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Design of Multidimensional Shinnar-Le Roux RF Pulses

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Abstract

Purpose—To generalize the conventional Shinnar-Le Roux (SLR) method for the design of multidimensional RF pulses.

Methods—Using echo-planar gradients, the multidimensional RF pulse design problem was converted into a series of 1D polynomial design problems. Each of the 1D polynomial design problems was solved efficiently. P_0 inhomogeneity compensation and design of spatial-spectral pulses were also considered.

Results—The proposed method was used to design 2D excitation and refocusing pulses. The results were validated through Bloch equation simulation and experiments on a 3.0 T scanner. Large-tip-angle, equiripple-error, multidimensional excitation was achieved with ripple levels closely matching the design specifications.

Conclusion—The conventional SLR method can be extended to design multidimensional RF pulses. The proposed method achieves almost equiripple excitation errors, allows easy control of the tradeoff among design parameters, and is computationally efficient.

Keywords

RF pulse design; Multidimensional RF pulse; Shinnar-Le Roux method

Introduction

Spatially selective RF pulses have found widespread use in MR imaging/spectroscopy experiments. One-dimensional RF pulses are widely used for slice selection (1–4). Multidimensional RF pulses (5, 6) provide more flexible spatial selectivity, and have a range of applications, including localized spectroscopic imaging (7, 8), simultaneous spatial-spectral selective excitation (9, 10), motion tracking (11, 12), and reduced field-of-view (FOV) imaging of region-of-interest (ROI) (12–13).

The one-dimensional RF pulse design problem has been well studied for several decades (19–25). Among the existing methods, the Shinnar-Le Roux (SLR) method (25–29) is often the “method-of-choice”, because it effectively handles the nonlinearity of the Bloch equation, allows easy control of tradeoff among design parameters, and has a fast algorithm.

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Multidimensional RF pulse design, however, is still a challenging problem, largely due to the difficulties in effectively handling the nonlinearity of the Bloch equation. Methods based on the linear approximate solutions of the Bloch equation have been successfully used to design small-tip-angle multidimensional excitation pulses (6, 30, 31). However, these pulses are not accurate for large-tip-angle excitations. Optimal control methods have been proposed for designing large-tip-angle multidimensional RF pulses, but have limited practical utility because of computational burden (32, 33). Therefore, it is desirable to generalize the SLR method to design multidimensional RF pulses.

Generalizing the SLR method to the multidimensional case requires finding a unique invertible mapping between a multidimensional RF pulse and the coefficients of multidimensional polynomials. Without such a mapping, it would be difficult to efficiently solve the multidimensional RF pulse design problem via polynomial design problems like the SLR method (34). Alternatively, a separable design method has been proposed for multidimensional RF pulse design (35). This method, using an echo-planar gradient, decomposes a multidimensional RF pulse into a series of one-dimensional inherently refocused pulses. The tip angles and waveforms of the subpulses are then designed separately using the conventional SLR method. However, the rotation generated by each subpulse is assumed to be about an axis in the transverse plane. The assumption holds when the tip-angle of each subpulse is small, otherwise distortions in the resulting excitation patterns may occur, especially for refocusing and inversion pulses.

This paper presents a new approach to generalizing the SLR method to multidimensional RF pulse design. Specifically, the design problem is converted to a series of 1D RF pulse design problems. Each of the 1D RF pulse design problems is then mapped into a 1D polynomial design problem, which is formulated as a convex optimization problem and thus solved efficiently. The excitation pattern achieved by the proposed method shows almost equal-ripple errors. The tradeoff among ripple level, transition bandwidth, and pulse length can be made directly and explicitly. The proposed method is further generalized to compensate the B_0 inhomogeneity effect, an important issue for a multidimensional RF pulse because of its typically long duration. Based on a similar treatment, the proposed method is extended to design spatial-spectral RF pulses as well. The MATLAB code for the proposed method is made available online at <http://mri.beckman.uiuc.edu/software.html>.

Methods

For simplicity, the proposed method is described for designing a 2D spatially selective RF pulse. Assume that the B_1 field is uniform and that an echo-planar gradient is used as shown in Fig. 1a. The pulse in Fig. 1a can be decomposed into a series of segments in Fig. 1b. Each segment consists of two parts: a subpulse with an inherently refocused z -gradient that generates rotation by a tip-angle varying along the z -axis, and an x -gradient blip that generates a precession about the main magnetic field by an angle varying along the x -axis.

The rest of this section is organized as follows. In Subsection A, we show that for a given τ , the rotation generated by each subpulse, which is represented by Cayley-Klein parameters $C_{d,n_x}(z)$ (α parameter, n_x is the index of subpulses) and $S_{d,n_x}(z)$ (β parameter), can be

bijectionally mapped to a 1D polynomial. We further show that the phase of $C_{d,n_x}(z)$ is a free parameter in the inverse transform and present a scheme to make the phase of $C_{d,n_x}(z)$ feasible for a minimum power subpulse. For each z , a 1D polynomial is then designed to approximate the desired excitation pattern along the x -axis by solving a minimax optimization problem. The desired 1D rotation profile of each subpulse ($C_{d,n_x}(z)$ and $S_{d,n_x}(z)$) is finally determined by applying the inverse transform to the designed polynomial at each z . In Subsection B, the conventional SLR method is modified to design the waveform of each subpulse to approximate the desired 1D rotation profile of each subpulse determined in Subsection A. We show that the phase of $C_{d,n_x}(z)$ specified in Subsection A is feasible for a minimum power subpulse. In Subsection C, we summarize the design procedure of the proposed method. In Subsection D, we analyze how the approximation errors in solving the nested 1D polynomial design problems (Subsection A and B) affect the approximation errors of the resulting 2D excitation pattern. In Subsection E, we generalize the proposed method to correct the B_0 inhomogeneity effects on the resulting excitation pattern. Following a similar treatment, we generalize the proposed method to design spatial-spectral RF pulses in Subsection F.

A. Determination of the desired rotation generated by each subpulse

The rotation generated by each segment in Fig. 1b can be written using the Cayley-Klein parameters (25) as

$$\mathbf{Q}_{n_x} = \begin{bmatrix} C_{d,n_x}(z) & -S_{d,n_x}^*(z) \\ S_{d,n_x}(z) & C_{d,n_x}^*(z) \end{bmatrix} \begin{bmatrix} \omega_x^{1/2} & 0 \\ 0 & \omega_x^{-1/2} \end{bmatrix}, \quad (1)$$

where $\mathbf{Q}_{n_x}$ represents the rotation matrix of the n_x -th segment. In Eq. [1], the first matrix on the right-hand side represents the desired rotation by the n_x -th subpulse, where $C_{d,n_x}(z)$ and $S_{d,n_x}(z)$ represent the corresponding Cayley-Klein parameters as a function of z (assuming only the z -gradient is turned on during each subpulse). The second matrix on the right-hand side represents the precession due to the x -gradient blip, where ω_x is a function of x :

$$\omega_x = e^{i\gamma \int_0^\tau G_x(s) ds}, \quad (2)$$

and τ is the duration of the x -gradient blip. Note that the Cayley-Klein parameters are defined following the right-handed rotation convention (i.e., a counterclockwise rotation has a positive rotation angle).

