Symmetries and the u -condition in Hom-Yetter-Drinfeld categories

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Let (H, S, α) be a monoidal Hom-Hopf algebra and ${}^H_H\mathcal{HYD}$ the Hom-Yetter-Drinfeld category over (H, α) . Then in this paper, we first find sufficient and necessary conditions for ${}^H_H\mathcal{HYD}$ to be symmetric and pseudosymmetric, respectively. Second, we study the u -condition in ${}^H_H\mathcal{HYD}$ and show that the Hom-Yetter-Drinfeld module $(H, \text{adjoint}, \Delta, \alpha)$ (resp., $(H, m, \text{coadjoint}, \alpha)$) satisfies the u -condition if and only if $S^2 = id$. Finally, we prove that ${}^H_H\mathcal{HYD}$ over a triangular (resp., cotriangular) Hom-Hopf algebra contains a rich symmetric subcategory. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4892081>]

I. INTRODUCTION

Hom-algebras and Hom-coalgebras were introduced by Makhlouf and Silvestrov in Ref. 9 as generalizations of ordinary algebras and coalgebras in the following sense: the associativity of the multiplication is replaced by the Hom-associativity and similar for Hom-coassociativity. They also defined the structures of Hom-bialgebras and Hom-Hopf algebras, and described some of their properties extending properties of ordinary bialgebras and Hopf algebras in Refs. 10 and 11. Later, Yau¹⁷ proposed the definition of quasitriangular Hom-Hopf algebras and shown that each quasitriangular Hom-Hopf algebra yields a solution of Hom-Yang-Baxter equation. In Ref. 1, Caenepeel and Goyvaerts studied Hom-bialgebras and Hom-Hopf algebras from a categorical view point, and called them monoidal Hom-bialgebras and monoidal Hom-Hopf algebras respectively, which are slightly different from the above Hom-bialgebras and Hom-Hopf algebras. Recently, many more properties and structures of Hom-Hopf algebras have been developed, see Refs. 2, 6, and 16 and references cited therein.

In Ref. 1, Caenepeel and Goyvaerts also introduced the notion of Hom-Hopf modules over a monoidal Hom-Hopf algebra and proved the fundamental theorem of Hom-Hopf modules. In Ref. 12, Makhlouf and Panaite defined Yetter-Drinfeld modules over Hom-bialgebras and shown that Yetter-Drinfeld modules over a Hom-bialgebra with bijective structure map provide solutions of the Hom-Yang-Baxter equation. Also Liu and Shen⁸ studied Yetter-Drinfeld modules over monoidal Hom-bialgebras and called them Hom-Yetter-Drinfeld modules, and shown that the category of Hom-Yetter-Drinfeld modules is a braided monoidal categories. Chen and Zhang³ defined the category of Hom-Yetter-Drinfeld modules in a slightly different way to Liu and Sheng,⁸ and shown that it is a full monoidal subcategory of the left center of left Hom-module category.

The notion of symmetric category is a classical concept in category theory. For a quasitriangular Hopf algebra (H, R) , the category ${}_H\mathcal{M}$ of left H -modules is symmetric if and only if (H, R) is triangular. Similarly, the category ${}^H\mathcal{M}$ of left H -comodules over a coquasitriangular Hopf algebra $(H, \langle | \rangle)$ is symmetric if and only if $(H, \langle | \rangle)$ is cotriangular. Let H be a Hopf algebra with a bijective antipode, Cohen and Westreich⁴ tested the symmetries and the u -condition in Yetter-Drinfeld category ${}^H_H\mathcal{YD}$. Pareigis¹⁴ proved that $\mathcal{YD}_H^H$ is symmetric if and only if $H = k$. Later, Panaite, Staic, and Oystaeyen¹³

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proposed the definition of pseudosymmetric braided categories which can be viewed as a kind of weakened symmetric braided categories, and shown that the category ${}_H\mathcal{YD}^H$ is pseudosymmetric if and only if H is commutative and cocommutative. In this paper, we mainly discuss the symmetries, the pseudosymmetries and the u -condition in Hom-Yetter-Drinfeld categories. The results in Refs. 4, 13, and 14 are slightly generalized and more properties are given.

This paper is organized as follows. In Sec. II, we recall some basic definitions and basic results related to monoidal Hom-Hopf algebras and Hom-Yetter-Drinfeld modules.

In Sec. III, we show that Hom-Yetter-Drinfeld category ${}_H^H\mathcal{HYD}$ is symmetric if and only if (H, α) is trivial (see Theorem 3.1), ${}_H^H\mathcal{HYD}$ is pseudosymmetric if and only if (H, α) is commutative and cocommutative (see Theorem 3.3).

In Sec. IV, for a monoidal Hom-Hopf algebra $H = (H, \alpha)$, we give a necessary and sufficient condition for H_1 and (H_2) to satisfy the u -condition (see Proposition 4.2), where $H_1 = (H, \text{adjoint}, \Delta, \alpha)$, $H_2 = (H, m, \text{coadjoint}, \alpha)$.

In Secs. V and VI, we prove that the category ${}_H\mathcal{M}$ over a quasitriangular Hom-Hopf algebra (resp., ${}_H^H\mathcal{M}$ over a coquasitriangular Hom-Hopf algebra) is a Hom-Yetter-Drinfeld subcategory of ${}_H^H\mathcal{HYD}$ (see Theorems 5.1 and 6.1). Conversely, assuming the braiding $C_{H,H}$ is a symmetry forces in many cases (H, α) to be triangular and cotriangular (see Theorems 5.3 and 6.3).

II. PRELIMINARIES

Throughout this paper, k is a fixed field. Unless otherwise stated, all vector spaces, algebras, coalgebras, maps, and unadorned tensor products are over k . For a coalgebra C , the coproduct will be denoted by Δ . We adopt a Sweedler's notation $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$, in which we often omit the summation symbols for convenience. Any unexplained definitions and notations can be found in Ref. 15.

Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ be the category of k -modules. There is a new monoidal category $\mathcal{H}(\mathcal{M}_k)$. The objects of $\mathcal{H}(\mathcal{M}_k)$ are pairs (M, μ) , where $M \in \mathcal{M}_k$ and $\mu \in \text{Aut}_k(M)$. The morphisms of $\mathcal{H}(\mathcal{M}_k)$ are morphisms $f: (M, \mu) \rightarrow (N, \nu)$ in $\mathcal{M}_k$ such that $\nu \circ f = f \circ \mu$. For any objects $(M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)$, the monoidal structure is given by

$$(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu) \text{ and } (k, Id_k).$$

Briefly speaking, all Hom-structures are objects in the monoidal category $\tilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, id), \tilde{a}, \tilde{l}, \tilde{r})$ introduced in Ref. 1, where the associator $\tilde{a}$ is given by the formula

$$\tilde{a}_{M,N,L} = a_{M,N,L} \circ ((\mu \otimes Id) \otimes \varsigma^{-1}) = (\mu \otimes (Id \otimes \varsigma^{-1})) \circ a_{M,N,L},$$

for any objects $(M, \mu), (N, \nu), (L, \varsigma) \in \tilde{\mathcal{H}}(\mathcal{M}_k)$, and the unitors $\tilde{l}$ and $\tilde{r}$ are

$$\tilde{l}_M = \mu \circ l_M = l_M \circ (Id \otimes \mu), \tilde{r}_M = \mu \circ r_M = r_M \circ (\mu \otimes Id).$$

The category $\tilde{\mathcal{H}}(\mathcal{M}_k)$ is called the Hom-category associated to the monoidal category $\mathcal{M}_k$.

In the following we recall some definitions about Hom-structures from Ref. 1.

Definition 2.1. A monoidal Hom-algebra is an object (A, α) in the category $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with an element $1_A \in A$ and a linear map $m: A \otimes A \rightarrow A$, $a \otimes b \mapsto ab$ such that

$$\alpha(a)(bc) = (ab)\alpha(c), \alpha(ab) = \alpha(a)\alpha(b), \quad (2.1)$$

$$a1_A = 1_A a = \alpha(a), \alpha(1_A) = 1_A, \quad (2.2)$$

for all $a, b, c \in A$.

As noted in Ref. 1, the definition of monoidal Hom-algebras is different from the definition of Hom-associative algebras defined in Ref. 11. Specifically, the same twisted associativity condition (2.1) holds in both cases. However, the unitality condition in Ref. 11 is the usual untwisted one: $a1_A = 1_A a = a$, for any $a \in A$, and the condition (1.2) is not desired there.

Definition 2.2. Let (A, α) be a monoidal Hom-algebra. A *left (A, α) -Hom-module* consists of $(M, \mu) \in \tilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\psi : A \otimes M \rightarrow M$, $\psi(a \otimes m) = a \cdot m$ such that

$$\alpha(a) \cdot (b \cdot m) = (ab) \cdot \mu(m), 1_A \cdot m = \mu(m), \mu(a \cdot m) = \alpha(a) \cdot \mu(m), \quad (2.3)$$

for all $a, b \in A$ and $m \in M$.

A monoidal Hom-algebra (A, α) can be considered as a left (A, α) -Hom-module by the Hom-multiplication. Let (M, μ) and (N, ν) be two left (A, α) -Hom-modules. A morphism $f : M \rightarrow N$ is called *left A -linear* if $f(am) = af(m)$, for any $a \in A$, $m \in M$ and $f \circ \mu = \nu \circ f$. We denote the category of left (A, α) -Hom-modules by $\tilde{\mathcal{H}}_A(\mathcal{M})$.

Definition 2.3. A *monoidal Hom-coalgebra* is an object (C, γ) in the category $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with linear maps $\Delta : C \rightarrow C \otimes C$, $\Delta(c) = c_1 \otimes c_2$ and $\epsilon : C \rightarrow k$ such that

$$g^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \gamma^{-1}(c_2), \Delta(\gamma(c)) = \gamma(c_1) \otimes \gamma(c_2), \quad (2.4)$$

$$c_1 \epsilon(c_2) = \epsilon(c_1) c_2 = \gamma^{-1}(c), \epsilon(\gamma(c)) = \epsilon(c), \quad (2.5)$$

for all $c \in C$.

Note that the first part of (2.4) is equivalent to $c_1 \otimes c_{21} \otimes \gamma(c_{22}) = \gamma(c_{11}) \otimes c_{12} \otimes c_2$, which we will use frequently in the following. Also the definition of monoidal Hom-coalgebras is different from the definition of Hom-coassociative coalgebras defined in Ref. 11. The coassociativity condition is twisted by some endomorphism, not necessarily by the inverse of the automorphism γ . The counitality condition in Ref. 11 is the usual untwisted one: $c_1 \epsilon(c_2) = \epsilon(c_1) c_2 = c$, for any $c \in C$, and the condition (2.5) is not needed there.

Definition 2.4. Let (C, γ) be a monoidal Hom-coalgebra. A *left (C, γ) -Hom-comodule* consists of $(M, \mu) \in \tilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\rho_M : M \rightarrow C \otimes M$, $\rho_M(m) = m_{(-1)} \otimes m_0$ such that

$$\Delta_C(m_{(-1)}) \otimes \mu^{-1}(m_0) = \gamma^{-1}(m_{(-1)}) \otimes (m_{0(-1)} \otimes m_{00}), \quad (2.6)$$

$$\rho_M(\mu(m)) = \gamma(m_{(-1)}) \otimes \mu(m_0), \epsilon(m_{(-1)}) m_0 = \mu^{-1}(m), \quad (2.7)$$

for all $m \in M$.

