Cluster-like coordinates in supersymmetric quantum field theory

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Recently it has become apparent that $\mathcal{N}=2$ supersymmetric quantum field theory has something to do with cluster algebras. I review one aspect of the connection: supersymmetric quantum field theories have associated hyperkähler moduli spaces, and these moduli spaces carry a structure that looks like an extension of the notion of cluster variety. In particular, one encounters the usual variables and mutations of the cluster story, along with more exotic extra variables and generalized mutations. I focus on a class of examples where the underlying cluster varieties are moduli spaces of flat connections on surfaces, as considered by Fock and Goncharov [Fock V, Goncharov A (2006) *Publ Math Inst Hautes Études Sci* 103:1–211]. The work reviewed here is largely joint with Davide Gaiotto and Greg Moore.

character varieties | Hitchin systems | supersymmetry

1. Introduction

Supersymmetric quantum field theory has been a rich source of mathematical ideas and insights over the last few decades. This paper is a review of one new facet of that story, which concerns a relation between $\mathcal{N}=2$ supersymmetric quantum field theories in four dimensions, or $\mathcal{N}=2$ theories for short, and cluster algebras.

Quantum field theory is far from a mathematically rigorous subject. Nevertheless, the study of quantum field theory sometimes leads to precise predictions about concrete mathematical objects. In particular, every $\mathcal{N}=2$ theory is expected to give rise to a family of moduli spaces $\mathcal{M}[R]$, parameterized by $R > 0$. From now on, we write $\mathcal{M}$ for any $\mathcal{M}[R]$.

$\mathcal{M}$ is expected to carry a number of interesting structures. For example, $\mathcal{M}$ is a complex integrable system, carrying a compatible hyperkähler metric. $\mathcal{M}$ is a Calabi–Yau space, which arises naturally as a special Lagrangian fibration. This setup is precisely the one introduced by Strominger et al. (1) for studying mirror symmetry of Calabi–Yau manifolds. Moreover $\mathcal{M}$ is essentially self-mirror.

In joint work of the author with Davide Gaiotto and Greg Moore (2, 3), it was appreciated that quantum field theory suggests that $\mathcal{M}$ has another important structure, a collection of canonical local Darboux coordinate systems $\{\mathcal{X}_\gamma\}$. These coordinate systems are related to one another by transformations that generalize the behavior of cluster coefficients under mutation (see Eq. 2.4). The main aim of this paper is to describe the expected properties of the functions $\mathcal{X}_\gamma$ and some examples of their construction. Most of the paper is written with a mathematical reader in mind; in section 5, I describe a bit more of the physical underpinnings of the story.

As we will describe, in some cases, the $\mathcal{X}_\gamma$ are essentially the “cluster $\mathcal{X}$ coordinates” on moduli spaces of local systems introduced by Fock and Goncharov in ref. 4. Even in these cases, the field theory perspective leads to a new geometric way of thinking about the cluster $\mathcal{X}$ coordinates. It appears, however, that there are other cases in which the $\mathcal{X}_\gamma$ are not cluster coordinates, but rather a generalization thereof. Moreover, even in the cases where $\mathcal{X}_\gamma$ are cluster coordinates, our perspective leads to considering coordinate transformations that are not cluster transformations (see Eq. 2.4 for the most general possibility and Eq. 4.4 for a concrete instance). Thus, we seem to be meeting an

enlargement of the usual cluster story, and we are hopeful that it may be of some interest for the cluster community.

The connection between $\mathcal{N}=2$ theories and the structures studied in the cluster community goes well beyond what I can describe here, and presumably there is much more still to be discovered. For some work in the physics literature, see refs. 5–9. There is also another way in which cluster algebras have been connected to supersymmetric quantum field theory, namely in perturbative scattering amplitudes in $\mathcal{N}=4$ theories (10, 11). I do not know any relation between that story and the one reviewed here, beyond the fact that cluster algebras show up in both places.

2. Expectations

We begin with a brief rundown of the expectations that come from quantum field theory: given an $\mathcal{N}=2$ theory (whatever that is), we expect to get all of the following objects and conditions. The relation to cluster algebras is most apparent in X5 and X6 below.

As we will see below, we expect in particular that we have a hyperkähler space $\mathcal{M}$, which, considered as a complex space, looks very different depending on which complex structure we use. In any complex structure J_ζ for $\zeta \in \mathbb{C}^\times$, $\mathcal{M}$ has a structure much like that of a cluster variety in the sense of ref. 4. Conversely, in complex structure J_0 , $\mathcal{M}$ looks very different: it is a complex integrable system, fibered by compact complex tori over a complex base $\mathcal{B}$.

For the rest of this paper, we will mainly focus on the complex structures J_ζ for $\zeta \in \mathbb{C}^\times$, because these are the ones that make direct contact with the cluster story. I emphasize, however, that some of the most interesting applications arise from the interplay between the cluster-like spaces at $\zeta \in \mathbb{C}^\times$ and the complex integrable system at $\zeta = 0$, expressed by Eq. 2.8. For example, this interplay is the key to a proposed scheme for describing the hyperkähler metric on $\mathcal{M}$ (2, 12).

The reader who knows a bit of quantum field theory should read this section in parallel with section 5 below, where we give an indication of where these expectations come from.

Significance

The subject of cluster algebras was born out of the study of concrete mathematical questions such as “how can we detect when a matrix will have all eigenvalues positive?” Recently it has turned out that cluster algebras show up in all kinds of unexpected places, even in the physicists’ playground of quantum field theory. This paper is a review of one way in which quantum field theory and cluster algebras interact. In particular, the paper argues that geometric ideas coming from quantum field theory lead to a natural extension of the theory of cluster algebras.

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- C1. There is a complex manifold $\mathcal{B}$. (In the best-studied examples, $\mathcal{B}$ is noncompact and indeed an affine-linear space.)
 C2. There is a local system of lattices Γ over the complement of some divisor $D \subset \mathcal{B}$, obtained as an extension

$$0 \rightarrow \Gamma_f \rightarrow \Gamma \rightarrow \Gamma_g \rightarrow 0, \quad [2.1]$$

where Γ_f is a fixed lattice (trivial local system). Γ carries an integral antisymmetric pairing $\langle \cdot, \cdot \rangle$ which vanishes on Γ_f and thus induces a pairing on Γ_g ; this induced pairing on Γ_g is unimodular (therefore, in particular, Γ_g has even rank).

- M1. For any $R > 0$, there is a hyperkähler space $\mathcal{M}[R]$.

