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Longitudinal Data Analysis Using Bayesian-frequentist Hybrid Random Effects Model

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Abstract

The mixed random effect model is commonly used in longitudinal data analysis within either frequentist or Bayesian framework. Here we consider a case, we have prior knowledge on partial-parameters, while no such information on rest. Thus, we use the hybrid approach on the random-effects model with partial-parameters. The parameters are estimated via Bayesian procedure, and the rest of parameters by the frequentist maximum likelihood estimation (MLE), simultaneously on the same model. In practices, we often know partial prior information such as, covariates of age, gender, and etc. These information can be used, and get accurate estimations in mixed random-effects model. A series of simulation studies were performed to compare the results with the commonly used random-effects model with and without partial prior information. The results in hybrid estimation (HYB) and Maximum likelihood estimation (MLE) were very close each other. The estimated θ values in with partial prior information model (HYB) were more closer to true θ values, and shown less variances than without partial prior information in MLE. To compare with true θ values, the mean square of errors (MSE) are much less in HYB than in MLE. This advantage of HYB is very obvious in longitudinal data with small sample size. The methods of HYB and MLE are applied to a real longitudinal data for illustration.

Keywords

Hybrid; Longitudinal data; Simulation

Introduction

The longitudinal data is frequently used in data analysis (Fitzmaurice,2008), the random effects model (Laird and Ware, 1982), or called mixed effects model in some literatures, is commonly used, but either the frequentist or the Bayesian method is often used (Hong et al, 2013). In practice, for longitudinal data model, there are situations in which we have good

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knowledge, summarized into a prior distribution, on part of the model parameters, while no such information on the rest parameters. Although the Bayes estimation and the frequentist maximum likelihood estimation (MLE) are asymptotically equivalent, two methods differ in many aspects. With good partial prior information, the Bayes procedure has small sample advantage; with a bad partial prior information, or even non-informative prior, it may yield miss-leading results for data with small sample size. The Bayesian method also is often much computationally demanding, requires various numerical methods such as the Markov chain Monte Carlo, and may subject to additional approximation errors. On the other hand, although the frequentist MLE is computationally simpler, it does not enjoy the small sample advantage with the valuable sound prior information wasted. There are some advocates for some kind of Bayesian frequentist combined methods, such as in Bayarri and Berger (2004), and the Bayesian frequentist compromises (Good, 1992). However, these considerations do not treat the problem in the same framework. Recently, Yuan (2009) proposed a Bayesian frequentist hybrid model, in which part of the parameters are estimated by the Bayesian and the rest parameters estimated by the MLE, simultaneously in one model. Yuan et al. (2011) used this method in the analysis of gene copy number changes, Zheng et al. (2013) extended this method for the 0–1 loss case and with covariates, and applied it for genome-wide case-control association study. In this report, we use the Bayesian frequentist method on the random effects model with partial prior information in a series of simulation studies with individuals of 5, 10, 25, 50, 100, 300, and 500, respectively, to estimate parameters and comparing estimated parameters with a commonly used random effects model without prior information. The method was applied to a real data for illustration.

The Model and Method

We first give a brief review of the commonly used random effects (mixed effects) model (e.g. Laird and Ware, 1982) and the hybrid model of Zheng et al. (2013), then describe our proposed method.

1. The random effects model

Let $y_i = (y_{i1}, \dots, y_{i,n_i})'$ be the observed data from the i -th individual, $X_i = (X_{i1}, \dots, X_{i,n_i})'$ and $Z_i = (Z_{i1}, \dots, Z_{i,n_i})'$ be the corresponding covariates (each X_{ij} is a d -dimensional row vector and each Z_{ij} is a k -dimensional row vector), ($i = 1, \dots, m$). The aim is a regression analysis of the y_i 's on the X_i 's. Let $\theta = (\theta_1, \dots, \theta_d)'$ be the regression parameters (fixed effects), and $b_i = (b_{i1}, \dots, b_{i,k})'$ be unobserved individual random effects, and e_i be n_i -dimensional measurement error ($i = 1, \dots, m$). Often we assume (y_i, X_i, b_i) are independent over i , and e_i and b_i are independent. The commonly used mixed effects model is

$$y_i = X_i\theta + Z_ib_i + e_i, \quad (i=1, \dots, m). \quad (1)$$

The purpose is to estimate θ , not the random effects b_i 's. Assume $e_i \sim N(0_{n_i}, \sigma^2 I_{n_i})$, where 0_{n_i} is the n_i -dimensional column vector of zeros, I_{n_i} is the n_i -dimensional identity matrix; and $b_i \sim N(0_k, \tau^2 I_k)$. The constants σ^2 and τ^2 can be either known or unknown (to be estimated).