Note that (C, γ) is a Hom-comodule on itself via the Hom-comultiplication. Let (M, μ) and (N, ν) be two left (C, γ) -Hom-comodules. A morphism $g : M \rightarrow N$ is called *left C -colinear* if $g \circ \mu = \nu \circ g$ and $m_{(-1)} \otimes g(m_0) \equiv g(m)_{(-1)} \otimes g(m)_0$, for any $m \in M$. The category of left (C, γ) -Hom-comodules is denoted by $\tilde{\mathcal{H}}^C(\mathcal{M})$.

Definition 2.5. A *monoidal Hom-bialgebra* $H = (H, \alpha, m, 1_H, \Delta, \epsilon)$ is a bialgebra in the category $\tilde{\mathcal{H}}(\mathcal{M}_k)$. This means that $(H, \alpha, m, 1_H)$ is a monoidal Hom-algebra and $(H, \alpha, \Delta, \epsilon)$ is a monoidal Hom-coalgebra such that Δ and ϵ are Hom-algebra maps, that is, for any $h, g \in H$,

$$\begin{aligned} \Delta(hg) &= \Delta(h)\Delta(g), \quad \Delta(1_H) = 1_H \otimes 1_H, \\ \epsilon(hg) &= \epsilon(h)\epsilon(g), \quad \epsilon(1_H) = 1_k. \end{aligned}$$

A monoidal Hom-bialgebra (H, α) is called a *monoidal Hom-Hopf algebra* if there exists a morphism (called the antipode) $S : H \rightarrow H$ in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e., $S \circ \alpha = \alpha \circ S$), which is the convolution inverse of the identity morphism Id_H (i.e., $S * Id_H = \eta_H \circ \epsilon_H = Id_H * S$), this means for any $h \in H$,

$$S(h_1)h_2 = \epsilon(h)1_H = h_1 S(h_2). \quad (2.8)$$

Note that a monoidal Hom-Hopf algebra is just a usual Hopf algebra in $\tilde{\mathcal{H}}(\mathcal{M}_k)$. The antipode of monoidal Hom-Hopf algebras have similar properties of those of Hopf algebras such as they are

morphisms of Hom-anti-(co)algebras. Since α is bijective and commutes with antipode S , $S\alpha^{-1} = \alpha^{-1}S$.

Now, we recall the notion of the category of Hom-Yetter-Drinfeld modules given by Liu and Shen in Ref. 8.

Definition 2.6. Let (H, α) be a monoidal Hom-Hopf algebra. A *left-left (H, α) -Hom-Yetter-Drinfeld module* is an object $(M, \beta) \in \tilde{\mathcal{H}}(\mathcal{M}_k)$, such that (M, β) is both a left (H, α) -Hom-module and a left (H, α) -Hom-comodule with the following compatibility condition:

$$h_1 m_{(-1)} \otimes h_2 \cdot m_0 = (h_1 \cdot \beta^{-1}(m))_{(-1)} h_2 \otimes \beta((h_1 \cdot \beta^{-1}(m))_0), \quad (2.9)$$

for all $h \in H$ and $m \in M$.

Condition (2.9) is equivalent to the following equation:

$$\rho(h \cdot m) = (h_{11} \alpha^{-1}(m_{(-1)})) S(h_2) \otimes \alpha(h_{12}) \cdot m_0, \quad (2.10)$$

for all $h \in H$ and $m \in M$.

Example 2.7. Let (H, α) be a monoidal Hom-Hopf algebra. Then (H, α) is a Hom-Yetter-Drinfeld module with left (H, α) -action $h \cdot g = (h_1 \alpha^{-1}(g)) S(\alpha(h_2))$ and left (H, α) -coaction by the Hom-comultiplication Δ , note it by $H_1 = (H_1, \text{adjoint}, \Delta, \alpha)$. Dually, (H, α) is a Hom-Yetter-Drinfeld module with left (H, α) -action by the Hom-multiplication m and left (H, α) -coaction $\rho(h) = h_{11} \alpha^{-1}(S(h_2)) \otimes \alpha(h_{12})$, and note it by $H_2 = (H_2, m, \text{coadjoint}, \alpha)$.

Definition 2.8. Let (H, α) be a monoidal Hom-Hopf algebra. A *Hom-Yetter-Drinfeld category* ${}^H_H \mathcal{HYD}$ is a braided monoidal category whose objects are left-left (H, α) -Hom-Yetter-Drinfeld modules, morphisms are both left (H, α) -linear and (H, α) -colinear maps, and its braiding $C_{-, -}$ is given by

$$C_{M,N}(m \otimes n) = m_{(-1)} \cdot v^{-1}(n) \otimes \mu(m_{(0)}), \quad (2.11)$$

for all $m \in (M, \mu) \in {}^H_H \mathcal{HYD}$ and $n \in (N, v) \in {}^H_H \mathcal{HYD}$.

III. SYMMETRIC HOM-YETTER-DRINFELD CATEGORIES

In this section, we obtain necessary and sufficient conditions for a Hom-Yetter-Drinfeld category ${}^H_H \mathcal{HYD}$ over a monoidal Hom-Hopf algebra (H, α) to be symmetric and pseudosymmetric, respectively.

Let $\mathcal{C}$ be a monoidal category and C a braiding on $\mathcal{C}$. The braiding C is called a *symmetry*⁷ if $C_{X,Y} C_{Y,X} = id_{X \otimes Y}$ for all $X, Y \in \mathcal{C}$, and the category $\mathcal{C}$ is called *symmetric*. The braiding C is called a *pseudosymmetry* if the following condition holds, for all $X, Y, Z \in \mathcal{C}$:

$$(id_Z \otimes C_{X,Y})(C_{Z,X}^{-1} \otimes id_Y)(id_X \otimes C_{Y,Z}) = (C_{Y,Z} \otimes id_X)(id_Y \otimes C_{Z,X}^{-1})(C_{X,Y} \otimes id_Z). \quad (3.1)$$

In this case, $\mathcal{C}$ is called a *pseudosymmetric braided category*.¹³

If C is a symmetry, that is, $C_{Z,X}^{-1} = C_{X,Z}$, then obviously C is a pseudosymmetry.

Theorem 3.1. Let (H, α) be a monoidal Hom-Hopf algebra such that the canonical braiding of the Hom-Yetter-Drinfeld category ${}^H_H \mathcal{HYD}$ is a symmetry. Then $H = k$.

Proof. By Example 2.7, $H_1 = (H, \text{adjoint}, \Delta, \alpha)$ and $H_2 = (H, m, \text{coadjoint}, \alpha)$ are two Hom-Yetter-Drinfeld modules. Let $1_H \otimes h \in H_2 \otimes H_1$ and apply the square of the braiding C to it. Then we have

$$\begin{aligned} C_{H_2, H_1}(1_H \otimes h) &\stackrel{(1.11)}{=} 1_{H_{(-1)}} \cdot \alpha^{-1}(h) \otimes \alpha(1_{H_0}) \\ &= 1_H \cdot \alpha^{-1}(h) \otimes \alpha(1_H) = (1_H \alpha^{-2}(h)) S(\alpha(1_H)) \otimes 1_H \\ &= \alpha^{-1}(h) 1_H \otimes 1_H = h \otimes 1_H. \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned} C_{H_1, H_2} C_{H_2, H_1} (1_H \otimes h) &\stackrel{(1.11)}{=} C_{H_1, H_2} (h \otimes 1_H) \\ &= h_{(-1)} \cdot \alpha^{-1}(1_H) \otimes \alpha(h_0) = (h_{(-1)} \cdot 1_H) \otimes \alpha(h_0) \\ &= \alpha(h_1) \otimes \alpha(h_2). \end{aligned}$$

Since $C_{H_1, H_2} C_{H_2, H_1} = id_{H_2 \otimes H_1}$, it follows that $1_H \otimes h = \alpha(h_1) \otimes \alpha(h_2)$. Apply $id \otimes \epsilon$ to the identity, we have

$$\epsilon(h)1_H = \epsilon(\alpha(h_2))\alpha(h_1) = \epsilon(h_2)\alpha(h_1) = \alpha(h_1\epsilon(h_2)) = \alpha(\alpha^{-1}(h)) = h.$$

Therefore, $H = k$, as required. $\square$

In particular, if $\alpha = id$, Theorem 3.1 is nothing but the famous conclusion in Ref. 14: the canonical braiding of the category ${}^H_H\mathcal{YD}$ over a Hopf algebra is a symmetry only in the degenerate case $H = k$.

Next, we will discuss the pseudosymmetry on ${}^H_H\mathcal{YD}$ over a monoidal Hom-Hopf algebra (H, α) . For this purpose, we let H_1 and H_2 defined as above and give a useful lemma in the following.

Lemma 3.2. Let (H, α) be a cocommutative monoidal Hom-Hopf algebra. Then the canonical braiding C_{H_2, H_1} is the usual flip map.

Proof. For any $x \otimes y \in H_2 \otimes H_1$, we do a calculation as follows:

$$\begin{aligned} C_{H_2, H_1} (x \otimes y) &\stackrel{(1.11)}{=} x_{(-1)} \cdot \alpha^{-1}(y) \otimes \alpha(x_0) \\ &= (x_{11}\alpha^{-1}(S(x_2))) \cdot \alpha^{-1}(y) \otimes \alpha^2(x_{12}) \\ &= (x_{12}S(\alpha^{-1}(x_2))) \cdot \alpha^{-1}(y) \otimes \alpha^2(x_{11}) \\ &\stackrel{(1.4)}{=} (x_{21}S(x_{22})) \cdot \alpha^{-1}(y) \otimes \alpha x_1 \\ &= 1_H \cdot \alpha^{-1}(y) \otimes x = (1_H \alpha^{-2}(y))1_H \otimes x \\ &= y \otimes x. \end{aligned}$$

The second equality holds since $\rho(x) = x_{11}\alpha^{-1}(S(x_2)) \otimes \alpha(x_{12})$ and the third equality holds since H is cocommutative. $\square$

Theorem 3.3. Let (H, α) be a monoidal Hom-Hopf algebra. Then the canonical braiding of the category ${}^H_H\mathcal{YD}$ is pseudosymmetric if and only if (H, α) is commutative and cocommutative.

Proof. Assume that (H, α) is commutative and cocommutative. We first claim that the compatibility condition, Eq. (2.10), becomes the identity:

$$\rho(h \cdot m) = \alpha(m_{(-1)}) \otimes \alpha^{-1}(h) \cdot m_0,$$

for all $h \in H$ and $m \in (M, \mu) \in {}^H_H\mathcal{YD}$. In fact,

$$\begin{aligned} \rho(h \cdot m) &\stackrel{(1.10)}{=} (h_{11}\alpha^{-1}(m_{(-1)}))S(h_2) \otimes \alpha(h_{12}) \cdot m_0 \\ &\stackrel{(1.4)}{=} (\alpha^{-1}(h_1)\alpha^{-1}(m_{(-1)}))S(\alpha(h_{22})) \otimes \alpha(h_{21}) \cdot m_0 \\ &= (\alpha^{-1}(h_1)\alpha^{-1}(m_{(-1)}))S(\alpha(h_{21})) \otimes \alpha(h_{22}) \cdot m_0 \\ &\stackrel{(1.4)}{=} (h_{11}\alpha^{-1}(m_{(-1)}))S(\alpha(h_{12})) \otimes h_2 \cdot m_0 \\ &= (\alpha^{-1}(m_{(-1)})h_{11})S(\alpha(h_{12})) \otimes h_2 \cdot m_0 \\ &= m_{(-1)}(h_{11}S(h_{12})) \otimes h_2 \cdot m_0 \\ &\stackrel{(1.8)}{=} m_{(-1)}\epsilon(h_1)1_H \otimes h_2 \cdot m_0 \\ &= \alpha(m_{(-1)}) \otimes \alpha^{-1}(h) \cdot m_0. \end{aligned}$$

The third equality holds since (H, α) is cocommutative, the fifth equality holds since (H, α) is commutative.