For background on hyperkähler spaces, see refs. 13 and 14. Here we quickly review the basics. As with any hyperkähler space, $\mathcal{M}$ has a family of complex structures J_ζ parameterized by $\zeta \in \mathbb{CP}^1$ and a family of holomorphic symplectic forms ϖ_ζ . The complex structures J_ζ are organized into a single twistor space $\mathcal{Z}$. $\mathcal{Z}$ is diffeomorphic to $\mathcal{M} \times \mathbb{CP}^1$ and has a complex structure that restricts on each $\mathcal{M} \times \{\zeta\}$ to J_ζ , such that the projection $\mathcal{Z} \rightarrow \mathbb{CP}^1$ is holomorphic. We will mainly be interested in the J_ζ with $\zeta \in \mathbb{C}^\times \subset \mathbb{CP}^1$; thus, let $\mathcal{Z}'$ be the subset of $\mathcal{Z}$ corresponding to $\mathcal{M} \times \mathbb{C}^\times \subset \mathcal{M} \times \mathbb{CP}^1$. We denote points of $\mathcal{Z}'$ by $(x, \zeta) \in \mathcal{M} \times \mathbb{C}^\times$.

- M2. Viewed as a complex space using complex structure J_0 , $\mathcal{M}$ has a holomorphic projection $\pi: \mathcal{M} \rightarrow \mathcal{B}$, where $\dim \mathcal{B} = (1/2)\dim \mathcal{M}$. Each generic fiber $\mathcal{M} = \pi^{-1}(u)$ is a compact complex torus.

- X1. For a generic point $(u, \vartheta) \in \mathcal{B} \times S^1$ there is a canonical collection of $\mathbb{C}^\times$ -valued functions $\{\mathcal{X}_\gamma^{u, \vartheta}\}_{\gamma \in \Gamma_u}$. Each $\mathcal{X}_\gamma^{u, \vartheta}$ is defined on some dense open set in $\mathcal{Z}'$.

We sometimes suppress the labels (u, ϑ) ; whenever they are suppressed, they should be considered to be fixed.

- X2. The $\mathcal{X}_\gamma$ are multiplicative, in the sense that

$$\mathcal{X}_\gamma \mathcal{X}_{\gamma'} = (-1)^{\langle \gamma, \gamma' \rangle} \mathcal{X}_{\gamma + \gamma'}. \quad [2.2]$$

In particular, for any basis $\{\gamma_i\}_{i=1}^n$ of Γ , all of the $\mathcal{X}_\gamma$ are determined by $\{\mathcal{X}_{\gamma_i}\}_{i=1}^n$.

- X3. For fixed x , $\mathcal{X}_\gamma$ is holomorphic in ζ . For fixed ζ , $\mathcal{X}_\gamma$ is holomorphic in x , with respect to complex structure J_ζ on $\mathcal{M}$. In other words, $\mathcal{X}_\gamma$ is holomorphic on $\mathcal{Z}' \subset \mathcal{Z}$.

- X4. For any fixed $\zeta \times \mathbb{C}^\times$, the $\mathcal{X}_\gamma$ form a Darboux coordinate system on the holomorphic symplectic manifold $(\mathcal{M}, J_\zeta, \varpi_\zeta)$, in the following sense. The holomorphic symplectic structure induces a holomorphic Poisson bracket, and this bracket is very simple in terms of the $\mathcal{X}_\gamma$

$$\{\mathcal{X}_\gamma, \mathcal{X}_{\gamma'}\} = \langle \gamma, \gamma' \rangle \mathcal{X}_{\gamma + \gamma'}. \quad [2.3]$$

- X5. The coordinate system $\mathcal{X}_\gamma^{u, \vartheta}$ depends on (u, ϑ) in a piecewise constant fashion, but jumps when (u, ϑ) reach some real-codimension-1 walls in $\mathcal{B} \times S^1$.

Each wall is transverse to the S^1 factor and therefore has a + side and a − side. The relation between the $\mathcal{X}_\gamma^{u, \vartheta}$ on the two sides is computable in terms of data attached to the wall, namely an element $\gamma_h \in \Gamma$, integers $\{\Omega(\gamma)\}_{\gamma \in \Gamma}$, and signs $\sigma_n = \pm 1$

$$\mathcal{X}_\gamma^+ = \mathcal{X}_\gamma^- \prod_{n=1}^{\infty} (1 - \sigma_n \mathcal{X}_{n\gamma_h})^{n\Omega(n\gamma_h)\langle \gamma, \gamma_h \rangle}. \quad [2.4]$$

An important special case arises when $\Omega(\gamma_h) = 1$, $\Omega(n\gamma_h) = 0$ for $n > 1$, and $\sigma_1 = -1$. In this case

$$\mathcal{X}_\gamma^+ = \mathcal{X}_\gamma^- (1 + \mathcal{X}_{\gamma_h})^{\langle \gamma, \gamma_h \rangle}, \quad [2.5]$$

for some fixed $\gamma_h \in \Gamma$. This rule will look familiar to cluster enthusiasts: it is very close to the mutation law for coefficients in the sense of ref. 15 or, equivalently, the coordinate change under mutation in a cluster $\mathcal{X}$ variety in the sense of ref. 4.

Eq. 2.5 is not quite identical to the mutation in the $\mathcal{X}$ variety, but this difference arises for an easy reason: the cluster coordinates x_i are labeled by the elements i of some seed, whereas our functions $\mathcal{X}_\gamma$ are labeled by elements $\gamma \in \Gamma$; thus, to relate the two, we need to assign a basis element $\gamma_i \in \Gamma$ to each i ; after this is done, the two transformation laws become literally identical.

The changes of coordinates (Eq. 2.4) are a generalization of the mutation in the $\mathcal{X}$ variety; therefore, in the rest of this paper, we will call them generalized mutations.

As an aside, the integers $\Omega(\gamma)$ in Eq. 2.4 are expected to be generalized Donaldson–Thomas invariants, in the sense of refs. 16 and 17. In the case $K = 2$, this expectation has been made precise in ref. 18. The celebrated wall-crossing formula for generalized Donaldson–Thomas invariants, in the form written in ref. 16, has a simple interpretation in our setting: it is the fact that if (u, ϑ) travel around a contractible loop in $\mathcal{B} \times S^1$, the $\mathcal{X}_\gamma^{u, \vartheta}$ undergo a sequence of jumps of the form in Eq. 2.4, whose product must be the identity.

- X6. There exists a distinguished collection $\{L_a\}$ of regular functions on $\mathcal{Z}'$, such that for each (u, ϑ) and each a , one can expand L_a in the form

$$L_a = \sum_\gamma \overline{\Omega}^{u, \vartheta}(a, \gamma) \mathcal{X}_\gamma^{u, \vartheta}, \quad [2.6]$$

where $\overline{\Omega}^{u, \vartheta}(a, \gamma) \in \mathbb{Z}$. The coefficients $\overline{\Omega}^{u, \vartheta}(a, \gamma)$ jump when (u, ϑ) cross a wall. [The precise jump can be computed from the fact that $\mathcal{X}_\gamma^{u, \vartheta}$ jumps by Eq. 2.4 when (u, ϑ) cross a wall, whereas L_a is independent of (u, ϑ) .]