When σ^2 and τ^2 are known, let $V_i = \sigma^2 I_{n_i} + \tau^2 Z_i Z_i'$ (a $n_i \times n_i$ matrix), then $y_i - X_i\theta \sim N(0_{n_i}, V_i)$, and the MLE $\hat{\theta}$ of θ under the above model is

$$\hat{\theta} = \left(\sum_{i=1}^m X_i' V_i^{-1} X_i \right)^{-1} \sum_{i=1}^m X_i' V_i^{-1} y_i. \quad (2)$$

The above estimate $\hat{\theta}$ is a.s. consistent and asymptotic normal.

2. The hybrid model

Consider the observed data as $y_1, \dots, y_n$ iid density $f(\cdot|\theta)$, with covariates $X_1, \dots, X_n$, and $\theta = (\alpha, \beta)$ be all the parameters in the model, and assume that we have sound prior knowledge on β , but no prior information on α . Let $\pi(\cdot)$ be the prior density for β , $w(\cdot, \cdot)$ be the loss function for inferring β , and $d = d(Y^n|X^n)$ be the decision for β based on $Y^n = (y_1, \dots, y_n)$ given $X^n = (X_1, \dots, X_n)$. Denote the parameter spaces for α and β as Λ and Γ , respectively.

Let $f(Y^n|X^n, \theta) = \prod_{i=1}^n f(y_i|X_i, \theta)$ be the likelihood and $l(\theta) = \log f(Y^n|X^n, \theta)$ be the log-likelihood. For a fixed α , the Bayes estimate β^* for β is given by

$$\begin{aligned} \beta^* &= \beta^*(Y^n|X^n, \alpha) \\ &= \arg \inf_{d \in \Gamma} \int_{\Gamma} w(d, \beta) f(Y^n|X^n, \theta) \pi(\beta) d\beta, \end{aligned}$$

and for a fixed β , the maximum likelihood estimate (MLE) α^* for α is given by

$$\alpha^* = \alpha^*(Y^n|X^n, \beta) = \arg \sup_{\alpha \in \Lambda} f(Y^n|X^n, \theta).$$

Following Yuan (2009), the hybrid estimate of $\theta = (\alpha^T, \beta)$, denoted by $\tilde{\theta} = (\hat{\alpha}, \hat{\beta})$, satisfies

$$(\hat{\alpha}, \hat{\beta}) = \arg \sup_{\alpha \in \Lambda, d \in \Gamma} \int_{\Gamma} w(d, \beta) f(Y^n|X^n, \theta) \pi(\beta) d\beta. \quad (3)$$

Denote $H(\alpha, d) = w(d, \beta) f(Y^n|X^n, \theta) \pi(\beta)$. Then $H(\alpha, \hat{\beta}) \sim H(\alpha, \hat{d})$ for $\forall d \in \Gamma$ and $H(\alpha, \hat{\beta}) \sim H(\alpha, \hat{\beta})$ for $\forall \alpha \in \Lambda$. The hybrid estimate $\tilde{\theta}$ generally exists and is locally unique because it can be formulated as a Bayesian estimator under the 0–1 loss with a constant prior for α . Note that the operations $\inf$ and $\sup$ in the previous expression are applied jointly and simultaneously. Generally, $\sup_{\alpha \in \Lambda} \inf_{d \in \Gamma}$ is not equivalent to $\sup_{\alpha \in \Lambda} (\inf_{d \in \Gamma})$ or $\inf_{d \in \Gamma} (\sup_{\alpha \in \Lambda})$. Zheng et al. (2013) give the formulations of $(\hat{\alpha}, \hat{\beta})$ under commonly used loss functions.