Now we check that the canonical braiding is pseudosymmetric, for this purpose, we take any $x \in (X, \beta) \in {}^H_H\mathcal{HYD}$, $y \in (Y, \mu) \in {}^H_H\mathcal{HYD}$, $z \in (Z, \nu) \in {}^H_H\mathcal{HYD}$. On the one hand, we have

$$\begin{aligned}
 & (id_Z \otimes C_{X,Y})(C_{Z,X}^{-1} \otimes id_Y)(id_X \otimes C_{Y,Z})(x \otimes y \otimes z) \\
 \stackrel{(1.11)}{=} & (id_Z \otimes C_{X,Y})(C_{X,Y}^{-1} \otimes id)(x \otimes y_{(-1)} \cdot \mu^{-1}(z) \otimes \nu(y_0)) \\
 = & (id_Z \otimes C_{X,Y})(\mu(\alpha^{-1}(y_{(-1)}) \cdot \mu^{-1}(z_0)) \otimes S^{-1}(z_{(-1)}) \cdot \beta^{-1}(x) \otimes \nu(y_0)) \\
 = & (id_Z \otimes C_{X,Y})(y_{(-1)} \cdot z_0 \otimes S^{-1}(z_{(-1)}) \cdot \beta^{-1}(x) \otimes \nu(y_0)) \\
 = & y_{(-1)} \cdot z_0 \otimes x_{(-1)} \cdot y_0 \otimes \beta(S^{-1}(\alpha^{-1}(z_{(-1)})) \cdot \beta^{-1}(x_0)) \\
 = & y_{(-1)} \cdot z_0 \otimes x_{(-1)} \cdot y_0 \otimes S^{-1}(z_{(-1)}) \cdot x_0.
 \end{aligned}$$

The fourth equality holds since $\rho(S^{-1}(z_{(-1)}) \cdot \beta^{-1}(x)) = \alpha(\alpha^{-1}(x_{(-1)}) \otimes \alpha^{-1}(S^{-1}(z_{(-1)}))) \cdot \beta^{-1}(x_0) = x_{(-1)} \otimes S^{-1}(\alpha^{-1}(z_{(-1)})) \cdot \beta^{-1}(x_0)$. On the other hand, we have

$$\begin{aligned}
 & (C_{Y,Z} \otimes id_X)(id_Y \otimes C_{Z,X}^{-1})(C_{X,Y} \otimes id_Z)(x \otimes y \otimes z) \\
 = & (C_{Y,Z} \otimes id_X)(id_Y \otimes C_{Z,X}^{-1})(x_{(-1)} \cdot \nu^{-1}(y) \otimes \beta(x_0) \otimes z) \\
 = & (C_{Y,Z} \otimes id_X)(x_{(-1)} \cdot \nu^{-1}(y) \otimes \mu(z_0) \otimes S^{-1}(z_{(-1)}) \cdot x_0) \\
 = & y_{(-1)} \cdot \mu^{-1}(\mu(z_0)) \otimes \nu(\alpha^{-1}(x_{(-1)}) \cdot \nu^{-1}(y_0)) \otimes S^{-1}(z_{(-1)}) \cdot x_0 \\
 = & y_{(-1)} \cdot z_0 \otimes x_{(-1)} \cdot y_0 \otimes S^{-1}(z_{(-1)}) \cdot x_0.
 \end{aligned}$$

The third equality holds since $\rho(\nu^{-1}(y)) = \alpha^{-1}(y_{(-1)}) \otimes \nu^{-1}(y_0)$ and $\rho(x_{(-1)} \cdot \nu^{-1}(y)) = y_{(-1)} \otimes (\alpha^{-1}(x_{(-1)}) \cdot \nu^{-1}(y_0))$.

According to the arbitrary of x, y, z , it follows that $(id_Z \otimes C_{X,Y})(C_{Z,X}^{-1} \otimes id_Y)(id_X \otimes C_{Y,Z}) = (C_{Y,Z} \otimes id_X)(id_Y \otimes C_{Z,X}^{-1})(C_{X,Y} \otimes id_Z)$, that is, the canonical braiding C is pseudosymmetric.

Conversely, assume that the canonical braiding C is pseudosymmetric, we will first show that (H, α) is cocommutative. For any $1_H \otimes x \otimes 1_H \in H_2 \otimes H_1 \otimes H_2$, on the one side, we have

$$\begin{aligned}
 & (C_{H_1,H_2} \otimes id_{H_2})(id_{H_1} \otimes C_{H_2,H_2}^{-1})(C_{H_2,H_1} \otimes id_{H_2})(1_H \otimes x \otimes 1_H) \\
 = & (C_{H_1,H_2} \otimes id_{H_2})(id_{H_1} \otimes C_{H_2,H_2}^{-1})(x \otimes 1_H \otimes 1_H) \\
 = & (C_{H_1,H_2} \otimes id_{H_2})(x \otimes \alpha(1_H) \otimes S^{-1}(1_H) \cdot \alpha^{-1}(1_H)) \\
 = & x_{(-1)} \cdot \alpha^{-1}(1_H) \otimes \alpha(x_0) \otimes 1_H \\
 = & x_1 \cdot 1_H \otimes \alpha(x_2) \otimes 1_H \\
 = & \alpha(x_1) \otimes \alpha(x_2) \otimes 1_H.
 \end{aligned}$$

On the other side, we have

$$\begin{aligned}
 & (id_{H_2} \otimes C_{H_2,H_1})(C_{H_2,H_2}^{-1} \otimes id_{H_1})(id_{H_2} \otimes C_{H_1,H_2})(1_H \otimes x \otimes 1_H) \\
 = & (id_{H_2} \otimes C_{H_2,H_1})(C_{H_2,H_2}^{-1} \otimes id_{H_1})(1_H \otimes (x_{(-1)} \cdot \alpha^{-1}(1_H)) \otimes \alpha(x_0)) \\
 = & (id_{H_2} \otimes C_{H_2,H_1})(C_{H_2,H_2}^{-1} \otimes id_{H_1})(1_H \otimes (x_1 \cdot 1_H) \otimes \alpha(x_2)) \\
 = & (id_{H_2} \otimes C_{H_2,H_1})(C_{H_2,H_2}^{-1} \otimes id_{H_1})(1_H \otimes \alpha(x_1) \otimes \alpha(x_2)) \\
 = & (id_{H_2} \otimes C_{H_2,H_1})(\alpha^3(x_{112}) \otimes (S^{-1}(\alpha(x_{111})S(x_{12}))) \cdot \alpha^{-1}(1_H)) \otimes \alpha(x_2)) \\
 = & \alpha^3(x_{112}) \otimes \alpha(x_2) \otimes (S^{-1}(\alpha(x_{111})S(x_{12}))) \cdot \alpha^{-1}(1_H)) \\
 = & \alpha^3(x_{112}) \otimes \alpha(x_2) \otimes \alpha(x_{12})S^{-1}(\alpha^2(x_{111})).
 \end{aligned}$$

By assumption, ${}^H_H\mathcal{HYD}$ is pseudosymmetric, it follows that

$$\alpha(x_1) \otimes \alpha(x_2) \otimes 1_H = \alpha^3(x_{112}) \otimes \alpha(x_2) \otimes \alpha(x_{12})S^{-1}(\alpha^2(x_{111})).$$

By applying $id \otimes \epsilon \otimes id$, we get $x \otimes 1_H = \alpha^2(x_{12}) \otimes x_2S^{-1}(\alpha(x_{11}))$. Therefore, we have

$$\begin{aligned} x_2 \otimes x_1 &= x_2 \otimes \alpha^{-1}(1x_1) \\ &= \alpha^2(x_{212}) \otimes \alpha^{-1}((x_{22}S^{-1}(\alpha(x_{211})))x_1) \\ &= \alpha^2(x_{212}) \otimes \alpha^{-1}(\alpha(x_{22})(S^{-1}(\alpha(x_{211}))\alpha^{-1}(x_1))) \\ &= \alpha^2(x_{122}) \otimes \alpha^{-1}(x_2(S^{-1}(\alpha(x_{121})))x_{11})) \\ &= \alpha(x_{12}) \otimes \alpha^{-1}(x_2(S^{-1}(\alpha(x_{112}))\alpha(x_{111}))) \\ &= \alpha(x_{12}) \otimes \alpha^{-1}(x_2\epsilon(\alpha(x_{11}))1_H) \\ &= x_1 \otimes x_2. \end{aligned}$$

So (H, α) is cocommutative. Next, we will prove that (H, α) is commutative. For any $1_H \otimes x \otimes y \in H_2 \otimes H_1 \otimes H_1$, on the one side, we have

$$\begin{aligned} &(C_{H_1, H_1} \otimes id_{H_2})(id_{H_1} \otimes C_{H_1, H_2}^{-1})(C_{H_2, H_1} \otimes id_{H_1})(1_H \otimes x \otimes y) \\ &= (C_{H_1, H_1} \otimes id_{H_2})(id_{H_1} \otimes C_{H_1, H_2}^{-1})(1_H \cdot \alpha^{-1}(x) \otimes 1_H \otimes y) \\ &= (C_{H_1, H_1} \otimes id_{H_2})(id_{H_1} \otimes C_{H_1, H_2}^{-1})((1_H \alpha^{-2}(x))S(\alpha(1_H)) \otimes 1_H \otimes y) \\ &= (C_{H_1, H_1} \otimes id_{H_2})(id_{H_1} \otimes C_{H_1, H_2}^{-1})(x \otimes 1_H \otimes y) \\ &= (C_{H_1, H_1} \otimes id_{H_2})(x \otimes \alpha(y_0) \otimes S^{-1}(y_{(-1)}) \cdot 1_H) \\ &= (C_{H_1, H_1} \otimes id_{H_2})(x \otimes \alpha(y_2) \otimes S^{-1}(y_1) \cdot 1_H) \\ &= (C_{H_1, H_1} \otimes id_{H_2})(x \otimes \alpha(y_2) \otimes S^{-1}(\alpha(y_1))) \\ &= x_{(-1)} \cdot y_2 \otimes \alpha(x_0) \otimes S^{-1}(\alpha(y_1)) \\ &= x_1 \cdot y_2 \otimes \alpha(x_2) \otimes S^{-1}(\alpha(y_1)) \\ &= (x_{11}\alpha^{-1}(y_2))S(\alpha(x_{12})) \otimes \alpha(x_2) \otimes S^{-1}(\alpha(y_1)). \end{aligned}$$