In some examples, the L_a resemble the “dual canonical basis,” which was part of the motivation for the definition of cluster algebra, e.g., compare the last section of ref. 19 with ref. 20. (I thank Hugh Thomas for explaining this to me.) The L_a are also close to the functions appearing in the duality conjectures of Fock and Goncharov (4).

- X7. If $\zeta \rightarrow 0$ while remaining in the half-space centered on ϑ , and if

$$u = \pi(x), \quad [2.7]$$

then each $\mathcal{X}_\gamma^{u, \vartheta}(x, \zeta)$ has asymptotics of the form

$$\mathcal{X}_\gamma^{u, \vartheta}(x, \zeta) \sim c_\gamma(x) \exp[Z_\gamma(u)/\zeta], \quad [2.8]$$

where $Z_\gamma(u)$ depends holomorphically on u .

3. Theories of Class S

We now specialize to the theories of class S, for which all of the story of section 2 has been developed. More specifically, we will consider theories associated to the Lie algebra $\mathfrak{u}(K)$, which is the only case for which the functions $\mathcal{X}_\gamma$ have been thoroughly investigated to date.

Thus, let $\mathfrak{g} = \mathfrak{u}(K)$ and let $\mathfrak{h}$ be the Cartan subalgebra consisting of diagonal matrices. Let C be a compact Riemann surface of genus g with $n > 0$ marked points $z_1, \dots, z_n$. Also fix parameters $m_i^{\mathbb{C}} \in \mathfrak{h}_{\mathbb{C}}$ and $m_i^{\mathbb{R}} \in \mathfrak{h}$ for each marked point, with the $m_i^{\mathbb{C}}$ chosen linearly independent over $\mathbb{Z}$. We think of the marked

points and parameters as included in the definition of the surface C . Given these data, there is an $\mathcal{N}=2$ theory $S[g, C]$ (3, 21, 22), which we call a theory of class S .

We now describe the data $\mathbf{C}$ and $\mathbf{M}$ in these theories; this amounts to a lightning review of the basic facts about Hitchin's integrable system.

C1. $\mathcal{B}$ is the space of all tuples

$$u = (\varphi_1, \dots, \varphi_K), \quad [3.1]$$

where φ_k is a meromorphic k -differential, i.e., a section of the k th power of T^*C , such that φ_k has poles only at the points z_i of order k , and the residue of φ_k at z_i is the coefficient of λ^{K-k} in the characteristic polynomial $\det(m_i^C - \lambda)$.

$\mathcal{B}$ is an affine-linear space: the difference between any two tuples in $\mathcal{B}$ is an element of the vector space of tuples $(\tilde{\varphi}_1, \dots, \tilde{\varphi}_K)$ where $\tilde{\varphi}_k$ is a meromorphic k -differential, with poles only at the z_i , of order $\leq k-1$.

C2. Given a point $u = (\varphi_1, \dots, \varphi_K) \in \mathcal{B}$, we define the corresponding spectral curve

$$\Sigma_u = \left\{ \sum_{k=0}^K \lambda^{K-k} \varphi_k = 0 \right\} \subset T^*C, \quad [3.2]$$

where we set $\varphi_0 = 1$. Σ_u is a K -fold branched cover of $C \setminus \{z_1, \dots, z_n\}$. It can be compactified to a branched cover $\bar{\Sigma}_u$ of C , unbranched over the z_i .

The singular divisor $D \subset \mathcal{B}$ is the locus for which Σ_u is a singular curve. For $u \notin D$, we define $\Gamma_u = H_1(\Sigma_u, \mathbb{Z})$. The quotient $(\Gamma_g)_u$ is $H_1(\bar{\Sigma}_u, \mathbb{Z})$, with the quotient map $\Gamma_u \rightarrow (\Gamma_g)_u$ induced by the inclusion $\Sigma_u \subset \bar{\Sigma}_u$.

M1. $\mathcal{M}[R]$ is a moduli space of harmonic G bundles on C , defined as follows.

A harmonic G bundle on C is a triple (E, D, φ) , where E is a rank K Hermitian vector bundle on C , D is a unitary connection in E , an φ is a section of $\text{End}(E) \otimes K_C$ obeying the Hitchin equations (23)

$$F_D - R^2[\varphi, \varphi^\dagger] = 0, \quad [3.3a]$$

$$\bar{\partial}_D \varphi = 0, \quad [3.3b]$$

such that D and φ are smooth on C except at the marked points z_i and in a neighborhood of z_i have first-order singularities of the form (in some gauge)

$$D = D_{\text{reg}} + m_i^{\mathbb{R}} \left(\frac{dz}{z - z_i} - \frac{d\bar{z}}{\bar{z} - \bar{z}_i} \right), \quad [3.4a]$$

$$\varphi = \varphi_{\text{reg}} + m_i^C \left(\frac{dz}{z - z_i} \right), \quad [3.4b]$$

where D_{reg} and φ_{reg} are regular at z_i .

By an application of the gauge-theoretic tools developed in ref. 24, it has been shown that solutions to Eq. 3.3 indeed exist (23, 25) and that one can define a generically smooth, finite-dimensional moduli space $\mathcal{M}$ parameterizing harmonic G bundles on C modulo gauge equivalence. Moreover, as explained in ref. 23, $\mathcal{M}$ is naturally hyperkähler.

As expected for a hyperkähler space, $\mathcal{M}$ admits a canonical family of complex structures J_ζ , $\zeta \in \mathbb{CP}^1$. Each fixed ζ

corresponds to a different lens through which to view the harmonic bundles, by “forgetting” part of their structure. We will consider $\zeta = 0$ in M2 below, and $\zeta \in \mathbb{C}^\times$ in section 4.

M2. Given a harmonic bundle (E, D, φ) , let $\bar{\partial}_D$ be the $(0, 1)$ part of D and let ϕ be the $(1, 0)$ part of φ . The pair $(E, \bar{\partial}_D)$ define a holomorphic vector bundle over C ; call this holomorphic bundle E_h . Then, Eq. 3.3b says that ϕ is a meromorphic section of $\text{End}(E_h) \otimes K_C$. Thus, the pair (E_h, ϕ) are a G -Higgs bundle over C , with first-order poles at z_i , whose residues are determined by m_i^C .

Let $\mathcal{M}^H$ denote the moduli space of G -Higgs bundles with these singularities. This space carries a natural complex structure. Remarkably, the forgetful map $(E, D, \varphi) \mapsto (E_h, \phi)$ induces a diffeomorphism $\mathcal{M} \simeq \mathcal{M}^H$ and thus induces a complex structure on $\mathcal{M}$. This induced complex structure is J_0 .