The computation of $(\hat{\alpha}, \hat{\beta})$ is the simplest under the 0–1 loss, and we concentrate on this loss in this report. For the 0–1 loss, $w(d, \beta) = 0$ if $d = \beta$, and $w(d, \beta) = 1$ otherwise. Then, for a fixed α , $\hat{\beta}$ is the posterior mode. Hence $\hat{\alpha}$ fixed given above $\hat{\beta}$ is given by

$$\hat{\beta} = \arg \sup_{\beta \in \Gamma} \pi(\beta|Y^n, \hat{\alpha}, X^n) = \arg \sup_{\beta \in \Gamma} \{f(Y^n|X^n, \hat{\alpha}, \beta) \pi(\beta)\}$$

Consequently,

$$(\hat{\alpha}, \check{\beta}) = \arg \sup_{\alpha \in \Lambda, \beta \in \Gamma} (l(\alpha, \beta) + \log \pi(\beta)). \quad (4)$$

Hence, $\tilde{\theta}$ is simpler to compute than using the previous two loss functions because $\tilde{\theta}$ can be regarded as the MLE from $l^*(\alpha, \beta) = l(\alpha, \beta) + \log \pi(\beta)$.

3. The proposed method

As in the random effects model (1) described before, we assume $e_i \sim N(0_{n_i}, \sigma^2 I_{n_i})$, where I_{n_i} is the n_i -dimensional identity matrix; and $b_i \sim N(0_k, \tau^2 I_k)$, with σ^2 and τ^2 be known. To use the hybrid method, write $\theta = (\alpha, \beta)$, α is d_1 -dimensional, β is d_2 -dimensional ($d_1 + d_2 = d$), and suppose all the prior information about β is summarized in the density $\pi(\beta) \sim N(\mu, \Omega)$ for β , with μ ($d_1 \times 1$) and Ω ($d_1 \times d_1$) are known. We use the 0–1 loss. Under this set up, the log-likelihood and the log-prior are (with some quantity independent of the parameters omitted)

$$l(\alpha, \beta) = -\frac{1}{2} \sum_{i=1}^m (y_i - X_i \theta)' V_i^{-1} (y_i - X_i \theta), \quad \log \pi(\beta) = \frac{1}{2} (\beta - \mu)' \Omega^{-1} (\beta - \mu),$$

and $(\hat{\alpha}, \check{\beta})$ of (4) is the solution of the following set of equations

$$0_d = \frac{\partial}{\partial \theta} [l(\alpha, \beta) + \log \pi(\beta)] = \sum_{i=1}^m X_i' V_i^{-1} (y_i - X_i \theta) - \Omega^{-1} (\beta - \mu),$$

Let $0_{d_1 \times d_2}$ be the $d_1 \times d_2$ matrix with all entries being 0's, the the hybrid estimate $(\hat{\alpha}, \check{\beta})$ of the true parameter $\theta_0 = (\alpha_0, \beta_0)$ under this Bayesian frequentist hybrid random effects model is

$$\bar{\theta} := (\hat{\alpha}, \check{\beta}) = \left(\sum_{i=1}^m X_i' V_i^{-1} X_i + B \right)^{-1} \left(\sum_{i=1}^m X_i' V_i^{-1} y_i + A \right), \quad (5)$$

where

$$A = \begin{pmatrix} 0_{d_1} \\ \Omega^{-1} \mu \end{pmatrix}, \quad B = \begin{pmatrix} 0_{d_1 \times d_1} & 0_{d_1 \times d_2} \\ 0_{d_2 \times d_1} & \Omega^{-1} \end{pmatrix}.$$

Comparing the estimate of θ_0 in the commonly used random effects model as in (2), here the prior information for β is added into the vector A and the matrix B from prior studies.