On the other side, we have

$$\begin{aligned} &(id_{H_1} \otimes C_{H_2, H_1})(C_{H_1, H_2}^{-1} \otimes id_{H_1})(id_{H_2} \otimes C_{H_1, H_1})(1_H \otimes x \otimes y) \\ &= (id_{H_1} \otimes C_{H_2, H_1})(C_{H_1, H_2}^{-1} \otimes id_{H_1})(1_H \otimes x_{(-1)} \cdot \alpha^{-1}(y) \otimes \alpha(x_0)) \\ &= (id_{H_1} \otimes C_{H_2, H_1})(C_{H_1, H_2}^{-1} \otimes id_{H_1})(1_H \otimes x_1 \cdot \alpha^{-1}(y) \otimes \alpha(x_2)) \\ &= (id_{H_1} \otimes C_{H_2, H_1})(C_{H_1, H_2}^{-1} \otimes id_{H_1})(1_H \otimes (x_1\alpha^{-2}(y))S(\alpha(x_{12})) \otimes \alpha(x_2)) \\ &= (id_{H_1} \otimes C_{H_2, H_1})(\alpha((x_{112}\alpha^{-2}(y_2))S(\alpha(x_{121}))) \otimes S^{-1}(\alpha((x_{111}\alpha^{-2}(y_1))S(\alpha(x_{122})))) \otimes \alpha(x_2)) \\ &= \alpha((x_{112}\alpha^{-2}(y_2))S(\alpha(x_{121}))) \otimes \alpha(x_2) \otimes S^{-1}(\alpha((x_{111}\alpha^{-2}(y_1))S(\alpha(x_{122})))) \end{aligned}$$

Since ${}^H_H\mathcal{HYD}$ is pseudosymmetric, it follows that

$$\begin{aligned} &((x_{11}\alpha^{-1}(y_2))S(\alpha(x_{12}))) \otimes \alpha(x_2) \otimes S^{-1}(\alpha(y_1)) \\ &= \alpha((x_{112}\alpha^{-2}(y_2))S(\alpha(x_{121}))) \otimes \alpha(x_2) \otimes S^{-1}(\alpha((x_{111}\alpha^{-2}(y_1))S(\alpha(x_{122})))) \end{aligned}$$

By applying $(\epsilon \otimes id \otimes id)(id \otimes id \otimes S)$, we get

$$1_k \otimes x \otimes y = 1_k \otimes \alpha(x_2) \otimes ((x_{11}\alpha^{-2}(y))S(\alpha(x_{12}))).$$

Hence, we have

$$\begin{aligned}
 yx &= ((x_{11}\alpha^{-2}(y))S(\alpha(x_{12})))\alpha(x_2) \\
 &= ((\alpha^{-1}(x_1)\alpha^{-2}(y))S(\alpha(x_{21})))\alpha^2(x_{22}) \\
 &= (x_1\alpha^{-1}(y))(S(\alpha(x_{21}))\alpha(x_{22})) \\
 &= (x_1\alpha^{-1}(y))\epsilon(x_2)1_H \\
 &= (\alpha^{-1}(x)\alpha^{-1}(y))1_H \\
 &= xy.
 \end{aligned}$$

So H is cocommutative, as required. And this completes the proof. $\square$

IV. THE u -CONDITION

The u -condition is a classical definition in Hopf algebras. It appeared firstly in Ref. 5 on $M \in {}_H\mathcal{M}$ given by: $h \cdot m = m$ for all $h \in H$ and $m \in M$. Later, the notion of the u -condition has been extended to Yetter-Drinfeld modules in Ref. 4. We define now its generalization to any monoidal Hom-Hopf algebra (H, α) and (H, α) -Hom-Yetter-Drinfeld module (M, μ) .

Definition 4.1. Let (H, α) be a monoidal Hom-Hopf algebra and $(M, \mu) \in {}^H_H\mathcal{HYD}$. Then M is said to satisfy the u -condition if

$$m_{(-1)} \cdot m_0 = m, \quad (4.1)$$

for all $m \in M$.

Proposition 4.2. Let (H, α) be a monoidal Hom-Hopf algebra and H_1, H_2 defined as above. Then

- (1) H_1 satisfies the u -condition if and only if $S^2 = id$.
- (2) H_2 satisfies the u -condition if and only if $S^2 = id$.

Proof. Similar to usual Hopf algebras, one may check routinely that $S^2 = id$ if and only if $S(h_2)h_1 = \epsilon(h)1_H$ or $h_2S(h_1) = \epsilon(h)1_H$ for all $h \in H$.

(1) Assume that $S^2 = id$, then for any $h \in H_1$, we have $h_{(-1)} \cdot h_0 = h_1 \cdot h_2 = (h_{11}\alpha^{-1}(h_2))S(\alpha(h_{12})) = (\alpha^{-1}(h_1)h_{22})S(\alpha(h_{21})) = h_1(h_{22}S(h_{21})) = h_1(\epsilon(h_2)1_H) = h$. So H_1 satisfies the u -condition.

Conversely, if H_1 satisfies the u -condition, then for any $h \in H_1$, we get

$$\begin{aligned}
 h_2S(h_1) &= \epsilon(h_1)\alpha(h_{22}S(h_{21})) \\
 &= (\epsilon(h_1)1_H)(h_{22}S(h_{21})) \\
 &\stackrel{(1.4)}{=} (\epsilon(\alpha(h_{11}))1_H)(\alpha^{-1}(h_2)S(h_{12})) \\
 &\stackrel{(1.8)}{=} (S(\alpha(h_{11}))\alpha(h_{12}))(\alpha^{-1}(h_2)S(h_{12})) \\
 &= S(\alpha^2(h_{11}))(\alpha(h_{12})\alpha^{-1}(\alpha^{-1}(h_2)S(h_{12}))) \\
 &\stackrel{(1.1)}{=} S(\alpha(h_{11}))(\alpha(h_{121})(\alpha^{-2}(h_2)S(h_{122}))) \\
 &\stackrel{(1.1)}{=} S(\alpha(h_{11}))((h_{121}\alpha^{-2}(h_2))S(\alpha(h_{122}))) \\
 &\stackrel{(1.4)}{=} S(h_1)((h_{211}\alpha^{-1}(h_{22}))S(\alpha(h_{212}))) \\
 &= S(h_1)h_2 = \epsilon(h)1_H.
 \end{aligned}$$

Hence $S^2 = id$.

(2) Assume that $S^2 = id$, then for any $h \in H_2$, we have $h_{(-1)} \cdot h_0 = (h_{11}\alpha^{-1}(S(h_2)))\alpha(h_{12}) = (\alpha^{-1}(h_1)S(h_{22}))\alpha(h_{21}) = h_1(S(h_{22})h_{21}) = h_1(\epsilon(h_2)1_H) = h$. So H_2 satisfies the u -condition.

Conversely, if H_2 satisfies the u -condition, then for any $h \in H_2$, we get

$$\begin{aligned}
 S(h_2)h_1 &= \epsilon(h_1)\alpha(S(h_{22})h_{21}) \\
 &= (\epsilon(h_1)1_H)(S(h_{22})h_{21}) \\
 &\stackrel{(1.4)}{=} (\epsilon(\alpha(h_{11})1_H))(S(\alpha^{-1}(h_2))h_{12}) \\
 &\stackrel{(1.8)}{=} (S(\alpha(h_{11}))\alpha(h_{12}))(S(\alpha^{-1}(h_2))h_{12}) \\
 &\stackrel{(1.1)}{=} S(\alpha^2(h_{11}))(\alpha(h_{12})(S(\alpha^{-2}(h_2))\alpha^{-1}(h_{12}))) \\
 &= S(\alpha(h_{11}))(\alpha(h_{12})(S(\alpha^{-2}(h_2))h_{122})) \\
 &= S(\alpha(h_{11}))((h_{121}S(\alpha^{-2}(h_2))\alpha(h_{122})) \\
 &\stackrel{(1.4)}{=} S(h_1)((h_{211}S(\alpha^{-1}(h_{22})))\alpha(h_{212})) \\
 &= S(h_1)h_2 = \epsilon(h)1_H.
 \end{aligned}$$

Hence $S^2 = id$. The proof is finished. $\square$

Proposition 4.3. Let (H, α) be a monoidal Hom-Hopf algebra and $(M, \mu), (N, \nu) \in {}^H_H\mathcal{HYD}$. Assume that M and N satisfy the u -condition, then $M \otimes N$ satisfies the u -condition if and only if $C_{M,N}$ is a symmetry.

Proof. Recall from⁸ $(M \otimes N, \mu \otimes \nu)$ is a left-left (H, α) -Hom-Yetter-Drinfeld module with structures:

$$\begin{aligned}
 h \cdot (m \otimes n) &= h_1 \cdot m \otimes h_2 \cdot n, \\
 \rho(m \otimes n) &= m_{(-1)}n_{(-1)} \otimes m_0 \otimes n_0,
 \end{aligned}$$

for all $m \in M, n \in N$, and $h \in H$. To show the necessary and sufficient condition in the proposition, we do the following computation:

$$\begin{aligned}
 (m_{(-1)}n_{(-1)}) \cdot (m_0 \otimes n_0) &= (m_{(-1)1}n_{(-1)1}) \cdot m_0 \otimes (m_{(-1)2}n_{(-1)2}) \cdot n_0 \\
 &= (m_{(-1)1}n_{(-1)1}) \cdot m_0 \otimes \alpha(m_{(-1)2}) \cdot (n_{(-1)2} \cdot \nu^{-1}(n_0)) \\
 &\stackrel{(1.6)}{=} (m_{(-1)1}\alpha^{-1}(n_{(-1)})) \cdot m_0 \otimes \alpha(m_{(-1)2}) \cdot (n_{0(-1)} \cdot n_{00}) \\
 &\stackrel{(3.1)}{=} (m_{(-1)1}\alpha^{-1}(n_{(-1)})) \cdot m_0 \otimes \alpha(m_{(-1)2}) \cdot n_0 \\
 &\stackrel{(1.3)}{=} \alpha(m_{(-1)1}) \cdot (\alpha^{-1}(n_{(-1)}) \cdot \mu^{-1}(m_0)) \otimes \alpha(m_{(-1)2}) \cdot n_0 \\
 &= \alpha(m_{(-1)}) \cdot (\alpha^{-1}(n_{(-1)}) \cdot \mu^{-1}(m_0)) \otimes \alpha(m_{(-1)2}) \cdot n_0 \\
 &\stackrel{(1.11)}{=} \alpha(m_{(-1)}) \cdot C_{N,M}(\nu^{-1}(n) \otimes m_0) \\
 &= C_{N,M}(\alpha(m_{(-1)}) \cdot (\nu^{-1}(n) \otimes m_0)) \\
 &= C_{N,M}(\alpha(m_{(-1)1}) \cdot \nu^{-1}(n) \otimes \alpha(m_{(-1)2}) \cdot m_0) \\
 &= C_{N,M}(m_{(-1)} \cdot \nu^{-1}(n) \otimes \alpha(m_{0(-1)}) \cdot \mu(m_{00})) \\
 &\stackrel{(3.1)}{=} C_{N,M}(m_{(-1)} \cdot \nu^{-1}(n) \otimes \mu(m_0)) \\
 &= C_{N,M}C_{M,N}(m \otimes n).
 \end{aligned}$$

Obviously, $M \otimes N$ satisfies the u -condition if and only if $C_{M,N}$ is a symmetry. This finishes the proof. $\square$

V. HOM-YETTER-DRINFELD CATEGORIES OVER QUASITRIANGULAR HOM-HOPF ALGEBRAS

In this section, we mainly consider symmetries in Hom-Yetter-Drinfeld category ${}^H_H\mathcal{HYD}$ over a monoidal Hom-Hopf algebra (H, α) . We show that the braiding $C_{H,H}$ is a symmetry forces H to be triangular in many cases.