Now suppose a given point of $\mathcal{M}^H$, represented by a Higgs bundle (E_h, ϕ) . The characteristic polynomial of ϕ is of the form

$$P_\phi(\lambda) = \sum_{i=1}^K \varphi_i \lambda^{K-i}, \quad [3.5]$$

where each φ_i is a meromorphic section of $K_C^{\otimes i}$, with poles at the marked points z_i . Passing from (E_h, ϕ) to the tuple $(\varphi_1, \dots, \varphi_K)$ gives the holomorphic projection $\pi: \mathcal{M}^H \rightarrow \mathcal{B}$.

The discussion above admits various possible generalizations:

- One could consider harmonic bundles where φ has poles of order > 1 (irregular punctures). The mathematical theory of such bundles has been worked out in ref. 26. In fact, some of the technically simplest examples of the whole story are of this sort. We suppress this generalization here only to lighten the notation.
- One could consider surfaces C with no punctures at all, or with punctures where the residue of φ is nilpotent. These cases are technically more difficult to treat, although all of the formal expectations from quantum field theory (QFT) are the same.

Finally, although in this paper we are focusing on the theories of class S , we should note that there is another class of $\mathcal{N}=2$ theories that has been studied extensively, namely the class of quiver gauge theories. For these theories, the data in C1, C2, M3, and M4 have been described comprehensively in a recent work (27) (and many special cases were known earlier). However, in these cases, the functions $\mathcal{X}_\gamma$ of X1–X7 have not yet been described. Thus, it seems that the cluster structure in quiver gauge theories is waiting to be discovered. It should be very interesting to do so.

4. Spectral Coordinates

In the next few sections, following refs. 3 and 28, we will describe the data in X1–X7 in the theories of class S , first in the special case of $G = U(2)$ and then more generally for $G = U(K)$.

As we have mentioned, these data are most directly related to the complex structures J_ζ on $\mathcal{M}$ for $\zeta \in \mathbb{C}^\times$. Thus, we first describe what those complex structures look like.

Fix some $\zeta \in \mathbb{C}^\times$. Given a harmonic bundle (E, D, φ) , we may then build the combination

$$\nabla = \zeta^{-1} \varphi + D + \zeta \varphi^\dagger. \quad [4.1]$$

The resulting ∇ is a flat $G_\mathbb{C}$ connection (as follows from Eq. 3.3). Its monodromy around z_i is given by $\mu_i = \exp 2\pi i (\zeta^{-1} m_i^C + m_i^{\mathbb{R}} + \zeta m_i^C)$. Let $\mathcal{M}^b$ denote the moduli space of such connections. $\mathcal{M}^b$ carries a natural complex structure (coming from the complex structure in $G_\mathbb{C}$).

Fig. 1. Behavior of the foliation $F(u, \vartheta)$ around a second-order pole of φ_2 .

The forgetful map $(E, D, \varphi) \mapsto \nabla$ induces a diffeomorphism $\mathcal{M} \simeq \mathcal{M}^b$. This diffeomorphism induces a complex structure J_ζ on $\mathcal{M}$ (depending on the parameter $\zeta \in \mathbb{C}^\times$, which entered through the Eq. 4.1). Moreover, $\mathcal{M}^b$ also carries a holomorphic symplectic form [the complexification of the one described by Atiyah and Bott for G connections (29)], so we obtain a family of holomorphic symplectic forms ϖ_ζ on $\mathcal{M}$.

4.1. Case $G = U(2)$. We now specialize to $G = U(2)$. In this case, the construction of the $\mathcal{X}_\gamma$ was given in ref. 3.

Suppose a generic $(x, \zeta) \in \mathcal{Z}'$. As just explained, (x, ζ) determine a flat $GL(2, \mathbb{C})$ connection ∇ up to equivalence, via Eq. 4.1. Roughly, the desired $\mathcal{X}_\gamma^{u, \vartheta}(x, \zeta)$ will be “complexified shear coordinates” of the connection ∇ . These coordinates have been studied by many authors; for us, the most relevant reference is ref. 4, where they were linked to cluster structure on moduli spaces of flat connections. In our setup, for each generic (u, ϑ) , we will consider a preferred shear coordinate system, from which we build the $\mathcal{X}_\gamma^{u, \vartheta}$.

First we should understand what (u, ϑ) mean in this case. As we have reviewed above, $\mathcal{B}$ is a space of pairs (φ_1, φ_2) . In what follows, φ_1 plays no role, so we focus on φ_2 : it is a meromorphic quadratic differential on C , with a second-order pole at each z_i , of residue m_i^2 . Fixing u means fixing one such φ_2 . Fixing u generic means that φ_2 should have only simple zeroes.

We consider a singular foliation $F(u, \vartheta)$ of C whose leaves are the trajectories along which $e^{2i\vartheta}\varphi_2$ is real. $F(u, \vartheta)$ has singularities only at the zeroes and poles of φ_2 . Around a pole z_i of φ_2 , the picture depends on $e^{-i\vartheta}m_i^2$, as indicated in Fig. 1. Around a zero of φ_2 , $F(u, \vartheta)$ looks like Fig. 2; there are three distinguished “critical trajectories” emerging from the zero. Let $CG(u, \vartheta)$ denote the union of the critical trajectories (critical graph). Using results from ref. 30, we can describe the global topology of $CG(u, \vartheta)$ for generic (u, ϑ) : it divides C up into cells of two types, illustrated in Fig. 3. From now on, for simplicity, we suppose all of the cells are like the one on the left of Fig. 3. In this case, the critical graph $CG(u, \vartheta)$ induces a corresponding ideal triangulation of C as shown in Fig. 4.

Each cell c in C is contained in a quadrilateral $Q_c \subset C$, as shown in Fig. 5. We will build one coordinate function x_c from each c . Because Q_c is simply connected, the space of flat sections of ∇ over Q_c is a 2D vector space V . V has four natural

Fig. 2. Behavior of the foliation $F(u, \vartheta)$ around a simple zero of φ_2 .

Fig. 3. Cells in the complement of the critical graph $CG(u, \vartheta)$.

endomorphisms M_i , given by monodromy of ∇ around the four marked points z_i on the boundary of Q_c . Because ∇ is generic, each M_i is semisimple and hence decomposes V into two eigenlines. Moreover, because in Eq. 3.4 the singular parts of D and φ commute, these two eigenlines are also the eigenlines of the residue m_i^C of φ . Let ℓ_i be the eigenline of M_i corresponding to the eigenvalue β of m_i^C with the largest $\text{Re}(e^{-i\vartheta}\beta)$. We thus have four distinguished lines $\ell_i \subset V$, i.e., four points of the projective line $\mathbb{P}(V)$. Let $x_c \in \mathbb{C}^\times$ be the cross-ratio of these four points.