The total rotation generated by the first n_x segments is given by

$$\begin{bmatrix} \alpha_{n_x} & -\beta_{n_x}^* \\ \beta_{n_x} & \alpha_{n_x}^* \end{bmatrix} = \mathbf{Q}_{n_x} \mathbf{Q}_{n_x-1} \cdots \mathbf{Q}_1. \quad (3)$$

Equations [1] and [3] describe a rotation synthesis procedure that is similar to the conventional SLR method. Moreover, the synthesis is accurate even when the tip-angle of

each subpulse is large, because the coefficient $C_{d,n_x}(z)$ is supposed to be complex in the following transform (contrarily to the hard pulse assumption in the SLR method). In the following, an invertible transform is derived to uniquely map the rotation generated by each subpulse to the coefficients of two complex polynomials.

The forward transform is given by

$$\begin{bmatrix} A_{n_x} \\ B_{n_x} \end{bmatrix} = \begin{bmatrix} C_{d,n_x}(z) & -S_{d,n_x}^*(z) \\ S_{d,n_x}(z) & C_{d,n_x}^*(z) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \omega_x^{-1} \end{bmatrix} \begin{bmatrix} A_{n_x-1} \\ B_{n_x-1} \end{bmatrix}, \quad (4)$$

where

$$A_{n_x} = \omega_x^{-(n_x-1)/2} a_{n_x}(\omega_x), \quad B_{n_x} = \omega_x^{-(n_x-1)/2} b_{n_x}(\omega_x). \quad (5)$$

A_{n_x} and B_{n_x} can be further written as:

$$A_{n_x} = \sum_{n=0}^{n_x-1} a_n^{(n_x)}(\omega_x) \omega_x^{-n}, \quad (6)$$

$$B_{n_x} = \sum_{n=0}^{n_x-1} b_n^{(n_x)}(\omega_x) \omega_x^{-n}. \quad (7)$$

For a given z , A_{n_x} and B_{n_x} become polynomials of ω_x . We therefore simply refer to A_{n_x} and B_{n_x} as polynomials (in ω_x) assuming z is given, unless otherwise specified.

In order to derive the inverse transform, consider two polynomials A_{n_x} and B_{n_x} in Eqs. [6] and [7], which represent a rotation at any spatial location:

$$|A_{n_x}|^2 + |B_{n_x}|^2 \equiv 1. \quad (8)$$

According to Eq. [4], the inverse recursion of the polynomials is given by

$$\begin{bmatrix} A_{n_x-1} \\ B_{n_x-1} \end{bmatrix} = \begin{bmatrix} C_{d,n_x}^*(z) & S_{d,n_x}^*(z) \\ \omega_x^{-1} S_{d,n_x}(z) & \omega_x C_{d,n_x}(z) \end{bmatrix} \begin{bmatrix} A_{n_x} \\ B_{n_x} \end{bmatrix}. \quad (9)$$

Following a derivation similar to the SLR method, the degree of A_{n_x-1} and B_{n_x-1} should be n_x-2 , yielding

$$\frac{c_{d,n_x}^{(n_x)}(z)}{c_{d,n_x}(z)} = \frac{b_0^{(n_x)}(z)}{a_0^{(n_x)}(z)}. \quad (10)$$

$C_{d,n_x}(z)$ and $S_{d,n_x}(z)$ should also satisfy the following magnitude constraints to represent a rotation:

$$|C_{d,n_x}(z)|^2 + |S_{d,n_x}(z)|^2 \equiv 1. \quad (11)$$

Note that Eqs. [10] and [11] cannot uniquely determine $C_{d,n_x}(z)$ and $S_{d,n_x}(z)$, as the phase of $C_{d,n_x}(z)$ can be freely chosen. We choose to make the phase of $C_{d,n_x}(z)$ depend on $S_{d,n_x}(z)$. More specifically, while satisfying Eqs. [10] and [11], we choose to determine $C_{d,n_x}(z)$ and $S_{d,n_x}(z)$ as the following:

$$c_{d,n_x}(z) = \sqrt{1 - |S_{d,n_x}(z)|^2} e^{i\mathcal{H}\{\log\sqrt{1 - |S_{d,n_x}(z)|^2}\}}, \quad (12)$$

$$s_{d,n_x}(z) = \mathcal{H}\{\log\sqrt{1 - |S_{d,n_x}(z)|^2}\} + z \left[\frac{b_0^{(n_x)}(z)}{a_0^{(n_x)}(z)} \right], \quad (13)$$

$$|S_{d,n_x}(z)| = \sin \left(\arctan \frac{|b_0^{(n_x)}(z)|}{|a_0^{(n_x)}(z)|} \right), \quad (14)$$

where $\mathcal{H}\{\cdot\}$ denotes the Hilbert transform operator of a one-dimensional function. As will be discussed in Section 3, this specific choice of the phase of $C_{d,n_x}(z)$ will result in RF subpulses with minimum RF power. Equations [9], [12] to [14] define the inverse transform.

Supposing the number of subpulses is N_x , the problem of determining the desired rotation generated by each subpulse is then equivalent to a problem of designing the two polynomials A_{N_x} and B_{N_x} . As in the SLR method, only the polynomial B_{N_x} needs to be designed, and the polynomial A_{N_x} is determined using spectral factorization. More specifically, for a given z the polynomial A_{N_x} is determined by the polynomial B_{N_x} by

$$A_{N_x}(x) = \sqrt{1 - |B_{N_x}(x)|^2} e^{i\mathcal{H}\{\log\sqrt{1 - |B_{N_x}(x)|^2}\}}. \quad (15)$$

Suppose the desired profile of $B_{N_x}(z, x)$ is given on a discrete grid, denoted as $B(z_l, x_m)$. At each z_l , the coefficients of B_{N_x} in Eq. [7] are determined by solving the following optimization problem:

$$\arg\min_{b_n^{(N_x)}(z_l, x_m)} \max_{(z_l, x_m)} |W(z_l, x_m)[B_{N_x}(z_l, x_m) - d(z_l, x_m)]|, \quad (16)$$

where $W(z_l, x_m)$ is a nonnegative weight and the maximum approximate error is minimized. The optimization problem in Eq. [16] is convex, and can be solved efficiently with global optimality (25).

By solving the polynomial design problem in Eq. [16] at each z_l , the polynomial B_{N_x} is designed first, the polynomial A_{N_x} is then calculated using Eq. [15], and finally the desired rotation of each subpulse is determined using the inverse transform in Eqs. [9], [12] to [14].

B. Subpulse design

Suppose the desired rotation generated by the n_x -th subpulse (C_{d,n_x} and S_{d,n_x} on a given grid) has been determined by the procedure in Section A. In the following, we discuss how to design each subpulse to approximate the desired rotation.