In Ref. 17, a quasitriangular Hom-Hopf algebra is a monoidal Hom-Hopf algebra (H, α) together with an invertible element $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$ such that the following conditions hold:

$$\begin{aligned} (QT1) \quad & \Delta(R^{(1)}) \otimes \alpha(R^{(2)}) = \alpha(R^{(1)}) \otimes \alpha(R^{(2)}) \otimes R^{(2)}r^{(2)}, \\ (QT2) \quad & \alpha(R^{(1)}) \otimes \Delta(R^{(2)}) = R^{(1)}r^{(1)} \otimes \alpha(R^{(2)}) \otimes \alpha(R^{(2)}), \\ (QT3) \quad & \epsilon(R^{(1)})R^{(2)} = 1_H, \quad R^{(1)}\epsilon(R^{(2)}) = 1_H, \\ (QT4) \quad & \Delta^{\text{cop}}(h)R = R\Delta(h), \end{aligned}$$

where $\Delta^{\text{cop}}(h) = h_2 \otimes h_1$ for all $h \in H$. A quasitriangular Hom-Hopf algebra (H, R, α) is called triangular if $R^{-1} = R^{(2)} \otimes R^{(1)}$.

Theorem 5.1. Let (H, R, α) be a quasitriangular Hom-Hopf algebra. Assume that α is an involution and R is α -invariant (i.e., $\alpha(R^{(1)}) \otimes \alpha(R^{(2)}) = R^{(1)} \otimes R^{(2)}$). Then the category $\tilde{\mathcal{H}}(H, \mathcal{M})$ of left (H, α) -Hom-modules is a Hom-Yetter-Drinfeld subcategory of ${}^H_H\mathcal{HYD}$ under the comodule coaction $\rho(m) = \alpha(R^{(2)}) \otimes R^{(1)} \cdot \mu^{-2}(m)$, where $\cdot$ is the module action of (M, μ) and $m \in (M, \mu) \in \tilde{\mathcal{H}}(H, \mathcal{M})$.

Proof. We first show that (M, ρ, μ) is a left (H, α) -Hom-comodule. For any $m \in (M, \mu)$, it is clear that $\epsilon(m_{(-1)})m_0 = \epsilon(\alpha(R^{(2)}))(R^{(1)} \cdot \mu^{-2}(m)) = \epsilon(R^{(2)})(R^{(1)} \cdot \mu^{-2}(m)) = 1_H \cdot \mu^{-2}(m) = \mu^{-1}(m)$. By the comodule coaction defined above, we have

$$\begin{aligned} \Delta(m_{(-1)}) \otimes \mu^{-1}(m_0) &= \Delta(\alpha(R^{(2)})) \otimes \mu^{-1}(R^{(1)} \cdot \mu^{-2}(m)) \\ &= (\alpha \otimes \alpha)\Delta(R^{(2)}) \otimes \alpha^{-1}(R^{(1)}) \cdot \mu^{-3}(m) \\ &\stackrel{(QT2)}{=} (\alpha \otimes \alpha)(\alpha(R^{(2)}) \otimes \alpha(R^{(2)})) \otimes \alpha^{-2}(R^{(1)}r^{(1)}) \cdot \mu^{-3}(m) \\ &= \alpha^2(R^{(2)}) \otimes \alpha^2(R^{(2)}) \otimes \alpha^{-2}(R^{(1)}r^{(1)}) \cdot \mu^{-3}(m) \end{aligned}$$

and

$$\begin{aligned} \alpha^{-1}(m_{(-1)}) \otimes \rho(m_0) &= \alpha^{-1}(\alpha(R^{(2)})) \otimes \rho(R^{(1)} \cdot \mu^{-2}(m)) \\ &= R^{(2)} \otimes \alpha(R^{(2)}) \otimes R^{(1)} \cdot \mu^{-2}(R^{(1)} \cdot \mu^{-2}(m)) \\ &\stackrel{(1.7)}{=} R^{(2)} \otimes \alpha(R^{(2)}) \otimes R^{(1)} \cdot (\alpha^{-2}(R^{(1)}) \cdot \mu^{-4}(m)) \\ &= \alpha^2(R^{(2)}) \otimes \alpha^2(R^{(2)}) \otimes \alpha(R^{(1)}) \cdot (R^{(1)} \cdot \mu^{-4}(m)) \\ &\stackrel{(1.3)}{=} \alpha^2(R^{(2)}) \otimes \alpha^2(R^{(2)}) \otimes (R^{(1)}R^{(1)}) \cdot \mu^{-3}(m). \end{aligned}$$

Since α is an involution, it follows that $\Delta(m_{(-1)}) \otimes \mu^{-1}(m_0) = \alpha^{-1}(m_{(-1)}) \otimes \rho(m_0)$. Similarly, we get

$$\begin{aligned} \rho(\mu(m)) &= \alpha(R^{(2)}) \otimes R^{(1)} \cdot \mu^{-2}(\mu(m)) \\ &= \alpha(R^{(2)}) \otimes R^{(1)} \cdot \mu^{-1}(m) \\ &= \alpha^2(R^{(2)}) \otimes \alpha(R^{(1)}) \cdot \mu^{-1}(m) \\ &\stackrel{(1.7)}{=} \alpha(\alpha(R^{(2)})) \otimes \mu(R^{(1)} \cdot \mu^{-2}(m)) \\ &= \alpha(m_{(-1)}) \otimes \mu(m_0). \end{aligned}$$

The third equality holds since R is α -invariant. Hence (M, ρ, μ) is a left (H, α) -Hom-comodule.

Now we prove that (M, ρ, μ) satisfies the compatible condition, Eq. (2.10). Indeed, for any $h \in H$ and $m \in M$, on the one hand, we have

$$\begin{aligned} \rho(h \cdot m) &= \alpha(R^{(2)}) \otimes R^{(1)} \cdot \mu^{-2}(h \cdot m) \\ &\stackrel{(1.7)}{=} \alpha(R^{(2)}) \otimes R^{(1)} \cdot (\alpha^{-2}(h) \cdot \mu^{-2}(m)) \\ &= \alpha^2(R^{(2)}) \otimes \alpha(R^{(1)}) \cdot (\alpha^{-2}(h) \cdot \mu^{-2}(m)) \\ &\stackrel{(1.3)}{=} \alpha^2(R^{(2)}) \otimes (R^{(1)} \alpha^{-2}(h)) \cdot \mu^{-1}(m). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &(h_{11} \alpha^{-1}(m_{(-1)})) S(h_2) \otimes \alpha(h_{12}) \cdot m_0 \\ &= (h_{11} \alpha^{-1}(\alpha(R^{(2)}))) S(h_2) \otimes \alpha(h_{12}) \cdot (R^{(1)} \cdot \mu^{-2}(m)) \\ &\stackrel{(1.3)}{=} (h_{11} R^{(2)}) S(h_2) \otimes (h_{12} R^{(1)}) \cdot \mu^{-1}(m) \\ &\stackrel{(QT4)}{=} (R^{(2)} h_{12}) S(h_2) \otimes (R^{(1)} h_{11}) \cdot \mu^{-1}(m) \\ &\stackrel{(1.4)}{=} (R^{(2)} h_{21}) S(\alpha(h_{22})) \otimes (R^{(1)} \alpha^{-1}(h_1)) \cdot \mu^{-1}(m) \\ &\stackrel{(1.1)}{=} \alpha(R^{(2)})(h_{21} S(h_{22})) \otimes (R^{(1)} \alpha^{-1}(h_1)) \cdot \mu^{-1}(m) \\ &\stackrel{(1.8)}{=} \alpha(R^{(2)}) \epsilon(h_2) 1 \otimes (R^{(1)} \alpha^{-1}(h_1)) \cdot \mu^{-1}(m) \\ &\stackrel{(1.5)}{=} \alpha^2(R^{(2)}) \otimes (R^{(1)} \alpha^{-2}(h)) \cdot \mu^{-1}(m). \end{aligned}$$

So $\rho(h \cdot m) = (h_{11} \alpha^{-1}(m_{(-1)})) S(h_2) \otimes \alpha(h_{12}) \cdot m_0$, that is, (M, ρ, μ) is a left-left (H, α) -Hom-Yetter-Drinfeld module.

Finally, we verify that any morphism in $\tilde{\mathcal{H}}(H\mathcal{M})$ is left (H, α) -colinear, too. For this purpose, we take any $m \in (M, \mu) \in \tilde{\mathcal{H}}(H\mathcal{M})$ and $n \in (N, \nu) \in \tilde{\mathcal{H}}(H\mathcal{M})$. Assume that $f: (M, \mu) \rightarrow (N, \nu)$ is a morphism in $\tilde{\mathcal{H}}(H\mathcal{M})$, then

$$\begin{aligned} (id_H \otimes f)\rho(m) &= \alpha(R^{(2)}) \otimes f(R^{(1)} \cdot \mu^{-2}(m)) \\ &= \alpha(R^{(2)}) \otimes R^{(1)} \cdot f(\mu^{-2}(m)) \\ &= \alpha(R^{(2)}) \otimes R^{(1)} \cdot \mu^{-2}(f(m)) \\ &= \rho(f(m)). \end{aligned}$$

The second and third equality hold since f is a left (H, α) -linear morphism of Hom-modules. So f is left (H, α) -colinear, as desired. And this finishes the proof. $\square$

Proposition 5.2. Let (H, R, α) be a triangular Hom-Hopf algebra. Assume that α is an involution and R is α -invariant. Then the Hom-Yetter-Drinfeld subcategory $\tilde{\mathcal{H}}(H\mathcal{M})$ as defined in Theorem 4.1 is symmetric.

Proof. For any $m \in (M, \mu) \in \tilde{\mathcal{H}}(H\mathcal{M})$ and $n \in (N, \nu) \in \tilde{\mathcal{H}}(H\mathcal{M})$,

$$\begin{aligned} C_{N,M} C_{M,N}(m \otimes n) &= C_{N,M}(\alpha(R^{(2)}) \cdot \nu^{-1}(n) \otimes \mu(R^{(1)} \cdot \mu^{-2}(m))) \\ &= \alpha(r^{(2)}) \cdot \mu^{-1}(\mu(R^{(1)} \cdot \mu^{-2}(m))) \otimes \nu(r^{(1)} \cdot \nu^{-2}(\alpha(R^{(2)}) \cdot \nu^{-1}(n))) \\ &= \alpha(r^{(2)}) \cdot (R^{(1)} \cdot \mu^{-2}(m)) \otimes \alpha(r^{(1)}) \cdot \nu^{-1}(\alpha(R^{(2)}) \cdot \nu^{-1}(n)) \\ &\stackrel{(1.7)}{=} \alpha(r^{(2)}) \cdot (R^{(1)} \cdot \mu^{-2}(m)) \otimes \alpha(r^{(1)}) \cdot (R^{(2)} \cdot \nu^{-2}(n)) \\ &\stackrel{(1.3)}{=} (r^{(2)} R^{(1)}) \cdot \mu^{-1}(m) \otimes (r^{(1)} R^{(2)}) \cdot \nu^{-1}(n) \\ &= 1_H \cdot \mu^{-1}(m) \otimes 1_H \cdot \nu^{-1}(n) \\ &= m \otimes n. \end{aligned}$$

The sixth equality holds since (H, R, α) is triangular. So the braiding is a symmetry, as required. $\square$

Theorem 5.3. Let (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode. Assume that (H, m, ρ, α) is a left-left (H, α) -Hom-Yetter-Drinfeld module and α an involution. Then $C_{H,H}$ is a symmetry if and only if there exists $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$ such that (H, R, α) is a triangular monoidal Hom-Hopf algebra. Furthermore, ρ is induced by R , that is, $\rho(h) = \alpha(R^{(2)}) \otimes R^{(1)} \mu^{-2}(h)$, for all $h \in H$.