Now we are ready to build the desired functions $\mathcal{X}_\gamma^{u, \vartheta}$. Given c , there is a canonical $\gamma_c \in \Gamma_u$, pictured in Fig. 6. We define

$$\mathcal{X}_{\gamma_c}^{u, \vartheta} = x_c. \quad [4.2]$$

The γ_c are a basis for the sublattice of Γ_u , which is odd under the automorphism σ of Σ exchanging the two sheets. Thus, Eqs. 4.2 and 2.2 together determine $\mathcal{X}_\gamma^{u, \vartheta}(x, \zeta)$ for all σ -odd γ .

The σ -even part of Γ_u plays a minor role and could be eliminated by considering $G = SU(2)$ instead of $G = U(2)$. We will not discuss it explicitly here (it is automatically incorporated in the more general formalism of section 4.2).

The functions $\mathcal{X}_\gamma^{u, \vartheta}$ so constructed have all of the properties of X1–X7. In particular

- The generalized mutations (X5) emerge when we consider the dependence of $\mathcal{X}_\gamma^{u, \vartheta}$ on (u, ϑ) . The data (u, ϑ) are entered in the definition of $CG(u, \vartheta)$ and in the choice of the eigenlines ℓ_i . We consider these two dependences in turn.

First, consider how $CG(u, \vartheta)$ depends on $(u, \vartheta) \in \mathcal{B} \times S^1$. Beginning at a generic (u, ϑ) , small variations of (u, ϑ) lead to $CG(u, \vartheta)$ varying by an isotopy. The construction of $\mathcal{X}_\gamma^{u, \vartheta}$ evidently depends on $CG(u, \vartheta)$ only up to isotopy. Hence $\mathcal{X}_\gamma^{u, \vartheta}$ is not changed by sufficiently small variations of (u, ϑ) . However, there are codimension-1 walls in $\mathcal{B} \times S^1$ where $CG(u, \vartheta)$ jumps discontinuously. These walls are the ones that appear in X5.

One such jump is illustrated in Fig. 7: at $\vartheta = \vartheta_c$, the critical graph $CG(u, \vartheta)$ degenerates, because of the presence of a saddle connection. The corresponding triangulations are related by a flip. Let $\mathcal{X}^\pm$ denote the value of $\mathcal{X}_\gamma^{u, \vartheta}$ for (u, ϑ) on the $\pm$

Fig. 4. Relation between the critical graph $CG(u, \vartheta)$ and the corresponding induced ideal triangulation of C .

Fig. 5. Quadrilateral Q_c containing a cell c .

side; then, a straightforward calculation shows $\mathcal{X}^+$ and $\mathcal{X}^-$ are related by

$$\mathcal{X}_\gamma^+ = \mathcal{X}_\gamma^- \left(1 + \mathcal{X}_{\gamma_h}^- \right)^{\langle \gamma, \gamma_h \rangle}. \quad [4.3]$$

This equation matches the mutation law (Eq. 2.5).

- There is a second jumping phenomenon, associated with a different way in which $CG(u, \vartheta)$ can degenerate: a closed trajectory can appear. The associated transformation is of the form

$$\mathcal{X}_\gamma^+ = \mathcal{X}_\gamma^- \left(1 - \mathcal{X}_{\gamma_h}^- \right)^{-2\langle \gamma, \gamma_h \rangle}. \quad [4.4]$$

This transformation differs from Eq. 4.3 by the factor -2 in the exponent, and hence, Eq. 4.4 cannot be interpreted as a mutation. It is a generalized mutation (Eq. 2.4).

We must explain precisely what is meant by the $\mathcal{X}^+$ and $\mathcal{X}^-$ in Eq. 4.4. Care is needed here, because if (u, ϑ_c) lies on the locus where a closed trajectory appears, the behavior of $F(u, \vartheta)$ in a neighborhood of (u, ϑ_c) is wild: there are phases ϑ_n^+ and ϑ_n^- , with $\vartheta_n^- < \vartheta_c < \vartheta_n^+$, such that $\lim_{n \rightarrow \infty} \vartheta_n^\pm = \vartheta_c$, and at each $(u, \vartheta_n^\pm)$, there is a saddle connection. Each of these two infinite sequences of saddle connections consists of trajectories winding more and more tightly around a cylinder, which becomes foliated by closed trajectories at (u, ϑ_c) (see ref. 3 for pictures or ref. 31 for a movie illustrating this phenomenon).

Thus, the walls in $\mathcal{B} \times \mathcal{S}^1$ associated with closed trajectories are never isolated: rather they are loci where walls coming from saddle connections accumulate. Then we may ask: what happens to the functions $\mathcal{X}_\gamma^{u, \vartheta}$ in the limit as $\vartheta \rightarrow \vartheta_c^\pm$? They infinitely undergo many mutations (Eq. 4.3) as we approach this locus, but nevertheless, $\mathcal{X}_\gamma^\pm = \lim_{\vartheta \rightarrow \vartheta_c^\pm} \mathcal{X}_\gamma^{u, \vartheta}$ exists. These $\mathcal{X}_\gamma^\pm$ are the

Fig. 6. Cycle $\gamma_c \in \Gamma_u$ corresponding to a cell c .

Fig. 7. A jump of the critical graph $CG(u, \vartheta)$, induced by the appearance of a saddle connection.

quantities that appear in Eq. 4.4. [They also have an independent interpretation as local coordinate systems that include complexified Fenchel–Nielsen coordinates on $\mathcal{M}$, as well as complexified shear coordinates (32).] This limit was analyzed directly in ref. 3 and more indirectly (but more elegantly) in refs. 28 and 32.

A closely related construction recently appeared in ref. 33: roughly our $\mathcal{X}_\gamma^\pm$ should be understood as monomials attached to their limit triangulations.

- There are also some walls in $\mathcal{B} \times \mathcal{S}^1$ where the eigenvalues β of m_c^∇ have equal $\text{Re}(e^{-\vartheta}\beta)$. Upon crossing one of these walls, the eigenline ℓ_i changes, and the critical graph $CG(u, \vartheta)$ jumps. When both of these phenomena are taken into account, the functions $\mathcal{X}_\gamma$ are invariant, i.e., $\mathcal{X}_\gamma^+ = \mathcal{X}_\gamma^-$ (3).
- The functions L_c of X6 correspond to multicurves c on C . If c is a single curve on C that is not isotopic to a loop around any z_i , carrying weight k , $L_c(x, \zeta)$ is the trace of the monodromy of ∇ around c , taken in the k th symmetric power of the defining representation of $U(2)$. If c consists of several curves, then L_c is a product of such traces for each curve.
- The asymptotic property in X7 comes from the Wentzel–Kramers–Brillouin approximation applied to the family of connections $\nabla(\zeta)$ as $\zeta \rightarrow 0$. The function Z_γ appearing in the asymptotic is a period

$$Z_\gamma = \oint_\gamma \lambda, \quad [4.5]$$

where λ is the Liouville 1-form on T^*C .