Suppose each subpulse is approximated by N_z hard pulses and the duration of each hard pulse is Δt . Performing the forward SLR transform, the corresponding Cayley-Klein parameters C_{n_x} and S_{n_x} of the n_x -th subpulse can be written as two complex polynomials (see Appendix A for detailed derivations):

$$C_{n_x} = \sum_{n_z=0}^{N_z-1} a_{n_z, n_x} \omega_{z, n_x}^{-n_z}, \quad (17)$$

$$S_{n_x} = \omega_{z, n_x}^{N_z/2} \sum_{n_z=0}^{N_z-1} b_{n_z, n_x} \omega_{z, n_x}^{-n_z}, \quad (18)$$

where ω_{z, n_x} is a function of z , given by

$$\omega_{z, n_x} = \begin{cases} e^{i\gamma G_z z \Delta t}, & n_x \text{ is odd,} \\ e^{-i\gamma G_z z \Delta t}, & n_x \text{ is even.} \end{cases} \quad (19)$$

The polynomials C_{n_x} and S_{n_x} need to be designed to approximate the desired profile C_{d,n_x} and S_{d,n_x} . As in the SLR method, we intend to only design the polynomial S_{n_x} , and determine the polynomial C_{n_x} by spectral factorization, which leads to a pulse of minimum RF power (25):

$$C_{n_x}(z) = \sqrt{1 - |S_{n_x}(z)|^2} e^{i\mathcal{H}\{\log \sqrt{1 - |S_{n_x}(z)|^2}\}}. \quad (20)$$

However, by doing that, the polynomials C_{n_x} and S_{n_x} cannot be designed to approximate an arbitrary pair of C_{d,n_x} and S_{d,n_x} since the phase of C_{n_x} is not independently controllable. Fortunately, this does not impose any limitation on the proposed method, because the phase of C_{d,n_x} is a free parameter, as discussed in Section A. According to Eqs. [12] to [14], if S_{n_x} is designed to approximate S_{d,n_x} , the polynomial C_{n_x} determined by Eq. [20] will automatically approximate C_{d,n_x} .

For each n_x , the coefficients of the polynomial S_{n_x} in Eq. [18] are determined by solving the following convex optimization problem:

$$\arg \min_{\mathbf{b}_{n_x, n_x}} \max_{z_l} W(z_l, n_x) |S_{n_x}(z_l) - S_{d, n_x}(z_l)|, \quad (21)$$

where $W(z_l, n_x)$ is a nonnegative weight and the maximum approximate error is minimized.

With the polynomial S_{n_x} designed by solving the optimization problem in Eq. [21], the polynomial C_{n_x} is calculated using Eq. [20]. The n_x -th RF subpulse is then calculated using the inverse SLR transform.

C. Design procedure

The procedure for designing a 2D RF pulse or N_x subpulses is summarized as follows.

Step 1: Suppose the desired profile of $B_{N_x}(z, x)$ is given on a discrete grid, denoted as $d(z_l, x_m)$. At each z_l , design a 1D polynomial $B_{N_x}(z_l, x)$ that best approximates the desired profile $d(z_l, x_m)$ by solving the optimization problem in Eq. [16]. The polynomial $A_{N_x}(z_l, x)$ is then determined using Eq. [15].

Step 2: For each n_x , determine the desired rotation generated by each subpulse (i.e., $C_{d, n_x}(z_l)$ and $S_{d, n_x}(z_l)$) using Eqs. [12] to [14], and then reduce the polynomial order by 1 using Eq. [9].

Step 3: For each n_x , design the polynomial $S_{n_x}(z)$ that best approximates the desired profile $S_{d, n_x}(z_l)$ by solving the optimization problem in Eq. [21]. The polynomial $C_{n_x}(z)$ is then determined using Eq. [20]. The waveform of the n_x -th RF subpulse is finally determined using the inverse SLR transform.

D. Specification of ripple levels

In the following, we discuss how to specify the ripple levels in the polynomial design ($B_{N_x}(z, x)$ and $S_{n_x}(z)$) to achieve desired effective ripple levels of the resulting excitation pattern, i.e., the ripple levels of the transverse or longitudinal component of the magnetization vector. We first investigate the relation between the ripples of the polynomials ($B_{N_x}(z, x)$ and $S_{n_x}(z)$) and the ripples of the resulting excitation pattern.

For simplicity, suppose the desired excitation pattern is a rectangular RCU. The field-of-excitation (FOX) can then be divided into four regions as shown in Fig. 2, where region 1 is the passband along both the x - and z -axis, region 2 is the stopband along the x -axis, region 3 is the stopband along the z -axis, and region 4 is the stopband along both the x - and z -axis. Suppose the desired tip-angle is θ , and the polynomial $B_{N_x}(z, x)$ approximates the desired

excitation pattern with passband ripple $\delta_{p,x}$ and stopband ripple $\delta_{s,x}$. The polynomial $B_{N_x}(z, x)$ can be written as

$$B_{N_x}(z, x) = \begin{cases} (1 + f_{p,x}(x))\sin\frac{\theta}{2}, & \text{region1,} \\ f_{s,x}(x)\sin\frac{\theta}{2}, & \text{region2,} \\ 0, & \text{regions3and4,} \end{cases} \quad (22)$$

where $f_{p,x}(x)$ and $f_{s,x}(x)$ denote the approximation error and are bounded by $\delta_{p,x}$ and $\delta_{s,x}$ respectively.

Suppose that the polynomial $S_{n_x}(z)$ approximates the desired rotation with passband ripple $\delta_{p,z}$ and stopband ripple $\delta_{s,z}$. Assuming the ripple is small, $S_{n_x}(z)$ and $C_{n_x}(z)$ can be approximated by

$$S_{n_x}(z) \approx \begin{cases} S_{d,n_x}(1 + f_{p,z}(z)), & \text{regions1and2,} \\ S_{c,n_x}f_{s,z}(z), & \text{regions3and4,} \end{cases} \quad (23)$$

$$C_{n_x}(z) \approx \begin{cases} C_{d,n_x}, & \text{regions1and2,} \\ 1, & \text{regions3and4,} \end{cases} \quad (24)$$

where $f_{p,z}(z)$ and $f_{s,z}(z)$ denote the approximation error and are bounded by $\delta_{p,z}$ and $\delta_{s,z}$ respectively.

Combining Eqs. [4], [22] to [24], the polynomial achieved by the designed RF pulse can be approximated by

$$B_{N_x}(z, x) \approx \begin{cases} (1 + f_{p,x}(x))(1 + f_{p,z}(z))\sin\frac{\theta}{2}, & \text{region1,} \\ f_{s,x}(x)(1 + f_{p,z}(z))\sin\frac{\theta}{2}, & \text{region2,} \\ f_{s,z}(z)(1 + f_{p,x}(x))\sin\frac{\theta}{2}, & \text{region3,} \\ f_{s,z}(z)f_{s,x}(x)\sin\frac{\theta}{2}, & \text{region4.} \end{cases} \quad (25)$$

Note that to obtain Eq. [25] one needs to assume the tip-angle of each subpulse to be small. However, this assumption is only used here for ripple analysis and does not limit the proposed method to the design of small-tip-angle subpulses. The ripple estimation in Eq. [25] will be less accurate when the tip angles of subpulses are large, but will be still useful as shown in the design examples in the Result section.