Simulation studies

Set $m = (5, 10, 25, 50, 100, 300, 500)$, $\theta_0 = (1.52, -0.68, 0.73, 1.12, -1.35)' = (\alpha_0, \beta_0)'$ with $\alpha_0 = (1.52, -0.68, 0.73)'$ and $\beta_0 = (1.12, -1.35)'$, $\tau^2 = 0.56$, $\sigma^2 = 1.10$, and $N = 100$. The

prior for β is $\pi(\beta) \sim N(\mu, \Omega)$, with $\mu = (1.11, -1.33)'$,
$$\Omega = \begin{pmatrix} 0.11 & -0.06 \\ -0.06 & 0.12 \end{pmatrix}.$$

For $i = 1, \dots, m$, we sample the y_i 's by the following way:

1. Sample $r \sim U[0, 1]$, set $n_i = [Nr]$, the integer part of Nr .
2. For $j = 1, \dots, n_i$, sample $X_{ij} \sim N(\mu_x, \Omega_x)$, with $\mu_x = (1.25, 0.33, -0.75, 1.11, 0.34)'$ and $\Omega_x = \text{diag}(0.5, 0.1, 0.23, 0.4, 0.11)$, and set $X_i = (X_{i1}, \dots, X_{i,n_i})'$.
3. For $j = 1, \dots, n_i$, sample $Z_{ij} \sim N(\mu_z, \Omega_z)$, with $\mu_z = (-0.85, 0.66, 0.45)$ and $\Omega_z = \text{diag}(0.22, 0.18, 0.12)$. Set $Z_i = (Z_{i1}, \dots, Z_{i,n_i})'$.
4. Sample $b_i \sim N(0, \tau^2 I_3)$.
5. For $j = 1, \dots, n_i$, sample $e_{ij} \sim N(0, \sigma^2)$, and set $e_i = (e_{i1}, \dots, e_{i,n_i})'$.
6. Set $y_i = X_i \theta + Z_i b_i + e_i$.

Then use formula (5) to compute $\bar{\theta}$ to estimate θ_0 , and use formula (2) to compute $\hat{\theta}$. Then compare $\bar{\theta}$ and $\hat{\theta}$, and their errors $\|\bar{\theta} - \theta_0\|$ and $\|\hat{\theta} - \theta_0\|$.

Table 1 presents the average of estimation of θ_0 for 5, 10, 25, 50, 100, 300, and 500 individuals in a total of $7 \times 3,000$ times simulations. For a true $\theta_{0,1} = 1.52$, the average of estimated values is 1.5211, 1.5160, 1.5183, 1.5216, 1.5194, 1.5191, and 1.5201 in MLE (without partial prior information) models, 1.5205, 1.5159, 1.5184, 1.5115, 1.5195, 1.5192, and 1.5201 in HYB (with partial prior information models) for 5, 10, 25, 50, 100, 300, and 500 individuals, respectively. The HYB and MLE models shown less bias (more closer to true values) as increasing the number of individuals. As increasing the number of individuals, the more accuracy (smaller variance) is also observed for both of MLE and HYB models. The variances is 0.0480, 0.0217, 0.0079, 0.0041, 0.0021, 0.0006, and 0.0004 in MLE models, 0.0428, 0.0201, 0.0076, 0.0040, 0.0020, 0.0006, and 0.0004 in HYB models for number of individuals of 5, 10, 25, 50, 100, 300, and 500, respectively. The variances of estimated $\theta_{0,1}$ from HYB model is smaller than from MLE model when number of individuals were small. The obvious difference of variances were displayed in Figure 1. The points with gray color presented estimation of $\theta_{0,1}$ values from MLE model, and the points with blue color presented estimation of $\theta_{0,1}$ values from HYB models. The points with blue colour shown a narrow range than points with gray color. The character of a narrow range was very obvious when number of individuals size was small. As number of individuals was great than 50, the estimation of parameters become very stable.

To compare with true $\theta_{0,1}$ value, the values of Mean Square Errors (MSE) in MLE are 0.0480, 0.0218, 0.0080, 0.0041, 0.0020, 0.0007, and 0.0004, and values of MSE in HYB are 0.0428, 0.0201, 0.0076, 0.0039, 0.0020, 0.0006, and 0.0004 for 5, 10, 25, 50, 100, 300, and 500 individuals, respectively. The MSE value is smaller in HYB than in MLE models, especially in number of individuals is small, such as 5 individuals (Table 2). As number of

individuals increased to 50, the values of MSE is near to 0 for both of MLE and HYB models (Figure 2). The characters of less bias, more accuracy, and a narrow range from HYB models were also observed for $\theta_{0,2}$, $\theta_{0,3}$, $\theta_{0,4}$, $\theta_{0,5}$ estimations.