This whole story is closely related to the cluster $\mathfrak{X}$ variety structure on $\mathcal{M}$ (or more precisely a covering space of $\mathcal{M}$) in ref. 4 and the “cluster algebras from surfaces” in ref. 34. Each generic (u, ϑ) induces a seed in the corresponding cluster $\mathfrak{X}$ variety, and the functions $\mathcal{X}_\gamma^{u, \vartheta}$ are the coordinates on the cluster torus. The mutations described above, which occur when a saddle connection appears in $CG(u, \vartheta)$, and the corresponding triangulation flips are the same ones that appear in refs. 4 and 34.

4.2. Case $G = U(K)$.

Now we consider the case $G = U(K)$ where $K \geq 2$. Fix a generic $(x, \zeta) \in \mathcal{Z}'$. (x, ζ) determine a flat $GL(K, \mathbb{C})$ connection ∇ up to isomorphism, via Eq. 4.1. The desired $\mathcal{X}_\gamma^{u, \vartheta}(x, \zeta)$ are a new kind of coordinates associated with ∇ . Their construction has not been completely worked out in general, but some special cases are understood.

The construction involves an auxiliary gadget $\mathcal{W}(u, \vartheta)$, which we call a spectral network, generalizing the critical graphs $CG(u, \vartheta)$ that appeared in the $K = 2$ case. We now describe the construction of $\mathcal{W}(u, \vartheta)$.

Over any contractible $U \subset C$ containing no branch points of $\Sigma \rightarrow C$, we may trivialize the covering Σ , labeling the sheets by $i = 1, \dots, K$. The 1-form λ on Σ then induces holomorphic 1-forms $\lambda_1, \dots, \lambda_K$ on U . We want to consider paths $r(t)$ on C that locally obey an equation of the form $[\lambda_i(r) - \lambda_j(r)]\dot{r} = 1$. More precisely, let a trajectory be an oriented ray $r: [0, \infty) \rightarrow C$ with a closed

Fig. 8. Three distinguished trajectories emerging from a branch point.

image, such that for any $t \in (0, \infty)$, there exists a neighborhood N of t and a pair (i, j) of sheets of Σ over $r(N)$, with

$$\dot{r}(t) = 1/(\lambda_i - \lambda_j)[r(t)] \quad \text{for } t \in N. \quad [4.6]$$

Given initial data, consisting of a point $z \in C$ that is not a branch point and a pair (i, j) of sheets of Σ over a neighborhood of z , there is a unique trajectory r determined by these initial data.

To draw concrete pictures, it is convenient to trivialize Σ away from a collection of “branch cuts” on C . On the complement of the cuts, we can label each trajectory by a pair (i, j) according to which equation it obeys. When a trajectory of type (i, j) crosses a branch cut, its label changes to (i^σ, j^σ) , where σ is the permutation of sheets associated to the cut.

Consider a simple branch point b of $\Sigma \rightarrow C$ at which two sheets collide. Monodromy around b exchanges those two sheets. A short computation shows that there are three distinguished trajectories emerging from b (Fig. 8).

Now we can define the spectral network $\mathcal{W}(u, \vartheta)$: it is the smallest collection of trajectories on C such that the three distinguished trajectories emerging from each branch point belong to $\mathcal{W}(u, \vartheta)$ and if $\mathcal{W}(u, \vartheta)$ contains two trajectories that intersect, and near the intersection point these trajectories carry labels (i, j) and (j, k) , then $\mathcal{W}(u, \vartheta)$ contains another trajectory, which originates from the intersection point and carries the label (i, k) , as shown in Fig. 9.

In the case $K = 2$, trajectories never cross, and in that case, $\mathcal{W}(u, \vartheta)$ is just the critical graph $CG(u, \vartheta)$ that we considered in section 4.1. For $K > 2$, the structure of $\mathcal{W}(u, \vartheta)$ can be much more intricate. There are examples of C for which $\mathcal{W}(u, \vartheta)$ is always well behaved—in particular, it is locally finite on C . Fig. 10 shows an example. However, for general C , even with punctures, $\mathcal{W}(u, \vartheta)$ may be dense in parts of C .

Having defined $\mathcal{W}(u, \vartheta)$, we return to our original goal: the construction of the functions $\mathcal{X}_\gamma^{u, \vartheta}$. The construction works by associating to the $GL(K)$ connection ∇ over C a corresponding $GL(1)$ connection ∇^{ab} over Σ , in a way depending on the network $\mathcal{W}(u, \vartheta)$. The desired $\mathcal{X}_\gamma^{u, \vartheta}(x, \zeta) \in \mathbb{C}^\times$ will simply be the holonomies of ∇^{ab} around cycles γ .

The connections ∇^{ab} and ∇ are supposed to be related by $\mathcal{W}(u, \vartheta)$ abelianization, which means an additional datum ι as follows:

- Consider the pushforward connection $\pi_* \nabla^{\text{ab}}$, defined on the complement of the branch locus of $\Sigma \rightarrow C$. ι is an isomorphism $\iota: \nabla \simeq \pi_* \nabla^{\text{ab}}$ on $C \setminus \mathcal{W}(u, \vartheta)$.
- The limits $\iota_\pm$ of ι as we approach $\mathcal{W}(u, \vartheta)$ need not be equal, but their difference is constrained, as follows. Fix a point z on $\mathcal{W}(u, \vartheta)$, a contractible neighborhood U of z , and a trivialization

Fig. 9. The rule by which colliding trajectories in $\mathcal{W}(u, \vartheta)$ give birth to new ones.

Fig. 10. Example of a spectral network.

of $\Sigma \rightarrow C$ over U . Let L denote the line bundle over Σ underlying ∇^{ab} . On U , the vector bundle underlying $\pi_* \nabla^{\text{ab}}$ is canonically decomposed as

$$\pi_* L = \bigoplus_{k=1}^K L_k. \quad [4.7]$$

We require

$$\iota_+ = (\text{Id} + S) \circ \iota_-, \quad [4.8]$$

where $S: \pi_* L \rightarrow \pi_* L$ maps $L_i \rightarrow L_j$ and maps all of the other summands to zero, and (i, j) is the labeling of the trajectory where z sits. [Thus, in a basis respecting Eq. 4.7, S is a matrix whose only nonzero entry is in the (i, j) position.]

Refs. 28 and 32 provide more details on abelianization. We expect (but have not proven!) that a generic connection ∇ can be $\mathcal{W}(u, \vartheta)$ abelianized only in finitely many ways and that fixing (u, ϑ) also gives a way of fixing this discrete choice to get a preferred $\mathcal{W}(u, \vartheta)$ abelianization. (If $K = 2$, this discrete choice is the choice of lines ℓ_i we used in section 4.1.)