According to Eq. [25], the ripple of the resulting excitation pattern, i.e., the ripple of the Cayley-Klein parameter (β), can be estimated as

$$\delta_{p,z} = \begin{cases} \delta_{p,x} + \delta_{p,z}, & \text{region1,} \\ \delta_{s,x}, & \text{region2,} \\ \delta_{s,z}, & \text{region3,} \\ 0, & \text{region4,} \end{cases} \quad (26)$$

where the high-order terms of $\delta_{p,x}$, $\delta_{s,x}$, $\delta_{p,z}$, and $\delta_{s,z}$ are ignored.

Given the desired effective ripple levels of an excitation pattern, the desired ripple levels of the Cayley-Klein parameter can be determined based on the RF pulse type as in the SLR method (25). The desired ripple levels of the passband and stopband of the polynomial $B_{N_x}(z, x)$ and $S_{n_x}(z)$ can then be determined using Eq. [26]. The transition bandwidth can be determined in a way similar to the SLR method as well, using the empirical relationship among passband and stopband ripple, transition bandwidth, and filter order in digital filter design (25). After specifying the ripple levels and transition bandwidth, the weighting matrix in Eqs. [16] and [21] can be easily determined as in digital filter design (37).

E. B_0 inhomogeneity effect analysis and correction

Assuming that the B_0 inhomogeneity is relatively small compared to the bandwidth of the subpulse, the rotation matrix of the n_x -th segment in the presence of B_0 inhomogeneity can be approximated by (for detailed derivations, please refer to Appendix B)

$$\mathbf{Q}_{n_x}(z, x) \approx \begin{bmatrix} e^{i\psi_f/2} & 0 \\ 0 & e^{-i\psi_f/2} \end{bmatrix} \mathbf{Q}_{n_x}(z, x') \begin{bmatrix} e^{-i\psi_f/2} & 0 \\ 0 & e^{i\psi_f/2} \end{bmatrix}. \quad (27)$$

In Eq. [27], $\mathbf{Q}_{n_x}$ is the rotation matrix in the absence of B_0 inhomogeneity defined in Eq. [1], and ω_f and x' are given by

$$\psi_f = \pi N_z \Delta t \Delta f(z, x) \quad (28)$$

$$x' = x + \frac{2\pi(N_z \Delta t + \tau) \Delta f(z, x)}{\gamma \int_0^1 G_x(s) ds}, \quad (29)$$

where $\Delta f(z, x)$ denotes the B_0 field inhomogeneity in Hertz and $N_z \Delta t + \tau$ is the length of each segment assuming the transition time of z-gradient is the same as the duration of x-gradient blip.

Cascading the rotation generated by each segment as in Eq. [3], the total rotation generated by the first n_x segments in the presence of the B_0 inhomogeneity is given by

$$\begin{bmatrix} \tilde{\alpha}_{n_x}(z, x) & -\beta_{n_x}^*(z, x) \\ \beta_{n_x}(z, x) & \tilde{\alpha}_{n_x}^*(z, x) \end{bmatrix} = \begin{bmatrix} e^{i\psi_f/2} & 0 \\ 0 & e^{-i\psi_f/2} \end{bmatrix} \begin{bmatrix} \alpha_{n_x}(z, x') & -\beta_{n_x}^*(z, x') \\ \beta_{n_x}(z, x') & \alpha_{n_x}^*(z, x') \end{bmatrix} \begin{bmatrix} e^{-i\psi_f/2} & 0 \\ 0 & e^{i\psi_f/2} \end{bmatrix}, \quad (30)$$

where α_{n_x} and β_{n_x} are the Cayley-Klein parameters in the absence of B_0 inhomogeneity defined in Eq. [31]. Equation [30] shows that the major effect of the B_0 inhomogeneity on the resulting excitation pattern is a shift along the x -axis.

Assuming the B_0 inhomogeneity map $\Delta f(z, x)$ is known, one can design an RF pulse using the proposed method to excite a pre-distorted excitation profile compensating for the distortion caused by B_0 inhomogeneity. For a given desired excitation profile $d(z, x)$, the pre-distorted excitation profile $\tilde{d}(z, x)$ is

$$\tilde{d}(z, x) = d\left(z, x - \frac{2\pi(N_z \Delta t + \tau)\Delta f(z, x)}{\gamma \int_0^\tau G_x(s) ds}\right). \quad (31)$$

F. Generalization to spatial-spectral RF pulse design

The proposed method can be extended to design spatial-spectral RF pulses as well. For simplicity, consider a type I spatial-spectral RF pulses as shown in Fig. 3a (38). The pulse can also be decomposed into a series of segments as shown in Fig. 3b. Following a derivation similar to Section E, the rotation matrix of the n_f -th segment can be written as

$$\mathbf{C}_{n_f}(\cdot, \Delta f) = \begin{bmatrix} e^{i\psi_f/2} & 0 \\ 0 & e^{-i\psi_f/2} \end{bmatrix} \begin{bmatrix} C_{n_x}(z') - S_{n_x}^*(z') \\ S_{n_x}(z') - C_{n_x}^*(z') \end{bmatrix} \begin{bmatrix} \omega_f^{1/2} & 0 \\ 0 & \omega_f^{-1/2} \end{bmatrix} \begin{bmatrix} e^{-i\psi_f/2} & 0 \\ 0 & e^{i\psi_f/2} \end{bmatrix}, \quad (32)$$

where

$$z' = z + \frac{2\pi\Delta f}{\gamma G_x}, \quad (33)$$

$$\omega_f = e^{i2\pi T \Delta f}, \quad (34)$$

and T is the length of each segment as labeled in Fig. 3.

The first and last matrix in Eq. [32] are not critical as they will be canceled by neighboring segments. The similarity between Eq. [32] and Eq. [1] indicates that the proposed method can be easily extended to design spatial-spectral RF pulses.

Results

In this section, we present several representative design examples, including a 2D linear-phase 90° excitation pulse, a 2D linear-phase 180° refocusing pulse, a 2D linear-phase 180° spatial-spectral refocusing pulse, a 2D linear-phase 90° excitation pulse with B_0 inhomogeneity correction, and a 2D maximum-phase 90° saturation pulse. The proposed method was implemented with MATLAB. The optimization problems in Eqs. [16] and [21] were solved using a MATLAB-based convex optimization solver called CVX (39). The proposed method was validated both through Bloch equation simulation and experiments. The experiments were performed on a Siemens 3.0 T MAGNETOM Trio Scanner (Siemens,

Erlangen, Germany) with a uniform spherical phantom of 18 cm diameter. All the gradient waveforms were designed with a maximum gradient slew rate of 13000 G/cm/s.

Experiment I: A 2D linear-phase 90° excitation pulse

In this example, the proposed method was used to design a 2D linear-phase 90° excitation pulse with a non-separable excitation pattern, i.e., a circular ROI of 3.5 cm radius in a 20 cm × 20 cm field-of-excitation (FOV) with 5% ripples in both the passband and the stopband. The RF pulse was designed with an echo-planar gradient of 11 lobes (Fig. 4a). The remaining parameters were: pulse length = 9.9 ms and time-bandwidth product of the subpulse = 5.

