction, we may consider the likelihood ratio for testing the null hypothesis that $\Sigma_1 = \Sigma_2$. Writing $\Lambda = F(I+F)^{-1}$, we have

$$L(\Delta; \Lambda) = \frac{p(\Lambda; \Delta)}{p(\Lambda; I)} = |\Delta|^{-n_1/2} {}_1F_0\left(\frac{n}{2}; I - \Delta^{-1}, \Lambda\right).$$

Turning now to apply the result of this paper, suppose that Σ_1 is a rank one perturbation of Σ_2 , so that $\Sigma_1 = (I + \psi h \psi') \Sigma_2$ for real h and for ψ a unit vector in $\mathbb{R}^p$. In this case, $\Sigma_1 = I + \psi h \psi'$, so that $I - \Sigma_1^{-1}$ has rank one, with nonzero eigenvalue $\tau = h/(1 + h)$.

Since all components of $\Lambda = F(I + F)^{-1}$ are less than one, we may apply the contour formula (2). Since ${}_1F_0(a; x) = (1 - x)^{-a}$, we obtain

$$L(\tau; \Lambda) = \frac{n-p}{2} B\left(\frac{p}{2}, \frac{n-p}{2}\right) \frac{(1-\tau)^{n_1/2}}{\tau^{p/2-1}} \frac{1}{2\pi i} \int_{\Gamma} \frac{(1-\tau s)^{-(n-p+2)/2}}{\prod_j (s - \lambda_j)^{1/2}} ds.$$

where $B(a, \beta)$ is the usual beta function. This is a form suitable for asymptotic approximation, the details of which will be reported elsewhere.

Remark—A useful check on this last formula is obtained by letting the error degrees of freedom $n_2 \rightarrow \infty$ while keeping p and n_1 fixed. This limit corresponds to the case where Σ_2 is known, say $\Sigma_2 = I$ for convenience here, and we consider the single matrix rank one model $\Sigma_1 = I + \psi h \psi'$ and test the hypothesis that $h = 0$. To compare with the formula of Onatski et al. (2013, Lemma 3), let (μ_j) be the eigenvalues of $n_1^{-1} A_1 (n_2^{-1} A_2)^{-1}$, so that $\lambda_j = f_j / (1 + f_j) = n_1 \mu_j / (n_2 + n_1 \mu_j)$. With the change of variables $s = n_1 z / n_2$, the previous display converges to

$$L(\tau; \mu) = \Gamma\left(\frac{p}{2}\right) \left(\frac{2}{n_1}\right)^{p/2-1} \frac{(1-\tau)^{n_1/2}}{\tau^{p/2-1}} \frac{1}{2\pi i} \int_{\Gamma} e^{n_1 \tau z / 2} \prod_j (z - \mu_j)^{-1/2} dz,$$

which is the cited expression for the ${}_0F_0$ likelihood ratio.

4. Real Case, integer r

Here we prove Proposition 1, for *real* matrices with integer dimension r , *not necessarily even*. A similar result, with proof extending that of Onatski et al. (2013, Lemma 2) has been obtained by Alexei Onatski (personal communication) and will appear elsewhere.

Our goal is to prove the validity of the following expression for $0 \leq p \leq q + 1$:

$${}_pF_q^2(a, b; X, Y) = \frac{\Gamma(m+1)}{x^m \rho'_m} \frac{1}{2\pi i} \int_{\Gamma} {}_pF_q(a-m, b-m; xs) \Delta_y(s) ds \quad (7)$$

where we have defined $\Delta_y(s) = \prod_{j=1}^r (s - y_j)^{-\frac{1}{2}}$. The contour Γ starts from $-\infty$ and encircles $y_1, y_2, \dots, y_r$ in the positive direction (i.e., counter-clockwise) and goes back to $-\infty$.

In what follows, we provide an inductive proof for the above claim. First we establish the initial cases: ${}_0F_q$ for $q \geq 0$ and, separately, ${}_1F_0$. The inductive step establishes truth

for ${}_{p+1}F_{q+1}$ given truth for ${}_pF_q$. Also, it is worth mentioning that we assume all powers have their principal values and all angles in the range $[-\pi, \pi)$.

The following alternative representation of the hypergeometric function of two matrix arguments is useful in the sequel. Let $\mathcal{O}(r)$ be the orthogonal group and let (dQ) be the invariant measure on $\mathcal{O}(r)$ normalized to make the total measure unity. Then, following James (1964), we can write

$${}_pF_q^2(a, b; X, Y) = \int_{\mathcal{O}(r)} {}_pF_q^2(a, b; XQ'YQ) (dQ). \quad (8)$$

Moreover, let us assume, without loss of generality, that $Y = \text{diag}(y_1, y_2, \dots, y_r)$. Since X is rank-1, we can further simplify (8) to yield

$${}_pF_q^2(a, b; X, Y) = \int_{\mathcal{S}(r)} {}_pF_q(a, b; xq_r'Yq_r) (dq_r). \quad (9)$$

where $\mathcal{S}(r)$ is the $r - 1$ dimensional sphere embedded in $\mathbb{R}^r$, q_r is the first column of Q and (dq_r) is the invariant measure on $\mathcal{S}(r)$ normalized such that the total measure is one.