One class of examples of $\mathcal{W}(u, \vartheta)$ where everything works as expected was considered in ref. 35. One can construct them by beginning with the critical graph for a quadratic differential and then replacing each branch point by a cluster of $K(K - 1)/2$ branch points, in one of two standard structures called “Yin” and “Yang”; the Yin case is illustrated in Fig. 11. Each of the trajectories that appear in the $K = 2$ case then gets replaced by a “cable” of $K(K - 1)/2$ trajectories. The global structure is governed by an ideal triangulation of C just as in the $K = 2$ case (Fig. 12). When all triangles are of the Yin type, it is shown in ref. 35 that for particular choices of cycles $\gamma_i \in \Gamma$, the spectral coordinates $\mathcal{X}_\gamma^{u, \vartheta}$ are the coordinate functions in the $\mathcal{X}$ cluster variety introduced by Fock and Goncharov (4). Thus, we get a new way of thinking about these cluster coordinates: they can be explained

Fig. 11. The local structure of a spectral network around a cluster of branch points, in the Yin case, where we took $K = 4$.

Fig. 12. (Left) Relation between a spectral network with $K = 2$ and corresponding ideal triangulation. (Right) Corresponding picture with $K = 3$, where each branch point has been replaced by a cluster of three branch points like that of Fig. 11, and each trajectory has been fattened into a cable of three trajectories.

by the abelianization construction. We believe that this is likely to be a useful point of view.

It is argued in ref. 28 that the $\mathcal{X}_\gamma^{u,\vartheta}$ have the properties of X1–X7, generalizing what we said in section 4.1 in the case $G = U(2)$. In particular, for sufficiently small variations of (u, ϑ) , the network $\mathcal{W}(u, \vartheta)$ varies by a generalization of isotopy, called “equivalence” in ref. 28. As before, there are codimension-1 walls in $\mathcal{B} \times S^1$ where $\mathcal{W}(u, \vartheta)$ undergoes a topology change. The jump of the $\mathcal{X}_\gamma^{u,\vartheta}$ at such a wall can always be written as a generalized mutation (Eq. 2.4). In the case $G = U(2)$, we had just two possible kinds of topology change for $CG(u, \vartheta)$ and two corresponding jump formulas (Eqs. 4.3 and 4.4). For $G = U(K)$, $K > 2$, the story seems to be much wilder: there is a recipe in ref. 28 for determining the jump associated to any particular topology change of $\mathcal{W}(u, \vartheta)$, but there is no classification of all possible topology changes.

The simplest topology change that can occur for $U(K)$ but not for $U(2)$ is shown in Fig. 13. This particular change leads to a mutation. At the moment when the mutation occurs, a new finite subnetwork appears in $\mathcal{W}(u, \vartheta)$; this three-pronged subnetwork is a generalization of the saddle connections from the $K = 2$ case. A few other simple examples are discussed in ref. 28. A more elaborate topology change and its corresponding generalized mutation are considered in ref. 36.

We emphasize that, in contrast to the $G = U(2)$ case, for $G = U(K)$, a generic $\mathcal{W}(u, \vartheta)$ is not governed by an ideal triangulation. Correspondingly, the spectral coordinates $\mathcal{X}_\gamma^{u,\vartheta}$ are more general than the ones assigned to triangulations. We do not know whether they exhaust the set of coordinate systems in the cluster atlas of ref. 4.

In addition, it seems possible that some of the coordinate systems $\mathcal{X}_\gamma^{u,\vartheta}$ are not part of the cluster atlas at all. The reason is that there can be domains in $\mathcal{B} \times S^1$ where the walls are actually dense (36). To escape from such a domain, one would have to cross infinitely many walls; thus, if (u, ϑ) sit in such a domain, then there seems to be no reason why the coordinate system $\mathcal{X}_\gamma^{u,\vartheta}$ should be connected to a cluster coordinate system by finitely many mutations. The role of these coordinate systems should be similar to that of the limiting coordinate systems we described in the $U(2)$ case above: those limiting coordinate systems also did not belong to the cluster atlas, because they were separated from the cluster coordinate systems by infinite chains of mutations.

Fig. 13. A jump in the topology of a spectral network with $K = 3$.

5. Some QFT

Finally, for readers with an interest in quantum field theory, we briefly revisit the list of key properties from section 2 and describe where they come from.

- C1. $\mathcal{B}$ is the Coulomb branch of the $\mathcal{N} = 2$ theory. This is one branch of the moduli space of Poincaré invariant quantum vacua of the theory. Its general structure was described in ref. 37, where one particularly important example was worked out, namely pure $SU(2)$ super Yang–Mills theory. [That theory is in fact also an example of a theory of class $\mathcal{S}$ with $G = SU(2)$, as explained in ref. 21 and later revisited in a context close to that of this paper in ref. 3.]
- C2. Γ is the lattice of charges carried by states in the low-energy approximation to the full $\mathcal{N} = 2$ theory. The quotient Γ_g consists of electromagnetic charges in the usual sense. The sublattice Γ_f consists of pure flavor charges, related by Noether’s theorem to a global symmetry rather than a gauge symmetry.
- M1. $\mathcal{M}[R]$ is the target of a 3D sigma model obtained by compactifying our $\mathcal{N} = 2$ theory on a circle of length R and considering the physics at energies $E \ll 1/R$. Such a target space is necessarily hyperkähler as explained in refs. 38 and 39. The various complex structures J_ζ correspond to various subalgebras of the supersymmetry algebra of this sigma model.
- M2. The existence and holomorphy of the projection π follow from the fact that supersymmetric local operators of the original $\mathcal{N} = 2$ theory also give supersymmetric local operators in the theory on $\mathbb{R}^3 \times S^1$ (invariant under a subalgebra of the supersymmetry algebra, corresponding to $J_\zeta = 0$).
- X1. The $\mathcal{X}_\gamma^{u,\vartheta}$ are the vacuum expectation values of certain supersymmetric line defects in the low energy approximation to the $\mathcal{N} = 2$ theory, when these defects are placed at the locus $\{\text{point}\} \times S^1 \subset \mathbb{R}^3 \times S^1$.
- X2. The product law for $\mathcal{X}_\gamma^{u,\vartheta}$ follows from the operator product algebra of line defects in the low energy $\mathcal{N} = 2$ theory with the abelian gauge group.
- X3. The holomorphy of $\mathcal{X}_\gamma$ for fixed x is a consequence of the supersymmetry of the corresponding line defect.
- X4. The simplicity of the Poisson brackets is shown in refs. 2 and 40 but in a somewhat indirect way. It should follow from the fact that the Poisson bracket can be expressed in terms of topological field theory (Rozansky–Witten theory), although I believe this has not been fully worked out.
- X5. $\Omega(\gamma)$ is an index counting Bogomolny–Prasad–Sommerfield (BPS) states of charge γ in the $\mathcal{N} = 2$ theory. The generalized mutations (Eq. 2.4) are most easily understood in terms of framed wall-crossing (see below).
- X6. The functions L_a are vacuum expectation values of line defects of the original $\mathcal{N} = 2$ theory. The coefficients $\overline{\Omega}(a, \gamma)$ govern the decomposition of the line defects of the original theory into line defects of the low energy theory. They can also be understood as dimensions of certain vector spaces: the spaces of framed BPS states. The jumps of $\overline{\Omega}(a, \gamma)$ arise from framed wall-crossing: as the parameters (x, ζ) of the field theory and line defect vary, framed BPS states may enter or leave the Hilbert space from infinity. The picture is an infinitely heavy particle sitting at the origin of $\mathbb{R}^3$, which decays by emitting a bulk particle or forms a bound state by attracting a bulk particle.
- X7. This asymptotic property was deduced indirectly in refs. 2 and 40, but (as far as I know) has not yet been explained in a physically satisfying way.