Nine images were first collected using a gradient recalled echo (GRE) sequence with nominal flip angles varying from 10° to 90°. The receiving inhomogeneity was estimated as a byproduct of B_1 mapping as in (40). The GRE sequence was then modified by replacing the original sinc RF pulse with the designed multidimensional RF pulse for excitation. An image was collected using the modified GRE sequence, and the excitation pattern was finally determined by dividing the GRE image by the receiving inhomogeneity. The images were collected using the following parameters: TE = 10 ms, TR = 500 ms, matrix size = 64 × 64 × 18, and FOV = 18 cm × 18 cm × 18 cm.

The designed RF pulse is shown in Fig. 4b. The simulated and experimental excitation pattern (the center slice) are shown in Fig. 4c. Two representative excitation profile plots are shown in Fig. 4d and Fig. 4e, where the red dashed lines and the blue solid lines represent the simulated and experimental excitation pattern respectively. For the simulated excitation pattern, the achieved passband ripple was 5.2%, which was close to the desired value, while the achieved stopband ripple was 6.3%, which was higher than the desired value. For the experimental excitation pattern, the achieved passband ripple was 6.9%, and the achieved stopband ripple was 5.4%. The experimental excitation pattern closely matched the simulation prediction, and the slight mismatch was likely due to transmit B_1 inhomogeneity. Also note that the transition band along the z-axis (Fig. 4e) is narrower than that along the x-axis (Fig. 4d). This is because the time-bandwidth product of the subpulse that determines the transition bandwidth along the z-axis is larger than the effective time-bandwidth product of the pulse envelope that determines the transition bandwidth along the x-axis.

Experiment II: A 2D linear-phase 180° refocusing pulse

The separable design method in (35) and the proposed method were used to design a 2D linear-phase 180° pulse that refocused spins in an 8 cm × 2 cm rectangular ROI in a 20 cm × 10 cm FOV with 5% passband ripple and 1% stopband ripple. The RF pulses were designed with an echo-planar gradient of nine lobes (Fig. 5a) where the center lobe was stretched to reduce the peak B_1 value of the designed RF pulse. The remaining parameters were: pulse length = 8.3 ms and time-bandwidth product of the subpulse = 3.

A spin-echo (SE) sequence was modified to collect experimental data. The slice-selective excitation RF pulse of the SE sequence (along the y-axis) was kept unchanged, while the slice selective refocusing RF pulse was replaced with the designed multidimensional refocusing RF pulses. A reference image was first collected while turning off the refocusing

RF pulse and crusher gradients. SE images were then collected while turning on the refocusing RF pulse and crusher gradients. The excitation patterns ($|\beta|^2$) were finally determined by normalizing the SE images to the reference image. The images were collected using the following parameters: TE = 30 ms, TR = 1000 ms, matrix size = 128×128 , slice thickness = 5 mm, and FOV = $18 \text{ cm} \times 18 \text{ cm}$.

Figure 5b shows the RF pulses designed by the separable design method (black dashed line) and the proposed method (red solid line). The simulated and experimental excitation patterns are shown in Fig. 5c. Representative excitation profile plots are shown in Fig. 5d and Fig. 5e. The experimental results match well with simulation results. While significant geometric distortions appear in the excitation pattern achieved by the separable design method, they are significantly reduced in the excitation pattern achieved by the proposed method. In the simulation results, the achieved ripples for the separable design method and the proposed method were 7.2% and 4.9% respectively in the passband, and 2.1% and 1.2% respectively in the stopband. In the experimental results, the achieved ripples for the separable design method and the proposed method were 8.6% and 7.0% respectively in the passband, and 3.0% and 3.0% respectively in the stopband. In both cases, the proposed method achieved smaller ripples.

In the proposed method, two 1D min-max polynomial approximation problems (Eq. [16] and Eq. [21]) are solved sequentially to design a 2D RF pulse. Figure 6 shows the performance of the two polynomial approximations while designing the 2D refocusing pulse in Fig. 5. Figure 6a shows the performance of the first approximation, which is approximating the desired excitation profile while assuming ideal rotation of each subpulse. Figure 6b (the magnitude), Fig. 6c (the real part) and Fig. 6d (the imaginary part) show the performance of the second approximation, which is approximating the desired rotation (the β parameter in Eqs. [13] and [14]) of the fifth subpulse. Both approximations show equal-ripple errors as expected.

Experiment III: A 2D linear-phase 180° spatial-spectral refocusing pulse

The proposed method was used to design a 2D linear-phase 180° spatial-spectral refocusing pulse that refocused spins in a $1 \text{ cm} \times 350 \text{ Hz}$ ROI in a $1 \text{ cm} \times 770 \text{ Hz}$ FOV in the z - f plane with 10% passband ripple and 5% stopband ripple. The RF pulse was designed with an echo-planar gradient of 15 lobes (Fig. 7a). The remaining parameters were: pulse length = 19.1 ms and time-bandwidth product of the subpulse = 2.

In the experiments, the modified SE sequence in Experiment I was used. First, a reference image was collected while turning off the spatial-spectral refocusing pulse and crusher gradients. Second, a $180.7 \text{ } \mu\text{T/m}$ gradient was added by modifying the value of the x shimming gradient to mimic the spectral dimension of 1.38 kHz width in the phantom. Third, a SE image was collected while turning on the spatial-spectral refocusing pulse and crusher gradients. The excitation pattern ($|\beta|^2$) was finally determined by normalizing the SE image to the reference image. The images were collected using the following parameters: TE = 38 ms, TR = 1000 ms, matrix size = 256×256 , slice thickness = 5 mm, and FOV = $18 \text{ cm} \times 18 \text{ cm}$.

The designed RF pulse is shown in Fig. 7b. The simulated and experimental excitation patterns in the center $5\text{ cm} \times 578\text{ Hz}$ ROI are shown in Fig. 7c. The experimental excitation pattern is close to the simulation prediction. The excitation profile plots along the frequency axis and the z-axis are shown in Fig. 7d and Fig. 7e, where the red dashed lines and the blue solid lines represent the simulated and experimental excitation pattern, respectively. In Fig. 7d, the frequency is converted to ppm (123 Hz/ppm for our 3T scanner and assuming proper frequency shift of the transmitter). The downward-pointing triangles from left to right indicate the positions of water (4.7 ppm), Choline (3.22 ppm), Creatine (3.03 ppm), NAA (2.02 ppm) and lipid (the main peak around 1.2 ppm) signals. Figure 7d indicates that the designed spatial-spectral refocusing pulse can be used in chemical shift-imaging for simultaneous water and fat suppression. For the simulated excitation pattern, the achieved passband and stopband ripple was 10.5% and 5.0%, respectively. For the experimental excitation pattern, the achieved passband and stopband ripple was 11.9% and 5.3%, respectively. We also designed a spatial-spectral refocusing pulse using the separable design method. Distortions on the achieved excitation patterns similar to those in Experiment II were found. The results are therefore not repeated here.

Experiment IV: A 2D linear-phase 90° excitation pulse with B_0 inhomogeneity correction

Two-dimensional linear-phase 90° excitation pulses were designed to excite a $6\text{ cm} \times 2\text{ cm}$ rectangular ROI in $20\text{ cm} \times 10\text{ cm}$ FOV with 5% passband ripple and 5% stopband ripple. The B_0 inhomogeneity was created by introducing a $-50\text{ }\mu\text{T/m}$ z gradient, a $20\text{ }\mu\text{T/m}$ x gradient and a 30 Hz center frequency offset. The RF pulses were then designed with and without B_0 inhomogeneity correction using the proposed method. The gradient was an echo-planar gradient of 13 lobes as shown in Fig. 8a. The remaining parameters were: pulse length = 9.5 ms and time-bandwidth product of the subpulse = 3.