The ratio of MSE values in MLE and HYB models are 1.1215 (0.0480/0.0428), 1.0686 (0.2834/0.2652), 1.1001 (0.1077/0.00979), 2.6453 (0.0619/0.0234), and, 11.2067(0.2656/0.0237) for $\theta_{0,1-5}$ (Figure 3), respectively, in simulations with 5 individuals. As samples size is small, such as 5, both values of MSE in MLE and HYB are closer each other in ($\theta_{0,1}$, $\theta_{0,2}$, $\theta_{0,3}$) (without prior information) and ratio values are near to 1 to compare the smaller MSE values in HYB model in $\theta_{0,4}$, $\theta_{0,5}$ (with prior information) and ratio values are much larger than 1 (Figure 3). As increasing sample size to 50, this character is disappear (Table 2, Figure 2, and Figure 3).

Real applications

We applied MLE and Hybrid methods into a real public data (Riesby4.dat, tiger.uic.edu/~hedeker/long.html). The data was using as introduction to growth modelling in longitudinal continuous Kaplan (2004). This Longitudinal data contains 66 individuals in a total of 250 measures. 14 individuals (0.21) were with 3 measures and 52 (0.79) were with 4 measures. Six variables of Intcpt, week, mlnimi, mlndmi, Inidev, and Lnddev are in a fixed effect model. Variable of week is also in a random effect model. In MLE model, we estimated θ_0 . However, in Hybrid model, we are assuming we know partial prior θ information of Inidev and Lnddev in

$$A = \begin{pmatrix} 2.55 \\ -1.80 \end{pmatrix}, \text{ and, } B = \begin{pmatrix} 0.0300 & 0.0167 \\ 0.0167 & 0.0690 \end{pmatrix}$$

, and estimated these six θ values. Table 3 presented estimations of θ values in MLE and Hybrid models. Both of MLE and HYB yield similar θ values, both of variance are same, because variance was estimated by asymmetrical estimation, when n is going to infinity.

Discussion

The longitudinal studies play a key role in epidemiology, clinical research, and therapeutic evaluation. The longitudinal studies are tracking the same individuals. The most longitudinal studies are observational, and have more power than cross sectional observational study. Because repeated measures on each subject, the intra subject correlation of response measures must be properly account, otherwise, statistical inferences can be grossly invalid. The first considered hybrid estimates with both frequentist and Bayesian components is Yuan (2009), and the concept was extended in genetic association studies (Yuan et al 2011). In epidemiology longitudinal studies, the partial parameters had been reported in other studies, how to use these information still unknown. In this paper, we extend hybrid method in a special longitudinal data, and incorporated the estimated parameters from past studies into analysis of a current longitudinal data in a hybrid fashion. The marginal, mixed effect, and transition models are used in longitudinal data analysis (Fitzmaurice et al., 2008). The

marginal is used to describe variation in population means of subgroups, it is not attempt or not directly attempt to explain or model correlation among repeated observation for an individual. Transition model is less commonly used. Mixed effect model directly incorporate natural individual variability, and is most useful to make inference about individuals. The common method of parameter estimation in mixed effect model is maximum likelihood estimation (MLE). In here, we used MLE in mixed effects model as an example in longitudinal data analysis to compare with Hybrid model.

Our hybrid method required a correct partial prior information from past studies. In epidemiology studies, the age, gender, and Body Mass Index (BMI) etc are common covariates. These information is likely heterogeneous by race and geography, and easy to obtain. In longitudinal data analysis, we highly recommend to use these prior information, because using prior information has following benefits.

Study power

An existing approach of using the power prior (Ibrahim, 2000) was applied, which is equivalent to inflate the variance or reduce the actual sample size of the past study to a smaller one.