Proof. Assume that $C_{H,H}$ is a symmetry. Since $C_{H,H}(\alpha^{-1}(h) \otimes 1_H) = (\alpha^{-1}(h_{(-1)}) \cdot 1_H) \otimes \alpha(\alpha^{-1}(h_0)) = h_{(-1)} \otimes h_0$, it follows that $\rho(h) = C_{H,H}(\alpha^{-1}(h) \otimes 1_H)$ for all $h \in H$. For any $h, g \in H$, by the definition of $C_{H,H}$ and $C_{H,H}^{-1}$,

$$\begin{aligned} C_{H,H}(h \otimes g) &= h_{(-1)} \cdot \alpha^{-1}(g) \otimes \alpha(h_0) = h_{(-1)} \alpha^{-1}(g) \otimes \alpha(h_0), \\ C_{H,H}^{-1}(h \otimes g) &= \alpha(g_0) \otimes S^{-1}(g_{(-1)}) \cdot \alpha^{-1}(h) = \alpha(g_0) \otimes S^{-1}(g_{(-1)}) \alpha^{-1}(h). \end{aligned}$$

Since $C_{H,H}$ is a symmetry, we have

$$C_{H,H}(h \otimes g) = h_{(-1)} \alpha^{-1}(g) \otimes \alpha(h_0) = \alpha(g_0) \otimes S^{-1}(g_{(-1)}) \alpha^{-1}(h). \quad (5.1)$$

In particular, let $\rho(1_H) = x_i \otimes y_i$, then

$$x_i \otimes y_i = \rho(1_H) = C_{H,H}(\alpha^{-1}(1_H) \otimes 1_H) = C_{H,H}(1_H \otimes 1_H) = \alpha(y_i) \otimes \alpha(S^{-1}(x_i)). \quad (5.2)$$

Applying $id \otimes S$ to Eq. (5.2), we obtain

$$x_i \otimes S(y_i) = \alpha(y_i) \otimes \alpha(x_i).$$

Since $\alpha(x_i) \otimes \alpha(y_i) = \rho(\alpha(1_H)) = \rho(1_H) = x_i \otimes y_i$, it is clearly that

$$x_i \otimes S(y_i) = y_i \otimes x_i. \quad (5.3)$$

Set $R = y_i \otimes x_i$. We will show that (H, R, α) is a triangular monoidal Hom-Hopf algebra and that ρ is induced by R . For this purpose, we first note that $\rho(h) = \alpha(x_i) \otimes y_i \alpha^{-2}(h)$ for all $h \in H$. In fact, $\rho(h) = C_{H,H}(\alpha^{-1}(h) \otimes 1_H) = \alpha(y_i) \otimes S^{-1}(x_i) \alpha^{-2}(h) = \alpha(x_i) \otimes y_i \alpha^{-2}(h)$.

Second, we check that R is invertible as follows:

$$\begin{aligned} 1_H \otimes 1_H &= C_{H,H}^2(1_H \otimes 1_H) = C_{H,H}(x_i \otimes y_i) \\ &= x_{i(-1)} \alpha^{-1}(y_i) \otimes \alpha(x_{i0}) \\ &= (x_{i(-1)} \otimes x_{i0})(\alpha^{-1}(y_i) \otimes 1_H) \\ &= \rho(x_i)(\alpha^{-1}(y_i) \otimes 1_H) \\ &= (\alpha(x_j) \otimes y_j \alpha^{-2}(x_i))(\alpha^{-1}(y_i) \otimes 1_H) \\ &= \alpha(x_j) \alpha^{-1}(y_i) \otimes (y_j \alpha^{-2}(x_i)) 1_H \\ &= \alpha(x_j) \alpha^{-1}(y_i) \otimes \alpha(y_j) \alpha^{-1}(x_i) \\ &= x_j y_i \otimes y_j x_i = (x_j \otimes y_j)(y_i \otimes x_i). \end{aligned}$$

Hence $R^{-1} = x_j \otimes y_j$. Next, we will verify that R satisfies (QT1), that is, $\Delta(y_i) \otimes \alpha(x_i) = \alpha(y_i) \otimes \alpha(y_j) \otimes x_i x_j$. Indeed,

$$\begin{aligned} \Delta(y_i) \otimes \alpha(x_i) &= \Delta(x_i) \otimes \alpha(S(y_i)) \\ &= (id_H \otimes id_H \otimes S)(\Delta \otimes \alpha)(x_i \otimes y_i) \\ &= (id_H \otimes id_H \otimes S \alpha^2)(\Delta \otimes \alpha^{-1}) \rho(1_H) \end{aligned}$$

$$\begin{aligned}
&= (id_H \otimes id_H \otimes S\alpha^2)(\alpha^{-1} \otimes \rho)(1_H) \\
&= (id_H \otimes id_H \otimes S\alpha^2)(\alpha^{-1}(x_i) \otimes \rho(y_i)) \\
&= (id_H \otimes id_H \otimes S\alpha^2)(\alpha^{-1}(x_i) \otimes \alpha(x_j) \otimes y_j \alpha^{-2}(y_i)) \\
&= (id_H \otimes id_H \otimes S\alpha^2)(\alpha(x_i) \otimes \alpha(x_j) \otimes y_j y_i) \\
&= (id_H \otimes id_H \otimes \alpha^2 S)(\alpha(y_i) \otimes \alpha(y_j) \otimes S^{-1}(x_j) S^{-1}(x_i)) \\
&= (id_H \otimes id_H \otimes \alpha^2)(\alpha(y_i) \otimes \alpha(y_j) \otimes x_i x_j) \\
&= \alpha(y_i) \otimes \alpha(y_j) \otimes x_i x_j.
\end{aligned}$$

The last equality holds since α an involution. Similarly, one may check that R satisfies (QT3).

Since (H, m, ρ, α) is a left-left (H, α) -Hom-Yetter-Drinfeld module, $\epsilon(h_{(-1)})h_0 = \alpha^{-1}(h)$ for all $h \in H$. Let $h = 1_H$, we have $\epsilon(x_i)y_i = \alpha^{-1}(1_H) = 1_H$ and $x_i\epsilon(y_i) = \epsilon(x_i)S(y_i) = S(\epsilon(x_i)y_i) = \alpha^{-1}(1_H) = 1_H$. So (QT2) and (QT4) hold.

Now we verify (QT5), that is, $h_1x_i \otimes h_2y_i = x_ih_2 \otimes y_ih_1$ for all $h \in H$. In fact,

$$\begin{aligned}
h_1x_i \otimes h_2y_i &= h \cdot (x_i \otimes y_i) = h \cdot \rho(1_H) \\
&= h \cdot C_{H,H}(1_H \otimes 1_H) \\
&= C_{H,H}(h \cdot (1_H \otimes 1_H)) \\
&= C_{H,H}(\alpha(h_1) \otimes \alpha(h_2)) \\
&= (\alpha(h_1)_{(-1)} \cdot \alpha^{-1}(\alpha(h_2))) \otimes \alpha(\alpha(h_1)_0) \\
&= \alpha(h_{1(-1)}) \cdot h_2 \otimes \alpha^2(h_{10}) \\
&= \alpha^2(x_i)h_2 \otimes \alpha^2(y_i\alpha^{-2}(h_1)) \\
&= \alpha^2(x_i)h_2 \otimes \alpha^2(y_i)h_1 \\
&= x_ih_2 \otimes y_ih_1.
\end{aligned}$$

The last equality holds since α an involution. And this completes the proof. $\square$

Corollary 5.4. Let (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode. Assume that $(H, m, \text{coadjoint}, \alpha)$ is a left-left (H, α) -Hom-Yetter-Drinfeld module and α an involution. Then $C_{H,H}$ is a symmetry if and only if (H, α) is cocommutative.

Proof. Assume that $C_{H,H}$ is a symmetry, then by Theorem 5.1 (H, R, α) is triangular with $\rho(1_H) = R^\tau$, where τ denotes the usual flip map. But now the coaction is defined by the coadjoint coaction, hence $\rho(1_H) = (1_H)_{11}\alpha^{-1}(S((1_H)_2)) \otimes \alpha((1_H)_{12}) = 1 \otimes 1_H$, so $R = 1_H \otimes 1_H$. Therefore QT3 implies that (H, α) is cocommutative.

Conversely, if (H, α) is cocommutative, then

$$\begin{aligned}
\rho(h) &= h_{11}\alpha^{-1}(S(h_2)) \otimes \alpha(h_{12}) \\
&= h_{12}\alpha^{-1}(S(h_2)) \otimes \alpha(h_{11}) \\
&= h_{21}S(h_{22}) \otimes h_1 = \epsilon(h_2)1_H \otimes h_1 \\
&= 1_H \otimes \alpha^{-1}(h).
\end{aligned}$$

So, $C_{H,H}(h \otimes g) = h_{(-1)}\alpha^{-1}(g) \otimes \alpha(h_0) = 1_H\alpha^{-1}(g) \otimes \alpha^{-1}(\alpha(h)) = g \otimes h$, i.e., $C_{H,H}$ is a symmetry, as required. $\square$

VI. HOM-YETTER-DRINFELD CATEGORIES OVER COQUASITRIANGULAR HOM-HOPF ALGEBRAS

Dual to quasitriangular Hom-Hopf algebras, a coquasitriangular Hom-Hopf algebra is a monoidal Hom-Hopf algebra (H, α) together with a bilinear form $\langle | \rangle$ on (H, α) (i.e., $\langle | \rangle \in \text{Hom}(H \otimes$

$H, k)$) such that the following axioms hold:

$$\begin{aligned} (BR1) \langle hg|\alpha(l) \rangle &= \langle \alpha(h)|l_2 \rangle \langle \alpha(g)|l_1 \rangle, \\ (BR2) \langle \alpha(h)|gl \rangle &= \langle h_1|\alpha(g) \rangle \langle h_2|\alpha(l) \rangle, \\ (BR3) \langle h_1|g_1 \rangle g_2 h_2 &= h_1 g_1 \langle h_2|g_2 \rangle, \\ (BR4) \langle 1|h \rangle &= \langle h|1 \rangle = \epsilon(h), \end{aligned}$$

for all $h, g, l \in H$. A coquasitriangular Hom-Hopf algebra $(H, \langle | \rangle, \alpha)$ is called cotriangular if $\langle | \rangle$ is convolution invertible in the sense of

$$(BR5) \langle h_1|g_1 \rangle \langle h_2|g_2 \rangle = \epsilon(h)\epsilon(g),$$

for all $h, g \in H$.