In the experiments, “good shimming” was first obtained using the automatic shimming of the scanner. An image was collected using the modified GRE sequence in Experiment I with the RF pulse designed without B_0 inhomogeneity correction. The B_0 inhomogeneity was then created as “bad shimming” by modifying the values of the z gradient, x gradient and center frequency offset in the shimming settings. Images were collected using the RF pulses designed with and without B_0 inhomogeneity correction. The images were collected using the following parameters: TE = 10 ms, TR = 500 ms, matrix size = $64 \times 64 \times 18$, and FOV = $18\text{ cm} \times 18\text{ cm} \times 18\text{ cm}$. The experimental excitation patterns of the experiments were determined in the same way as Experiment I.

The RF pulses designed with and without B_0 inhomogeneity correction are shown in Fig. 8c and Fig. 8b, respectively. The resulting simulated and experimental excitation patterns (the center slice) are shown in Fig. 8e. In the presence of the B_0 inhomogeneity, the excitation patterns achieved by the RF pulse designed without B_0 inhomogeneity correction (the middle column of Fig. 8e) show significant distortions, i.e., the notable shearing along the z-axis and shrinkage along the x-axis. These distortions were effectively corrected by the RF pulse designed with B_0 inhomogeneity correction as shown in the right column of Fig. 8e.

Experiment V: A 2D maximum-phase 90° saturation pulse

A linear-phase and a maximum-phase 90° saturation pulse were designed to excite/saturate spins in a 6 cm × 2 cm rectangular ROI in 20 cm × 10 cm FOV with 5% passband ripple and 1% stopband ripple. The RF pulse was designed with an echo-planar gradient of 11 lobes (Fig. 9a). The remaining parameters were: pulse length = 7.6 ms and time-bandwidth product of the subpulse = 3.

When designing the linear-phase RF pulse, linear-phase polynomials of length N_x ($B_{N_x, \text{lin}}(z, x)$) were designed to approximate the excitation profiles along the x -axis with passband ripple δ_1 and stopband ripple δ_2 . To design the maximum-phase RF pulse, linear-phase polynomials of length $2N_x-1$ ($B_{2N_x-1, \text{lin}}(z, x)$) were first designed to approximate the excitation profiles along the x -axis with passband ripple $2\delta_1$ and stopband ripple $\delta_2^2/2$. The magnitudes of the maximum-phase polynomial ($B_{N_x, \text{max}}(z, x)$) were the square-root of the profiles of the linear-phase polynomials $B_{2N_x-1, \text{lin}}(z, x)$. The maximum-phase polynomials were finally determined using spectral factorization. The rest design procedure for the maximum-phase RF pulse was the same as that for the linear-phase RF pulse. In the experiments, a CKE sequence was modified to test the designed saturation pulses.

In the sequence, the designed saturation pulse was first applied, immediately followed by spoiler gradients in three axes. A slice-selective excitation pulse (along the y -axis) and encoding gradients were then applied for imaging. A reference image was first collected while turning off the saturation RF pulse. Images were then collected while turning on the linear-phase and the maximum-phase saturation pulse, respectively. The excitation patterns ($|m_z|$) were finally determined by dividing the images by the reference image. The images were collected using the following parameters: TE = 5ms, TR = 500 ms, matrix size = 256 × 256, slice thickness = 5 mm, and FOV = 18 cm × 18 cm.

Figure 9b shows the RF waveforms of the linear-phase (black dashed line) and the maximum-phase (red solid line) saturation pulse. The simulated and experimental excitation patterns (m_z) are shown in Fig. 9c. Two representative excitation profiles are shown in Fig. 9d and Fig. 9e. The maximum-phase pulse achieved a sharper transition bandwidth along the x -axis. To achieve a similar transition bandwidth, a 10.4 ms linear-phase saturation pulse will be needed. This is 36.8% longer than the maximum-phase pulse, which is a significant difference for generally long multidimensional RF pulses.

Discussion

In this paper, a novel method is proposed to extend the conventional SLR method to the multidimensional case. The proposed method preserves almost all the desirable features of the SLR method. More specifically, the proposed method can effectively handle the nonlinearity of the Bloch equation, achieves almost equiripple excitation errors, and allows easy control of the tradeoff among ripple level, transition bandwidth and pulse length. Several of the features of the proposed method are also worth noting.

First, each of the 1D polynomial design problem is formulated as minimizing the maximum approximation error. The optimization problem is convex, and thus can be solved with

global optimality. However, note that the final multidimensional polynomial designed by the proposed method is not necessarily a global optimum in approximating the desired multidimensional excitation pattern, because the proposed method is not the only way to construct a multidimensional polynomial.

Second, the desired profiles are defined on discrete grids. One-dimensional polynomials are designed to approximate the desired profiles by solving a minimax optimization problem using a general convex optimization solver, e.g., CVX (39). This approach is flexible when designing polynomials to approximate general profiles. This is especially convenient when designing a subpulse to approximate its desired rotation profile (an analytical expression of which can be difficult to obtain). Increasing the number of discrete points will improve approximation accuracy at the cost of computational efficiency. In our experience, good RF pulse design was always achieved when the number of discrete points was about 15 times of the polynomial order.

Third, suppose two polynomials B_{N_x} and A_{N_x} are designed to approximate a 2D excitation pattern, and the desired rotation profiles are $C_{d,n_x}(z)$ and $S_{d,n_x}(z)$. If we ignore the phase of $C_{d,n_x}(z)$, using arguments similar to the conventional SLR method (25), we can show that determining the polynomial A_{N_x} using Eq. [15] will result in a set of subpulses with minimum tip-angles. This is desirable since smaller tip-angles generally lead to less RF power. We can further show that modifying the phase of $C_{d,n_x}(z)$ as in Eq. [12] and accordingly the phase of $S_{d,n_x}(z)$ as in Eq. [13] will not require extra RF energy of a subpulse for approximation. Intuitively, it is because only the magnitude of $S_{d,n_x}(z)$ determines the minimum amount of RF power of a subpulse for approximation (see Ref. (41) for more details). Each of the designed subpulses is a minimum energy 1D RF pulse, as it is designed similar to the conventional SLR method. However, also note that the resulting 2D RF pulse is not necessarily a minimum energy 2D RF pulse.

The proposed method assumes the same gradient waveform as the separable design method (35). In the separable design method, the desired tip-angle of each subpulse is first designed, followed by the design of the subpulses. The underlying assumption is that the rotation produced by each subpulse is about an axis in the transverse plane. This assumption holds when the tip-angle of each subpulse is small. In the proposed method, the desired rotation generated by each subpulse is first designed, which can be about an arbitrary axis. Subpulses are then designed to approximate such rotation profiles. For excitation or saturation pulses, the tip-angle of each subpulse is usually small. The proposed method is then reduced to the separable design method. For large-tip-angle pulses (e.g., refocusing or inversion pulses), the tip-angle of each subpulse can be large. In this case, the underlying assumption of the separable design method breaks down and can result in pulses that generate geometric distortions in the resulting excitation pattern. This problem is overcome with the proposed method.