Better for parameters estimation

In a series of simulation studies, both of estimations from MLE and HYB are very close to true values. However, the mean values of θ from HYB (with partial prior information) model is more closer to true θ values and shown less bias, and more accuracy (smaller variances) than MLE model (without partial prior information) (Table 1). For estimated θ values that with partial prior information (such as, $\theta_{4,5}$), these two characters of less bias, and more accuracy shown more obvious than estimated θ values that without partial prior information (such as, $\theta_{1,2,3}$). Both of mean square errors (MSE) with true θ values are very small and near to 0, but MSE in HYB model is smaller than in MLE model, and more obvious in parameter estimation of θ values with prior information (Table 2, Figure 2, Figure 3). The largest ratio of MSE values in MLE and HYB is 11.2 (Figure 3).

Better for smaller sample size

MSE values in MLE and HYB models are similar in a series of simulation studies, but MSE value is smaller in HYB (with prior information) than in MLE (without prior information). Especially, in smaller of number of individuals with prior information parameters (such as, $\theta_{4,5}$) (Table 2, Figure 2, Figure 3).

Conclusions

To compare with MLE model in simulation study, Hybrid model with partial prior information has less bias, more accuracy, and narrow range MSE characters in longitudinal data analysis. These characters is obviously in smaller sample size.

References

- Bayarri MJ, Berger JO. The interplay of Bayesian and Frequentist Analysis. *Statistical Science*. 2004; 19:58–80.
- Fitzmaurice, G.; Davidian, M.; Verbeke, G.; Molenberghs, G. *Longitudinal Data Analysis*. Oxford Statistical Science; 2008.
- Good IJ. The Bayes/non-Bayes compromise: A brief review. *Journal of the American Statistical Association*. 1992; 87:597–606.
- Hong H, Carlin PB, Shamliyan AT, Wyman FJ, Ramakrishnan R, Sinfort F, Kane LR. Comparing Bayesian and Frequentist Approaches for Multiple Outcome Mixed Treatment Comparisons. *Medical Decision Making*. 2013; 33:701–714.
- Ibrahim JG, Chen MH. Power prior distributions for regression models. *Statistics Science*. 2000; 15:46–60.
- Kaplan, D. *Quantitative Methodology for the social sciences*. Thousand Oaks CA: Sage Publications; 2004.
- Larid MN, Ware JH. Random-effects Models for Longitudinal data. *Biometrics*. 1982; 38:963–974. [PubMed: 7168798]
- Yuan A. Bayesian frequentist hybrid inference. *Annals of Statistics*. 2009; 37:2458–2501.
- Yuan A, Chen G, Rotimi C. Genetic network analysis by quasi-bayesian method. *J Bioinform Comput Biol*. 2009; 7:175–192. [PubMed: 19226666]
- Yuan A, Chen G, Xiong J, He W, Jin W, Rotimi C. Bayesian Frequentist hybrid Model with Application to the Analysis of Gene Copy Number Changes. *Journal of Applied Statistics*. 2011; 38(5):987–1005. [PubMed: 24014930]
- Zheng G, Li Q, Qin J, Yuan A. Estimation of genetic effects incorporating prior information. *Statistics and Its Interface*. 2013 In press.

Figure 1.

Estimated θ values (Y-axis) and Simulation data sets from 1 to 3000 (X-axis). The red dot lines presented true θ_0 , blue dots present estimated values in HYB model with partial prior information, Gray dots present estimated values in MLE model without prior information. From top to bottom presents number individuals of 5, 10, 25, 50, 100, 300, and 500, respectively

Figure 2.

Mean square Error values (Y-axis) with true θ_0 by using HYB (red) and MLE (blue) models. X-axis presents number of individuals in simulation studies. From left to right presents $(\theta_{0,1}, \dots, \theta_{0,5})$, respectively. $(\theta_1, \theta_2, \theta_3)$ without prior information, and (θ_4, θ_5) with prior information

Figure 3.

Ratio of MSE in MLE and MSE in HYB (Y-axis). X-axis presents $(\theta_1, \dots, \theta_5)$. $(\theta_1, \theta_2, \theta_3)$ without prior information, (θ_4, θ_5) with prior information

Table 1

Means, and Variances of θ values with and without Prior Information in Simulation data.