Theorem 6.1. Let $(H, \langle | \rangle, \alpha)$ be a coquasitriangular Hom-Hopf algebra. Assume that α is an involution and $\langle | \rangle$ is α -invariant (i.e., $\langle \alpha(h)|\alpha(g) \rangle = \langle h|g \rangle$ for all $h, g \in H$). Then the category $\tilde{\mathcal{H}}(H, \alpha)$ of left (H, α) -Hom-comodules is a Hom-Yetter-Drinfeld subcategory of ${}^H_H\mathcal{HYD}$ under the module action $h \cdot m = \langle \alpha(h)|m_{(-1)} \rangle \mu^2(m_0)$, for all $h \in H$ and $m \in (M, \mu) \in \tilde{\mathcal{H}}(H, \alpha)$.

Proof. We first show that $(M, \cdot, \mu)$ is a left (H, α) -Hom-module. For any $h, g \in H$, and $m \in (M, \mu) \in \tilde{\mathcal{H}}(H, \alpha)$, it is clear that $1_H \cdot m = \langle \alpha(1_H)|m_{(-1)} \rangle \mu^2(m_0) = \langle 1_H|m_{(-1)} \rangle \mu^2(m_0) = \epsilon(m_{(-1)})\mu^2(m_0) = \mu(m)$. Since $\langle | \rangle$ is α -invariant, it follows that

$$\alpha(h) \cdot \mu(m) = \langle \alpha^2(h)|\alpha(m_{(-1)}) \rangle \mu^3(m_0) = \langle \alpha(h)|m_{(-1)} \rangle \mu^3(m_0) = \mu(h \cdot m).$$

So it is sufficient to verify the equality $\alpha(h) \cdot (g \cdot m) = (hg) \cdot \alpha(m)$ as follows:

$$\begin{aligned} \alpha(h) \cdot (g \cdot m) &= \langle \alpha(g)|m_{(-1)} \rangle \alpha(h) \cdot \mu^2(m_0) \\ &= \langle \alpha(g)|m_{(-1)} \rangle \langle \alpha^2(h)|\alpha^2(m_{0(-1)}) \rangle \mu^4(m_{00}) \\ &\stackrel{(1.6)}{=} \langle \alpha(g)|\alpha(m_{(-1)1}) \rangle \langle \alpha^2(h)|\alpha^2(m_{(-1)2}) \rangle \mu^3(m_0) \\ &= \langle \alpha(g)|\alpha(m_{(-1)1}) \rangle \langle \alpha(h)|\alpha(m_{(-1)2}) \rangle \mu^3(m_0) \\ &\stackrel{(BR1)}{=} \langle hg|\alpha^2(m_{(-1)}) \rangle \mu^3(m_0) = \langle hg|m_{(-1)} \rangle \mu^3(m_0) \\ &= \langle \alpha(hg)|\alpha(m_{(-1)}) \rangle \mu^3(m_0) = (hg) \cdot \alpha(m). \end{aligned}$$

The fourth equality holds since $\langle | \rangle$ is α -invariant, and the sixth equality holds since α is an involution.

Next, we prove that $(M, \cdot, \rho, \mu)$ is a left-left (H, α) -Hom-Yetter-Drinfeld module. We should show that the module action satisfies the compatible condition, Eq. (2.10). For any $h \in H$ and $m \in (M, \mu) \in \tilde{\mathcal{H}}(H, \alpha)$, on the one side, we get

$$\begin{aligned} \rho(h \cdot m) &= \langle \alpha(h)|m_{(-1)} \rangle \rho(\mu^2(m_0)) \\ &= \langle \alpha(h)|m_{(-1)} \rangle \alpha^2(m_{0(-1)}) \otimes \mu^2(m_{00}) \\ &= \langle \alpha(h)|\alpha(m_{(-1)1}) \rangle \alpha^2(m_{(-1)2}) \otimes \mu(m_0) \\ &= \langle h|m_{(-1)1} \rangle m_{(-1)2} \otimes \mu(m_0). \end{aligned}$$

The last equality holds since $\langle | \rangle$ is α -invariant. On the other side, we have

$$\begin{aligned}
 & (h_{11}\alpha^{-1}(m_{(-1)}))S(h_2) \otimes \alpha(h_{12}) \cdot m_0 \\
 = & (h_{11}\alpha^{-1}(m_{(-1)}))S(h_2) \otimes \langle \alpha^2(h_{12})|m_{0(-1)} \rangle \mu^2(m_{00}) \\
 = & \langle h_{12}|m_{(-1)2} \rangle (h_{11}m_{(-1)1})S(h_2) \otimes \mu(m_0) \\
 \stackrel{(BR3)}{=} & \langle h_{11}|m_{(-1)1} \rangle (m_{(-1)2}h_{12})S(h_2) \otimes \mu(m_0) \\
 = & \langle \alpha^{-1}(h_1)|m_{(-1)1} \rangle (m_{(-1)2}h_{21})S(\alpha(h_2)) \otimes \mu(m_0) \\
 = & \langle \alpha^{-1}(h_1)|m_{(-1)1} \rangle \alpha(m_{(-1)2})(h_{21})S(h_2) \otimes \mu(m_0) \\
 = & \langle \alpha^{-1}(h_1)|m_{(-1)1} \rangle \alpha(m_{(-1)2})\epsilon(h_2)1 \otimes \mu(m_0) \\
 = & \langle \alpha^{-2}(h)|m_{(-1)1} \rangle \alpha^2(m_{(-1)2}) \otimes \mu(m_0) \\
 = & \langle h|m_{(-1)1} \rangle m_{(-1)2} \otimes \mu(m_0).
 \end{aligned}$$

Therefore, $(M, \cdot, \rho, \mu)$ is a left-left (H, α) -Hom-Yetter-Drinfeld module.

Finally, we show that any morphism in ${}^H\mathcal{M}$ is left (H, α) -linear, too. For this purpose, we take any $m \in (M, \mu) \in \tilde{\mathcal{H}}({}^H\mathcal{M})$, $n \in (N, \nu) \in \tilde{\mathcal{H}}({}^H\mathcal{M})$, and $h \in H$. Assume that $f: (M, \mu) \rightarrow (N, \nu)$ is a morphism in $\tilde{\mathcal{H}}({}^H\mathcal{M})$, then

$$\begin{aligned}
 h \cdot f(m) &= \langle \alpha(h)|(f(m))_{(-1)} \rangle \mu^2(f(m))_0 \\
 &= \langle \alpha(h)|m_{(-1)} \rangle \mu^2(f(m_0)), \\
 f(h \cdot m) &= \langle \alpha(h)|m_{(-1)} \rangle f(\mu^2(m_0)) \\
 &= \langle \alpha(h)|m_{(-1)} \rangle \mu^2(f(m_0)).
 \end{aligned}$$

The second equality holds since f is left (H, α) -colinear and the fourth equality holds since f is compatible with μ . So f is a left (H, α) -colinear morphism, as needed. The proof is completed. $\square$

Proposition 6.2. Let $(H, \langle | \rangle, \alpha)$ be a cotriangular Hom-Hopf algebra. Assume that α is an involution and $\langle | \rangle$ is α -invariant. Then the Hom-Yetter-Drinfeld subcategory $\tilde{\mathcal{H}}({}^H\mathcal{M})$ as defined in Theorem 6.1 is symmetric.

Proof. For any $m \in (M, \mu) \in \tilde{\mathcal{H}}({}^H\mathcal{M})$ and $n \in (N, \nu) \in \tilde{\mathcal{H}}({}^H\mathcal{M})$, it is easy to see that $\rho(v^{-1}(n)) = \alpha^{-1}(n_{(-1)}) \otimes v^{-1}(n_0)$, then

$$\begin{aligned}
 C_{N,M}C_{M,N}(m \otimes n) &= C_{N,M}(\langle \alpha(m_{(-1)})|\alpha^{-1}(n_{(-1)}) \rangle v(n_0) \otimes \mu(m_0)) \\
 &= \langle \alpha(m_{(-1)})|\alpha^{-1}(n_{(-1)}) \rangle \langle \alpha^2(n_{0(-1)})|m_{0(-1)} \rangle \mu^2(m_{00}) \otimes v^2(n_{00}) \\
 &\stackrel{(1.6)}{=} \langle \alpha^2(m_{(-1)1})|n_{(-1)1} \rangle \langle \alpha^2(n_{(-1)2})|m_{(-1)2} \rangle \mu(m_0) \otimes v(n_0) \\
 &= \langle m_{(-1)1}|n_{(-1)1} \rangle \langle n_{(-1)2}|m_{(-1)2} \rangle \mu(m_0) \otimes v(n_0) \\
 &\stackrel{(BR5)}{=} \epsilon(m_{(-1)})\mu(m_0) \otimes \epsilon(n_{(-1)})v(n_0) \\
 &= m \otimes n.
 \end{aligned}$$

Hence $C_{N,M}C_{M,N} = id_{M \otimes N}$, as required. $\square$

Theorem 6.3. Let (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode. Assume that $(H, \cdot, \Delta, \alpha)$ is a left-left (H, α) -Hom-Yetter-Drinfeld module and α an involution. Then $C_{H,H}$ is a symmetry if and only if there exists a bilinear form $\langle | \rangle$ on (H, α) such that $(H, \langle | \rangle, \alpha)$ is cotriangular. Furthermore, the (H, α) -Hom-module action $\cdot$ is induced by $\langle | \rangle$, that is, $h \cdot g = \langle h|g_1 \rangle g_2$, for all $h, g \in H$.

Proof. Assume that $C_{H,H}$ is a symmetry. Since the comodule coaction is defined by the Hom-comultiplication Δ , $C_{H,H}(h \otimes g) = h_1 \cdot \alpha^{-1}(g) \otimes \alpha(h_2)$, for all $h, g \in H$. By assumption, $C_{H,H}$ is a symmetry, then

$$C_{H,H}(h \otimes g) = \alpha(g_2) \otimes S^{-1}(g_1) \cdot \alpha^{-1}(h). \quad (6.1)$$

Define $\langle h|g \rangle = (\epsilon \otimes \epsilon)C_{H,H}(h \otimes g)$. Applying $\epsilon \otimes \epsilon$ to the expressions of $C_{H,H}(h \otimes g)$, we have

$$\begin{aligned} (\epsilon \otimes \epsilon)C_{H,H}(h \otimes g) &= \epsilon(h_1 \cdot \alpha^{-1}(g))\epsilon(\alpha(h_2)) \\ &= \epsilon(h_1 \cdot \alpha^{-1}(g))\epsilon(h_2) \\ &= \epsilon(\alpha^{-1}(h) \cdot \alpha^{-1}(g)). \end{aligned}$$

Furthermore, we obtain

$$\begin{aligned} (\epsilon \otimes \epsilon)C_{H,H}(h \otimes g) &= \epsilon(\alpha(g_2))\epsilon(S^{-1}(g_1) \cdot \alpha^{-1}(h)) \\ &= \epsilon(g_2)\epsilon(S^{-1}(g_1) \cdot \alpha^{-1}(h)) \\ &= \epsilon(\alpha^{-1}(S^{-1}(g)) \cdot \alpha^{-1}(h)) \\ &= \langle S^{-1}(g)|h \rangle. \end{aligned}$$

Thus $\langle h|g \rangle = \langle S^{-1}(g)|h \rangle$. By the arbitrary of h, g , $\langle g|S(h) \rangle = \langle S^{-1}(S(h))|g \rangle = \langle h|g \rangle$. So,

$$\langle h|g \rangle = \langle S^{-1}(g)|h \rangle = \langle g|S(h) \rangle. \quad (6.2)$$

Applying $id \otimes \epsilon$ to the equality $h_1 \cdot \alpha^{-1}(g) \otimes \alpha(h_2) = \alpha(g_2) \otimes S^{-1}(g_1) \cdot \alpha^{-1}(h)$. On the one hand, we have $(id \otimes \epsilon)(h_1 \cdot \alpha^{-1}(g) \otimes \alpha(h_2)) = \alpha^{-1}(h) \cdot \alpha^{-1}(g)$. On the other hand, we have $(id \otimes \epsilon)(\alpha(g_2) \otimes S^{-1}(g_1) \cdot \alpha^{-1}(h)) = \epsilon(\alpha^{-1}(S^{-1}(\alpha(g_1))) \cdot \alpha^{-1}(h))\alpha(g_2) = \langle h|\alpha(g_1)\rangle\alpha(g_2)$. It follows that $\alpha^{-1}(h) \cdot \alpha^{-1}(g) = \langle h|\alpha(g_1)\rangle\alpha(g_2)$. Since α is an involution, we have $h \cdot g = \langle \alpha(h)|g_1\rangle g_2$.