4.1. Initial cases

We first show that the statement (7) is true for ${}_0F_q$. With the standard notation

${}_0F_q(b; z) = {}_0F_q(b; z) / \prod_{j=1}^q \Gamma(b_j)$, this is equivalent to showing that, for $q \geq 0$,

$${}_0F_q^2(b; X, Y) = \frac{\Gamma(m+1)}{x^m} \frac{1}{2\pi i} \int_{\Gamma} {}_0F_q(b-m; xs) \Delta_y(s) ds. \quad (10)$$

Our tool is a contour representation of (Erdélyi, 1937, eq. (7.4)):

$${}_0F_q(b; z) = \frac{1}{(2\pi i)^q} \int_{-\infty}^{(0+)} \dots \int_{-\infty}^{(0+)} e^{\left(\sum_{j=1}^q w_j + \frac{z}{\prod_{j=1}^q w_j}\right)} \prod_{j=1}^q \frac{dw_j}{w_j^{b_j}} \quad (11)$$

where each contour starts from $-\infty$ and encircles the origin in the positive sense and goes back to $-\infty$. We use multi-index notation $w^b = \prod w_j^{b_j}$, $w = \prod w_j$ and $dw = \prod dw_j$.

We use the spherical average (9), then Erdélyi's representation, and change order of integration, to get

$${}_0F_q^2(b; X, Y) = \int_{\mathcal{S}(r)} {}_0F_q(b; xq_r'Yq_r) (dq_r) \quad (12)$$

$$= \frac{1}{(2\pi i)^q} \int_{-\infty}^{(0+)} \dots \int_{-\infty}^{(0+)} e^{\sum_{j=1}^q w_j} \int_{\mathcal{S}(r)} e^{\frac{x}{w} q_r'Yq_r} (dq_r) \frac{dw}{w^b}. \quad (13)$$

A change of variable in (Onatski et al., 2013, Lemma 2) shows that for $x, w > 0$,

$$\int_{\mathcal{S}(r)} e^{\frac{x}{w} q_r' Y q_r} (dq_r) = \frac{\Gamma(r/2)}{2\pi i} \left(\frac{w}{x}\right)^{r/2-1} \int_{\Gamma} e^{\frac{x}{w} s} \Delta_y(s) ds, \quad (14)$$

and the equality extends by analyticity to all nonzero $w \in \mathbb{C}$. Inserting this integral in (12) and noting that $\frac{r}{2} = m+1$ we obtain

$${}_0F_q^2(b; X, Y) = \frac{\Gamma(m+1)}{x^m (2\pi i)^{q+1}} \int_{-\infty}^{(0+)} \dots \int_{-\infty}^{(0+)} e^{\sum_{j=1}^q w_j} \int_{\Gamma} e^{\frac{x}{w} s} \Delta_y(s) ds \frac{dw}{w^{b-m}}$$

Finally, we change the order of integration and again make use of (11) to arrive at the desired equality (10). This proves the validity of the statement (7) for $p = 0$.

Now we show that, for $x \max\{y_j\} < 1$,

$${}_1F_0^2(a; X, Y) = \frac{\Gamma(m+1)}{x^m (a-m)_m} \frac{1}{2\pi i} \int_{\Gamma} {}_1F_0(a-m; xs) \Delta_y(s) ds. \quad (15)$$

We use identity (9), the special form ${}_1F_0(a; z) = (1-z)^{-a}$ and the relation

$$\frac{1}{s^a} = \frac{1}{\Gamma(a)} \int_0^\infty t^{a-1} e^{-st} dt, \quad \Re(s) > 0, \Re(a) > 0 \quad (16)$$

to obtain, after observing that $x \max\{y_j\} < 1$ implies $x q_r' Y q_r < 1$,

$${}_1F_0^2(a; X, Y) = \int_{\mathcal{S}(r)} \frac{1}{(1-x q_r' Y q_r)^a} (dq_r) = \int_0^\infty t^{a-1} e^{-t} \int_{\mathcal{S}(r)} e^{t x q_r' Y q_r} (dq_r) dt. \quad (17)$$