Methods	Num. of Inds	θ_1	θ_2	θ_3	θ_4	θ_5
MLE	5	1.5211(0.0480)	-0.6959(0.2830)	0.7363(0.1080)	1.1186(0.0619)	-1.3387(0.2660)
HYB	5	1.5205(0.0428)	-0.6943(0.2650)	0.7358(0.0979)	1.1210(0.0235)	-1.3471(0.0236)
MLE	10	1.5160(0.0217)	-0.6785(0.1280)	0.7319(0.0498)	1.1177(0.0269)	-1.3411(0.1130)
HYB	10	1.5159(0.0201)	-0.6793(0.1240)	0.7312(0.0469)	1.1188(0.0167)	-1.3488(0.0249)
MLE	25	1.5183(0.0079)	-0.6818(0.0484)	0.7309(0.0189)	1.1223(0.0103)	-1.3434(0.0413)
HYB	25	1.5184(0.0076)	-0.6818(0.0477)	0.7308(0.0184)	1.1215(0.0086)	-1.3471(0.0196)
MLE	50	1.5216(0.0041)	-0.6825(0.0239)	0.7340(0.0091)	1.1198(0.0049)	-1.3471(0.0213)
HYB	50	1.5115(0.0040)	-0.6824(0.0238)	0.7340(0.0089)	1.1199(0.0044)	-1.3500(0.0143)
MLE	100	1.5194(0.0021)	-0.6819(0.0113)	0.7289(0.0046)	1.1207(0.0025)	-1.3528(0.0106)
HYB	100	1.5195(0.0020)	-0.6818(0.0112)	0.7291(0.0045)	1.1206(0.0024)	-1.3510(0.0085)
MLE	300	1.5191(0.0006)	-0.6804(0.0039)	0.7301(0.0015)	1.1199(0.0008)	-1.3480(0.0035)
HYB	300	1.5192(0.0006)	-0.6804(0.0039)	0.7301(0.0015)	1.1199(0.0008)	-1.3481(0.0032)
MLE	500	1.5201(0.0004)	-0.6793(0.0023)	0.7300(0.0009)	1.1205(0.0005)	-1.3506(0.0021)
HYB	500	1.5201(0.0004)	-0.6794(0.0023)	0.7300(0.0009)	1.1205(0.0005)	-1.3502(0.0020)

True $\theta_0 = (1.52, -0.68, 1.12, 0.73, 1.12, -1.35)$, estimation results were presented with means(variances)

Table 2

Mean Square Errors with true θ Values in Simulations data with and without prior information.

Methods	Num. of Inds	θ_1	θ_2	θ_3	θ_4	θ_5
MLE	5	0.0480	0.2834	0.1077	0.0619	0.2656
HYB	5	0.0428	0.2652	0.0979	0.0234	0.0237
MLE	10	0.0218	0.1283	0.0498	0.0269	0.1133
HYB	10	0.0201	0.1242	0.0469	0.0167	0.0249
MLE	25	0.0080	0.0484	0.0189	0.0103	0.0413
HYB	25	0.0076	0.0477	0.0183	0.0085	0.0196
MLE	50	0.0041	0.0239	0.0092	0.0049	0.0213
HYB	50	0.0039	0.0237	0.0090	0.0044	0.0143
MLE	100	0.0020	0.0113	0.0046	0.0025	0.0106
HYB	100	0.0020	0.0112	0.0045	0.0024	0.0085
MLE	300	0.0007	0.0039	0.0015	0.0008	0.0035
HYB	300	0.0006	0.0039	0.0015	0.0008	0.0032
MLE	500	0.0004	0.0023	0.0009	0.0005	0.0021
HYB	500	0.0004	0.0023	0.0009	0.0005	0.0020

True $\theta_0 = (1.52, -0.68, 1.12, 0.73, 1.12, -1.35)$.

Table 3

Real data Analysis in with and without prior information.

Methods	Intcpt	Week	mlnlimi	mlndmi	lnidev	lnndev
MLE	6.509(3.2656)	-1.986(0.4276)	-0.295(0.6423)	-2.232(0.5136)	3.974(1.8932)	-3.038(1.3223)
Hybrid	6.458(3.2656)	-1.976(0.4276)	-0.302(0.6423)	-2.256(0.5136)	3.940(1.8932)	-3.135(1.3223)

Estimated θ (Standard Error)