We now show that $(H, \langle | \rangle, \alpha)$ is a cotriangular Hom-Hopf algebra. Obviously, $\langle hg|\alpha(l) \rangle = \epsilon(\alpha^{-1}(hg) \cdot l)$, for all $h, g \in (H, \alpha)$. To verify (BR1), we do the following calculation:

$$\begin{aligned} \langle \alpha(h)|l_2 \rangle \langle \alpha(g)|l_1 \rangle &= \langle \alpha(h)|\langle \alpha(g)|l_1 \rangle l_2 \rangle = \langle \alpha(h)|g \cdot l \rangle \\ &= (\epsilon \otimes \epsilon)C_{H,H}(\alpha(h) \otimes g \cdot l) \\ &= (\epsilon \otimes \epsilon)(\alpha(h_1) \cdot \alpha^{-1}(g \cdot l) \otimes \alpha^2(h_2)) \\ &= (\epsilon \otimes \epsilon)(\alpha(h_1) \cdot (\alpha^{-1}(g) \cdot \alpha^{-1}(l)) \otimes h_2) \\ &= (\epsilon \otimes \epsilon)((h_1 \alpha^{-1}(g)) \cdot l \otimes h_2) \\ &= \epsilon((h_1 \alpha^{-1}(g)) \cdot l)\epsilon(h_2) \\ &= \epsilon(\alpha^{-1}(hg) \cdot l). \end{aligned}$$

So $\langle hg|\alpha(l) \rangle = \langle \alpha(h)|l_2 \rangle \langle \alpha(g)|l_1 \rangle$, (BR1) holds. By Eq. (5.2), $\langle \alpha(h)|gl \rangle = \langle gl|S(\alpha(h)) \rangle = \langle \alpha(g)|S(h_1) \rangle \langle \alpha(l)|S(h_2) \rangle = \langle h_1|\alpha(g) \rangle \langle h_2|\alpha(l) \rangle$, (BR2) is satisfied.

Since the coaction is defined by the Hom-comultiplication, we have $\rho(h_1 \cdot \alpha(g)) = (h_{11}g_1)S(h_{12}) \otimes \alpha(h_{12}) \cdot \alpha(g_2)$, and

$$\begin{aligned} &(h_1 \cdot \alpha(g))_{(-1)}h_2 \otimes (h_1 \cdot \alpha(g))_0 \\ &= ((h_{11}g_1)S(h_{12}))h_2 \otimes \alpha(h_{12}) \cdot \alpha(g_2) \\ &\stackrel{(1.4)}{=} ((\alpha^{-1}(h_{11})g_1)S(h_{21}))\alpha(h_{22}) \otimes h_{12} \cdot \alpha(g_2) \\ &\stackrel{(1.1)}{=} (\alpha^{-1}(h_{11})g_1)(S(h_{21})h_{22}) \otimes h_{12} \cdot \alpha(g_2) \\ &\stackrel{(1.8)}{=} (\alpha^{-1}(h_{11})g_1)\epsilon(h_2)1_H \otimes h_{12} \cdot \alpha(g_2) \\ &= \alpha(h_1)g_1 \otimes \alpha^{-1}(h_2) \cdot \alpha^{-1}(g_2). \end{aligned}$$

Therefore, we get

$$\begin{aligned}
 h_1 g_1 \langle h_2 | g_2 \rangle &= h_1 g_1 \epsilon(\alpha^{-1}(h_2) \cdot \alpha^{-1}(g_2)) \\
 &= (id \otimes \epsilon)(h_1 g_1 \otimes \alpha^{-1}(h_2) \cdot \alpha^{-1}(g_2)) \\
 &\stackrel{(1.10)}{=} (id \otimes \epsilon)((h_1 \cdot \alpha(g))_{(-1)} h_2 \otimes (h_1 \cdot \alpha(g))_0) \\
 &= (id \otimes \epsilon)((h_1 \cdot \alpha(g))_1 h_2 \otimes (h_1 \cdot \alpha(g))_2) \\
 &= (h_1 \cdot \alpha(g))_1 h_2 \epsilon((h_1 \cdot \alpha(g))_2) \\
 &= \alpha^{-1}(h_1 \cdot \alpha(g)) h_2.
 \end{aligned}$$

So $h_1 g_1 \langle h_2 | g_2 \rangle = \alpha^{-1}(h_1 \cdot \alpha(g)) h_2 = (\alpha^{-1}(h_1) \cdot g) h_2 = (\alpha(h_1) \cdot g) h_2 = \langle h_1 | g_1 \rangle g_2 h_2$, (BR3) holds.

Clearly, (BR4) is true since $\langle h | 1_H \rangle = \epsilon(h \cdot 1_H) = \epsilon(h) = \langle 1_H | h \rangle$. For (BR5), we have

$$\begin{aligned}
 \langle h_1 | g_1 \rangle \langle g_2 | h_2 \rangle &= \langle \langle h_1 | g_1 \rangle g_2 | h_2 \rangle = \langle \alpha^{-1}(h_1) \cdot g | h_2 \rangle \\
 &= \epsilon(\alpha^{-1}(S^{-1}(h_2)) \cdot \alpha^{-1}(\alpha^{-1}(h_1) \cdot g)) \\
 &= \epsilon(\alpha^{-1}(S^{-1}(h_2)) \cdot (\alpha^{-2}(h_1) \cdot \alpha^{-1}(g))) \\
 &= \epsilon((S^{-1}(h_2) h_1) \cdot g) = \epsilon(h) \epsilon(g).
 \end{aligned}$$

So $(H, \langle | \rangle, \alpha)$ is a cotriangular Hom-Hopf algebra. $\square$

Corollary 6.4. Let (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode. Assume that $(H, \text{adjoint}, \Delta, \alpha)$ is a left-left (H, α) -Hom-Yetter-Drinfeld module and α an involution. Then $C_{H,H}$ is a symmetry if and only if (H, α) is commutative.

Proof. Assume that $C_{H,H}$ is a symmetry, then by Theorem 6.3 $(H, \langle | \rangle, \alpha)$ is cotriangular, where $\langle h | g \rangle = \epsilon(\alpha^{-1}(h) \cdot \alpha^{-1}(g)) = \epsilon((\alpha^{-1}(h_1) \alpha^{-2}(g)) S(h_{12})) = \epsilon(\alpha^{-1}(h_1) \alpha^{-2}(g)) \epsilon(S(h_{12})) = \epsilon(h) \epsilon(g)$, $h, g \in (H, \alpha)$. By (BR3), $\langle h_1 | g_1 \rangle g_2 h_2 = h_1 g_1 \langle h_2 | g_2 \rangle$, that is, $hg = gh$, (H, α) is commutative.

Conversely, if (H, α) is commutative, then $C_{H,H}(h \otimes g) = h_{(-1)} \alpha^{-1}(g) \otimes \alpha(h_0) = h_1 \alpha^{-1}(g) \otimes \alpha(h_2) = (h_{11} \alpha^{-2}(g)) S(\alpha(h_{12})) \otimes \alpha(h_2) = (\alpha^{-2}(g) h_{11}) S(\alpha(h_{12})) \otimes \alpha(h_2) = \alpha^{-1}(g) (h_{11} S(h_{12})) \otimes \alpha(h_2) = \alpha^{-1}(g) \epsilon(h_1) 1_H \otimes \alpha(h_2) = g \otimes h$, i.e., $C_{H,H}$ is a symmetry, as required. $\square$

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¹ S. Caenepeel and I. Goyvaerts, "Monoidal Hom-Hopf algebras," *Commun. Algebra* **39**(6), 2216–2240 (2011).

² Y. Y. Chen, Z. W. Wang, and L. Y. Zhang, "Quasi-triangular Hom-Lie bialgebras," *J. Lie Theory* **22**(4), 1075–1089 (2012).

³ Y. Y. Chen and L. Y. Zhang, "The category of Yetter-Drinfel'd Hom-modules and the quantum Hom-Yang-Baxter equation," *J. Math. Phys.* **55**(3), 031702 (2014).

⁴ M. Cohen and S. Westreich, "Determinants and symmetries in Yetter-Drinfeld categories," *Appl. Categ. Struct.* **6**(2), 267–289 (1998).

⁵ M. Cohen, S. Westreich, and S. L. Zhu, "Determinants, integrality and Noether's theorem for quantum commutative algebras," *Isr. J. Math.* **96**(1), 185–222 (1996).

⁶ A. Gohr, "On Hom-algebras with surjective twisting," *J. Algebra* **324**(7), 1483–1491 (2010).

⁷ A. Joyal and R. Street, "Braided tensor categories," *Adv. Math.* **102**(1), 20–78 (1993).

⁸ L. Liu and B. L. Sheng, "Radford's biproducts and Yetter-Drinfeld modules for monoidal Hom-Hopf algebras," *J. Math. Phys.* **55**(3), 031701 (2014).

⁹ A. Makhlouf and S. Silvestrov, "Hom-algebra structures," *J. Gen. Lie Theory* **3**(2), 51–64 (2008).

¹⁰ A. Makhlouf and S. Silvestrov, "Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras," *Journal of Generalized Lie Theory in Mathematics, Physics and Beyond* (Springer-Verlag, Berlin, 2009), pp. 189–206.

¹¹ A. Makhlouf and S. Silvestrov, "Hom-algebras and Hom-coalgebras," *J. Algebra Appl.* **09**(4), 553–589 (2010).

- ¹² A. Makhlouf and F. Panaite, “Yetter-Drinfeld modules for Hom-bialgebras,” *J. Math. Phys.* **55**(1), 013501 (2014).
- ¹³ F. Panaite, M. D. Staic, and F. V. Oystaeyen, “Pseudosymmetric braidings, twines and twisted algebras,” *J. Pure Appl. Algebra* **214**(6), 867–884 (2010).
- ¹⁴ B. Pareigis, “Symmetric Yetter-Drinfeld categories are trivial,” *J. Pure Appl. Algebra* **155**(1), 91–91 (2001).
- ¹⁵ M. E. Sweedler, *Hopf Algebras* (Benjamin, New York, 1969).
- ¹⁶ Z. W. Wang, Y. Y. Chen, and L. Y. Zhang, “The antipode and Drinfel’d double of Hom-Hopf algebras,” *Sci. Sin. Math.* **42**(11), 1079–1093 (2012).
- ¹⁷ D. Yau, “The Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras,” *J. Phys. A* **42**(16), 165202 (2009).