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1. Strominger A, Yau S-T, Zaslow E (1996) Mirror symmetry is T-duality. *Nucl Phys B* 479: 243–259.
2. Gaiotto D, Moore GW, Neitzke A (2010) Four-dimensional wall-crossing via three-dimensional field theory. *Commun Math Phys* 299:163–224.
3. Gaiotto D, Moore GW, Neitzke A (2009) Wall-crossing, Hitchin systems, and the WKB approximation.
4. Fock V, Goncharov A (2006) Moduli spaces of local systems and higher Teichmüller theory. *Publ Math Inst Hautes Études Sci* 103:1–211.
5. Cecotti S, Neitzke A, Vafa C (2010) R-Twisting and 4d/2d correspondences. arXiv: 1006.3435.
6. Alim M, et al. (2013) BPS quivers and spectra of complete N=2 quantum field theories. *Commun Math Phys* 323:1185–1227.
7. Alim M, et al. (2013) BPS quivers and spectra of complete $\mathcal{N}=2$ quantum field theories. *Commun Math Phys* 323(3):1185–1227.
8. Cecotti S (2013) Categorical tinkertoys for N=2 gauge theories. *Int J Mod Phys A* 28:1330006.
9. Cecotti S, Del Zotto M (2011) On Arnold's 14 'exceptional' N=2 superconformal gauge theories. *J High Energy Phys* 1110:99.
10. Arkani-Hamed N, et al. (2012) Scattering amplitudes and the positive Grassmannian. arXiv:1212.5605.
11. Golden J, Goncharov AB, Spradlin M, Vergu C, Volovich A (2013) Motivic amplitudes and cluster coordinates. *J High Energy Phys* 1401:91.
12. Neitzke A (2013) Notes on a new construction of hyperkahler metrics. *Proceedings of Mirror Symmetry and Tropical Geometry*, in press.
13. Hitchin N Hyper-Kähler manifolds. *Séminaire Nicolas Bourbaki 1991-1992* (Société Mathématique de France, Paris), Vol 34, No. 748, pp 137–166.
14. Hitchin NJ, Karlhede A, Lindstrom U, Roček M (1987) Hyperkähler metrics and supersymmetry. *Commun Math Phys* 108:535–589.
15. Fomin S, Zelevinsky A (2007) Cluster algebras. IV. Coefficients. *Compos Math* 143(2): 112–164.
16. Kontsevich M, Soibelman Y (2008) Stability structures, motivic Donaldson-Thomas invariants and cluster transformations. arXiv:0811.2435.
17. Joyce D, Song Y (2012) A theory of generalized Donaldson-Thomas invariants. *Mem Amer Math Soc* 217(1020).
18. Bridgeland T, Smith I (2013) Quadratic differentials as stability conditions. arXiv: 1302.7030.
19. Cordova C, Neitzke A (2013) Line defects, tropicalization, and multi-centered quiver quantum mechanics. arXiv:1308.6829.
20. Lampe P (2010) A quantum cluster algebra of Kronecker type and the dual canonical basis. *Int Math Res Notices* 2011(13):2970–3005.
21. Witten E (1997) Solutions of four-dimensional field theories via M-theory. *Nucl Phys B* 500:3–42.
22. Gaiotto D (2012) N=2 dualities. *J High Energy Phys* 1208:34.
23. Hitchin NJ (1987) The self-duality equations on a Riemann surface. *Proc London Math Soc* 55(1):59–126.
24. Uhlenbeck KK (1982) Connections with L^p bounds on curvature. *Commun Math Phys* 83(1):31–42.
25. Simpson CT (1990) Harmonic bundles on noncompact curves. *J Am Math Soc* 3(3): 713–770.
26. Biquard O, Boalch P (2004) Wild non-abelian Hodge theory on curves. *Compos Math* 140(1):179–204.
27. Nekrasov N, Pestun V (2012) Seiberg-Witten geometry of four dimensional N=2 quiver gauge theories. arXiv:1211.2240.
28. Gaiotto D, Moore GW, Neitzke A (2013) Spectral networks. *Ann Henri Poincaré* 14(7): 1643–1731.
29. Atiyah MF, Bott R (1983) The Yang-Mills equations over Riemann surfaces. *Philos Trans R Soc Lond Ser A* 308(1505):523–615.
30. Strebel K (1984) *Quadratic Differentials*, Results in Mathematics and Related Areas (Springer, Berlin), Vol 5, pp xii, 184.
31. Gaiotto D, Moore GW, Neitzke A, Spectral network movies. Available at <http://www.ma.utexas.edu/users/neitzke/spectral-network-movies/>. Accessed June 18, 2014.
32. Hollands L, Neitzke A (2013) Spectral networks and Fenchel-Nielsen coordinates. arXiv:1312.2979.
33. Baur K, Dupont G (2013) Compactifying exchange graphs I: Annuli and tubes. arXiv: 1303.3397.
34. Fomin S, Shapiro M, Thurston D (2008) Cluster algebras and triangulated surfaces. Part I: Cluster complexes. *Acta Mathematica* 201:83–146.
35. Gaiotto D, Moore GW, Neitzke A (2014) Spectral networks and snakes. *Ann Henri Poincaré* 15(1):61–141.
36. Galakhov D, Longhi P, Mainiero T, Moore GW, Neitzke A (2013) Wild wall crossing and BPS giants. *J High Energy Phys* 1311:46.
37. Seiberg N, Witten E (1994) Electric-magnetic duality, monopole condensation, and confinement in $\mathcal{N}=2$ supersymmetric Yang-Mills theory. *Nucl Phys B* 426:19–52.
38. Alvarez-Gaume L, Freedman DZ (1981) Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model. *Commun Math Phys* 80:443–451.
39. Seiberg N, Witten E (1996) Gauge dynamics and compactification to three dimensions. *The Mathematical Beauty of Physics: A Memorial Volume for Claude Itzykson Saclay*, Advanced Series in Mathematical Physics, eds Itzykson C, Zuber JB, Drouffe J-M (World Scientific, Singapore), pp 333–366.
40. Gaiotto D, Moore GW, Neitzke A (2010) Framed BPS states. *Advances in Theoretical and Mathematical Physics* 17(2):241–397.