The NUSLR method in (34) also extends the conventional SLR method to the multidimensional case. For 2D RF pulse design, the NUSLR method jointly designs two-dimensional polynomials B and A to approximate the desired excitation pattern. The resulting optimization problem is so far time-consuming to solve due to the nonlinear

dependence of the polynomial coefficients on the design parameters. A major advantage of the proposed method over NUSLR is computational efficiency. For a typical 2D RF pulse design, the proposed method only takes tens of seconds compared to a few hours using the NUSLR method reported in (34). However, the proposed method is limited to design of multidimensional RF pulses with echo-planar gradients. The NUSLR method can design multidimensional RF pulses with arbitrary time-varying gradients (e.g., spiral trajectories).

The optimal control approach (32, 33) can also design large tip-angle multidimensional RF pulses while accurately handling the nonlinearity of the Bloch equation. Compared to the optimal control approach, the proposed method has significant advantages in achieving equiripple excitation error, allowing easy control of the tradeoff among design parameters and fast computation.

B_0 inhomogeneity is a well-known problem in multidimensional RF pulse design. In conventional multidimensional RF pulse design methods, a B_0 map can be directly incorporated into the RF pulse design equation to correct the B_0 inhomogeneity effects on the resulting excitation pattern (31, 33). However, it is hard to do so in the proposed method because of difficulties in explicitly constructing the corresponding polynomials. Instead, assuming B_0 inhomogeneity is small compared to the bandwidth of each subpulse, we derive a closed-form expression describing the geometric distortions of the resulting excitation pattern caused by B_0 inhomogeneity and develop a correction scheme accordingly. In the small tip-angle regime, the proposed B_0 inhomogeneity correction method is analogous to image domain B_0 inhomogeneity correction methods in EPI-based imaging (42).

B_1 inhomogeneity is another problem with RF pulse design. A common practice is to incorporate B_1 inhomogeneity maps into RF pulse design to correct B_1 inhomogeneity. However, the proposed method has the same limitation as the conventional SLR method in doing so, because of difficulties in explicitly constructing the corresponding polynomials. An alternative approach is to design adiabatic pulses to gain robustness to B_1 inhomogeneity. It has been recently reported that SLR method can be used to design 1D pulses of adiabatic behavior i.e., insensitive to B_1 inhomogeneity, by imposing quadratic phase on the desired excitation pattern (43). Similar ideas may be adopted to design 2D adiabatic RF pulses using the proposed method.

Conclusions

This paper presents a new method to extend the conventional SLF method to the multidimensional case. The proposed method achieves equiripple excitation errors, allows easy control of the tradeoff among design parameters, and is computationally efficient. The proposed method is further generalized for B_0 inhomogeneity correction and spatial-spectral RF pulse design. The proposed method was validated through Bloch equation simulation and experiments on a 3.0 T MRI scanner. The proposed method may prove useful especially for designing multidimensional refocusing and inversion pulses.

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Appendix

A. Derivation of $C_{n_x}(z)$ and $S_{n_x}(z)$ in Eqs. [17] and [18]

The rotation matrix of the n_x -th subpulse can be decomposed as

$$\begin{bmatrix} C_{n_x}(z) & -v_{n_x}^*(z) \\ S_{n_x}(z) & C_{n_x}^*(z) \end{bmatrix} = \begin{bmatrix} e^{i\psi_{n_x}/2} & 0 \\ 0 & -e^{-i\psi_{n_x}/2} \end{bmatrix} \begin{bmatrix} \mu_{n_x}(z) & -v_{n_x}^*(z) \\ v_{n_x}(z) & \mu_{n_x}^*(z) \end{bmatrix} \begin{bmatrix} e^{i\psi_{n_x}/2} & 0 \\ 0 & -e^{-i\psi_{n_x}/2} \end{bmatrix}, \quad (35)$$

where the first and the third rotation matrix on the right-hand side represent the precession by the refocusing z gradient lobes, and the second rotation matrix represents the rotation of the n_x -th subpulse with a constant z gradient.

Applying the forward SLR transform, $\mu_{n_x}(z)$ and $v_{n_x}(z)$ can be written as

$$\mu_{n_x}(z) = \omega_{z,n_x}^{N_z/2} \sum_{n_z=0}^{N_z-1} a_{n_x,n_z} \omega_{z,n_x}^{-n_z}, \quad (36)$$

$$v_{n_x}(z) = \omega_{z,n_x}^{N_z/2} \sum_{n_z=0}^{N_z-1} v_{n_x,n_z} \omega_{z,n_x}^{-n_z}, \quad (37)$$

where ω_{z,n_x} is defined in Eq. [19].

For the refocusing z gradient lobe, the angle of the precession ψ_{n_x} in Eq. [35] is

$$e^{i\psi_{n_x}} = \omega_{z,n_x}^{-N_z/2}. \quad (38)$$

Substituting Eqs. [36] to [38] to Eq. [35], we obtain the expression of $C_{n_x}(z)$ and $S_{n_x}(z)$ in Eqs. [17] and [18].

B. The rotation matrix generated by each segment in the presence of B_0 inhomogeneity

In the presence of B_0 inhomogeneity, the rotation matrix generated by the n_x segment can be written as

$$\tilde{\mathbf{Q}}_{n_x}(z, x) = \begin{bmatrix} \hat{C}_{n_x}(z) & -\hat{S}_{n_x}^*(z) \\ \hat{S}_{n_x}(z) & \hat{C}_{n_x}^*(z) \end{bmatrix} \begin{bmatrix} \tilde{\omega}_x^{1/2} & 0 \\ 0 & \tilde{\omega}_x^{-1/2} \end{bmatrix}, \quad (39)$$

where

$$\tilde{\omega}_x = e^{i2\pi\tau\Delta f(z, x)} \omega_x. \quad (40)$$

The rotation matrix of the n_x th subpulse (the first matrix on the right-hand side of Eq. [39]) can still be calculated using Eqs. [35] to [38], except that ω_{z, n_x} in Eqs. [36] and [37] should be replaced by

$$\tilde{\omega}_{z, n_x} = e^{i\pi\Delta t\Delta f(z, x)} \omega_{z, n_x}. \quad (41)$$

Note that $e^{i\pi\Delta t\Delta f(z, x)}$ in Eqs. [35] and [38] does not change, because the refocusing lobe of the z -gradient will be canceled by neighboring segments, during which no phase accumulation caused by B_0 inhomogeneity will occur.