Now substitute the contour identity (14) with $t = 1/w$, and with the contour chosen to encircle $\{y_j\}$ and to lie to the left of $1/x$. We obtain

$$\begin{aligned} {}_1F_0^2(a; X, Y) &= \frac{\Gamma(r/2)}{\Gamma(a) x^{-\frac{r}{2}-1}} \frac{1}{2\pi i} \int_0^\infty \int_{\Gamma} t^{a-\frac{r}{2}} e^{-t(1-xs)} \Delta_y(s) ds dt \\ &= \frac{\Gamma(r/2) \Gamma(a+1-\frac{r}{2})}{\Gamma(a) x^{\frac{r}{2}-1}} \frac{1}{2\pi i} \int_{\Gamma} (1-xs)^{r/2-a-1} \Delta_y(s) ds, \end{aligned}$$

valid for $\Re(a) > \frac{r}{2}-1$, after changing order of integration and using (16) and the fact that $\Re(s) < 1/x$. Recalling that $m = r/2 - 1$ and ${}_1F_0(a; z) = (1-z)^{-a}$, the final form reduces to the right hand side of (15), under the condition $\Re(a) > m$. However, the both sides of the above equality, which we have established only in the domain $\Re(a) > m$ of complex plane, are analytic functions. Therefore, the equality must hold in the whole region of the analyticity of a . This establishes the claim (15).

4.2. Inductive step

First, some notation. We write $a_+ = (a, a_1, \dots, a_p)$ and $b_+ = (\beta, b_1, \dots, b_q)$ for the augmentations of a and b , and abbreviate ${}_{p+1}F_{q+1}$ by ${}_{p+}F_{q+}$. Thus, the induction step amounts to establishing the validity of the following statement, given the statement (7) is true

$${}_{p+}F_{q+}^2(a_+, b_+; X, Y) = \frac{\Gamma(m+1)}{x^m \rho_{m+}'} \frac{1}{2\pi i} \int \Gamma {}_{p+}F_{q+}(a_+ - m, b_+ - m; xs) \Delta_y(s) ds \quad (18)$$

where

$$\rho_{m+}' = \rho_m' \frac{\Gamma(\alpha)\Gamma(\beta-m)}{\Gamma(\alpha-m)\Gamma(\beta)}. \quad (19)$$

We use a reparametrized version of the beta density

$$\phi(t; \alpha, \beta) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} t^{\alpha-1} (1-t)^{\beta-\alpha-1},$$

and the integral representation of the generalized hypergeometric function (Erdélyi, 1937, eq. (3.2))

$${}_{p+}F_{q+}(a_+, b_+; x) = \int_0^1 \phi(t; \alpha, \beta) {}_pF_q(a, b; xt) dt \quad (20)$$

where $\Re(\beta) > \Re(\alpha) > 0$, along with (9), in order to write the left side of (18) as

$$\begin{aligned} {}_{p+}F_{q+}^2(a_+, b_+; X, Y) &= \int_{\mathcal{S}(r)} {}_{p+}F_{q+}(a_+, b_+; x q_r' Y q_r) (dq_r) \\ &= \int_{\mathcal{S}(r)} \int_0^1 \phi(t; \alpha, \beta) {}_pF_q(a, b; xt q_r' Y q_r) dt (dq_r) \\ &= \int_0^1 \phi(t; \alpha, \beta) \int_{\mathcal{S}(r)} {}_pF_q(a, b; xt q_r' Y q_r) (dq_r) dt \\ &= \int_0^1 \phi(t; \alpha, \beta) {}_pF_q^2(a, b; tX, Y) dt, \end{aligned}$$

where we have changed the order of integration and again used (9). The final expression can be rewritten with the help of our induction hypothesis (7) as

$$\frac{\Gamma(m+1)}{x^m \rho_m'} \frac{1}{2\pi i} \int_0^1 t^{-m} \phi(t; \alpha, \beta) \int \Gamma {}_pF_q(a-m, b-m; xts) \Delta_y(s) ds dt. \quad (21)$$

Now use the identity

$$t^{-m}\phi(t;\alpha,\beta)=\phi(t;\alpha-m,\beta-m)\frac{\Gamma(\beta)\Gamma(\alpha-m)}{\Gamma(\beta-m)\Gamma(\alpha)}$$

and note from (19) that the ratio of Gamma functions equals ρ'_{m+}/ρ'_{m+} . Inserting this into (21) and changing the order of integration, we obtain

$$\frac{\Gamma(m+1)}{x^m \rho'_{m+}} \frac{1}{2\pi i} \int_{\Gamma} \Delta_y(s) \int_0^1 \phi(t;\alpha-m,\beta-m)_p F_q(a-m,b-m;xts) dt ds.$$

Now again use (20), along with the restriction $\Re(\alpha) > m$, to yield (18) in the domain $\Re(\beta) > \Re(\alpha) > m$ of $\mathbb{C}$. Since both sides of equality (18) are analytic functions, the equality must hold in the whole region of the analyticity of α and β . This completes the induction step.

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