With the above preparation, it is easy to show that $\mathbf{Q}_{n_x}(z, x)$ can be rewritten as

$$\mathbf{Q}_{n_x}(z, x) = \begin{bmatrix} e^{i\psi_f/2} & 0 \\ 0 & e^{-i\psi_f/2} \end{bmatrix} \begin{bmatrix} \hat{C}_{n_x}(z) & -\hat{S}_{n_x}^*(z) \\ \hat{S}_{n_x}(z) & \hat{C}_{n_x}^*(z) \end{bmatrix} \begin{bmatrix} \omega_x^{1/2} & 0 \\ 0 & \omega_x^{-1/2} \end{bmatrix} \begin{bmatrix} e^{-i\psi_f/2} & 0 \\ 0 & e^{i\psi_f/2} \end{bmatrix}, \quad (42)$$

where ψ_f , ω'_x , $\tilde{C}_{n_x}(z)$ and $\tilde{S}_{n_x}(z)$ are given by

$$\psi_f = \pi N_z \Delta t \Delta f(z, x), \quad (43)$$

$$\omega_{x'} = e^{i2\pi(N\Delta t + \tau)\Delta f(z, x)} \omega_{x'}, \quad (44)$$

$$\hat{C}_{n_x}(z) = \sum_{n_z=0}^{N_z-1} a_{n_x, n_z} \tilde{\omega}_{z, n_x}^{-n_z} \quad (45)$$

$$\hat{S}_{n_x}(z) = \tilde{\omega}_{z, n_x}^{N_z/2} \sum_{n_z=0}^{N_z-1} b_{n_x, n_z} \tilde{\omega}_{z, n_x}^{-n_z}. \quad (46)$$

Assuming that the B_0 inhomogeneity is relatively small compared with the bandwidth of the subpulse, $\hat{C}_{n_x}(z)$ and $\hat{S}_{n_x}(z)$ can be approximated by $C_{n_x}(z)$ and $S_{n_x}(z)$ in Eqs. [17] and [18]. This assumption along with Eqs. [42] to [44] will directly lead to Eq. [27] to [29].

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Figure 1
A 2D RF pulse with an echo-planar gradient in (a) can be decomposed into a series of segments in (b). Note that the two refocusing z-gradient lobes in (b) will be canceled by neighboring segments.

Figure 2
The passband, transition band and stopband for a rectangular excitation pattern. The region within the red box is the desired excitation pattern. The shaded area is the transition band. Region 1: passband along both the x - and z -axes. Region 2: stopband along the x -axis. Region 3: stopband along the z -axis. Region 4: stopband along both the x - and z -axes.

Figure 3
A type I spatial-spectral RF pulse in (a) can be decomposed into a series of segments in (b).

Figure 4

A 2D linear-phase 90° excitation pulse was designed to excite a circular ROI of 3.5 cm radius in a $20\text{ cm} \times 20\text{ cm}$ FOV with 5% passband ripple and 5% stopband ripple. **a:** Echo-planar gradient (blue line: y -gradient; red line: x -gradient). **b:** RF pulse (blue line: real part, red line: imaginary part). **c:** Simulated and experimental excitation pattern ($|m_{xy}|$). **d** and **e:** Excitation profile plots along the dashed lines in (c), where the red dashed lines and the blue solid lines represent the simulated and experimental excitation pattern, respectively.

Figure 5

Two dimensional linear-phase 180° refocusing pulses were designed using the separable design method in (35) and the proposed method to refocus spins in an $8\text{ cm} \times 2\text{ cm}$ rectangular ROI in a $20\text{ cm} \times 10\text{ cm}$ FOV with 5% passband ripple and 1% stopband ripple. **a:** Echo-planar gradient (blue line: x -gradient; red line: y -gradient). **b:** RF pulses designed by the separable design method (black dashed line) and the proposed method (red solid line). **c:** Simulated and experimental excitation patterns ($|\beta|^2$) achieved by the separable design method and the proposed method. **d** and **e:** Excitation profile plots along the dashed lines in (c), where the black dashed lines and the red solid lines represent the excitation profiles achieved by the separable design method and the proposed method, respectively.

Figure 6
The performance of the two polynomial approximations (Eq. [16] and Eq. [21]) while designing the 2D refocusing pulse in Fig. 5. The approximation of the desired excitation profile (the β parameter) while assuming ideal rotation of each subpulse is shown in (a). The approximation of the desired rotation (the β parameter) of the 5th subpulse is shown in (b) (the magnitude), (c) (the real part) and (d) (the imaginary part). In these plots, the red dashed lines represent the desired profile and the blue solid lines represent the achieved profiles.

Figure 7

A linear-phase 180° refocusing spatial-spectral pulse was designed to refocusing spins in a $1 \text{ cm} \times 250 \text{ Hz}$ ROI in a $5 \text{ cm} \times 770 \text{ Hz}$ FOV in the z - f plane with 10% passband ripple and 5% stopband ripple. **a**: Echo planar gradient (the z -gradient). **b**: RF pulse (blue line: real part, red line: imaginary part). **c**: Simulated and experimental excitation pattern ($|\beta|^2$) in the center $5 \text{ cm} \times 678 \text{ Hz}$ ROI. **d** and **e**: Excitation profile plots along the frequency axis and the z -axis, where the red dashed lines and the blue solid lines represent the simulated and experimental excitation pattern, respectively. In **(a)**, the frequency is converted to ppm (123 Hz/ppm for our 3T scanner and assuming proper frequency shift of the transmitter). The downward-pointing triangles from left to right indicate the positions of water (4.7 ppm), Choline (3.22 ppm), Creatine (3.03 ppm), NAA (2.02 ppm) and lipid (the main peak around 1.2 ppm) signals.

Figure 8

Two dimensional linear-phase 90° excitation pulses were designed to excite a $6\text{ cm} \times 2\text{ cm}$ rectangular ROI in a $20\text{ cm} \times 10\text{ cm}$ FOV with 5% passband ripple and 5% stopband ripple. **a:** Echo-planar gradient (blue line: z -gradient; red line: x -gradient). **b** and **c:** RF pulses designed without and with B_0 inhomogeneity correction, respectively (blue line: real part, red line: imaginary part). **d:** The B_0 map in Hz, which consists of a $-50\text{ }\mu\text{T/m}$ z gradient, a $20\text{ }\mu\text{T/m}$ x gradient, and a 30 Hz offset. **e:** The simulated and experimental excitation patterns. Left column: excitation patterns achieved by the RF pulse in **(b)** without B_0 inhomogeneity. Middle column: excitation patterns achieved by the RF pulse in **(b)** with B_0 inhomogeneity. Right column: excitation patterns achieved by the RF pulse in **(c)** with B_0 inhomogeneity. The boxes of red dashed line in **(d)** and **(e)**, indicate the position of the desired ROI.

Figure 9

A linear-phase and a maximum-phase 90° saturation pulse were designed to excite/saturate spins in a $6\text{ cm} \times 2\text{ cm}$ rectangular ROI in a $20\text{ cm} \times 10\text{ cm}$ FOX with 5% passband ripple and 1% stop band ripple. **a:** Echo-planar gradient (blue line: z-gradient; red line: x-gradient). **b:** RF waveforms (black dashed line: linear-phase RF pulse, red solid line: maximum-phase RF pulse). **c:** Simulated and experimental excitation patterns ($|m_z|$) achieved by the linear-phase RF pulse and the maximum-phase RF pulse. **d** and **e:** Excitation profile plots along the horizontal (the x-axis) direction and the vertical (the z-axis) in (c), where the black dashed lines and the red solid lines represent the excitation profiles achieved by the linear-phase RF pulse and the maximum-phase RF pulse, respectively.