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Practical interior tomography with radial Hilbert filtering and a *priori* knowledge in a small round area

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Abstract

Purposes—Interior tomography problem can be solved using the so-called differentiated backprojection-projection onto convex sets (DBP-POCS) method, which requires *a priori* knowledge within a small area interior to the region of interest (ROI) to be imaged. In theory, the small area wherein the *a priori* knowledge is required can be in any shape, but most of the existing implementations carry out the Hilbert filtering either horizontally or vertically, leading to a vertical or horizontal strip that may be across a large area in the object. In this work, we implement a practical DBP-POCS method with radial Hilbert filtering and thus the small area with the *a priori* knowledge can be roughly round (e.g., a sinus or ventricles among other anatomic cavities in human or animal body). We also conduct an experimental evaluation to verify the performance of this practical implementation.

Methods—We specifically re-derive the reconstruction formula in the DBP-POCS fashion with radial Hilbert filtering to assure that only a small round area with the *a priori* knowledge be needed (namely radial DBP-POCS method henceforth). The performance of the practical DBP-POCS method with radial Hilbert filtering and *a priori* knowledge in a small round area is evaluated with projection data of the standard and modified Shepp-Logan phantoms simulated by computer, followed by a verification using real projection data acquired by a computed tomography (CT) scanner.

Results—The preliminary performance study show that, if *a priori* knowledge in a small round area is available, the radial DBP-POCS method can solve the interior tomography problem in a more practical way at high accuracy.

Conclusions—In comparison to the implementations of DBP-POCS method demanding the *a priori* knowledge in horizontal or vertical strip, the radial DBP-POCS method requires the *a priori* knowledge within a small round area only. Such a relaxed requirement on the availability of *a priori* knowledge can be readily met in practice, because a variety of small round areas (e.g., air-filled sinuses or fluid-filled ventricles among other anatomic cavities) exist in human or animal body. Therefore, the radial DBP-POCS method with *a priori* knowledge in a small round area is more feasible in clinical and preclinical practice.

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Keywords

CT; interior tomography; differentiated backprojection; projection onto convex sets; Hilbert filtering

I INTRODUCTION

With recourse to x-ray penetration, CT generates tomographic images corresponding to the spatial distribution of an object's linear attenuation coefficient (LAC) from a collection of its projection. The ionizing x-ray renders biological risk to patients and thereby has drawn increasing public concern in recent years¹⁻⁶. In clinical practice, the principle of "as low as reasonably achievable" (ALARA) has been advocated as the professional guidance for healthcare providers to reduce unnecessary radiation dose to patients⁷. Various hardware-based (e.g., automatic modulation of x-ray tube current in the manner conformal to a patient's habitus^{8,9}) and software-based (e.g., the diversified image reconstruction and processing algorithms^{10,11}) methods have been proposed to advance CT technology to achieve significant radiation dose reduction while maintaining image performance. Examples of the software-based attempts also include the schemes to reconstruct tomographic images from the projection data acquired with sparse angular positions or limited angular range¹². Another straightforward way to reduce CT's radiation dose is to acquire the projection data that encompasses only an ROI within a human or animal body, while the radiation to the surrounding tissues and organs is spared by collimation. Essentially, this is to solve the so-called interior tomography problem, which has challenged medical imaging scientists and researchers for decades¹³⁻¹⁹.

In classical analysis, it has been known that the interior tomography problem is seriously ill-posed, because the null space of the interior tomography problem is non-trivial, leading to severe difficulties in maintaining the solution's uniqueness and stability. In recent years, several regularization strategies have been proposed to select an optimal solution with improved stability over a set of candidates. Thus far, the total-variation (TV) minimization based method is showing promise, in which the regularization assumes that the image to be reconstructed is piecewise constant^{13,31} or polynomial¹⁴. Another one bearing the hope to solve the interior tomography problem is the DBP-based method, e.g., DBP-POCS^{15,16,29} and DBP-SVD (Singular Value Decomposition)^{18,19}. The stability of DBP-based method is further verified under the framework of complex analysis¹⁷, which has substantially advanced the insightful understanding of interior tomography problem and the way to find its best possible solution^{15-19,29}.

In particular, the DBP-based method demands *a priori* knowledge within a small area interior to the ROI to be reconstructed to improve the solution's uniqueness and stability^{15-19,29}. It should be pointed out that the satisfaction of such a requirement is not impossible in practice, since there exists a variety of air-filled sinus or fluid-filled ventricles among other anatomic cavities in human or animal body. If the implementation of DBP-based method is carried out either horizontally or vertically, i.e., along either the x_1 - or x_2 -axis in the two-dimensional (2D) Cartesian coordinate system, the area wherein the *a priori* knowledge is required has to be either a vertical or horizontal strip that may be across a large

area in human or animal body. To address this issue, the DBP-based method can be implemented with radial Hilbert filtering along the lines passing through the origin of a local polar coordinate system associated with the ROI¹⁶. In such a way, the area wherein the *a priori* knowledge is demanded can be a small round area interior to the ROI that is definitely local in contrast to either a horizontal or vertical strip.

In this work, we specifically re-derive the reconstruction formula in the DBP-POCS fashion with radial Hilbert filtering to assure that only a small round area with the *a priori* knowledge be needed. Using the projection data simulated by computer and acquired by a CT scanner, we evaluate and verify the performance of the radial DBP-POCS method, in which the requirement for the *a priori* knowledge in a small round area can be readily met in practice.

II. MATERIALS AND METHODS

The radial DBP-POCS method can be obtained with adequate variable substitution from the results in references^{15,16}. However, for the sake of completeness, we specifically re-derive it here.

In the 2D real space R^2 with a Cartesian coordinate system, an object function can be denoted as

$$f(\mathbf{x})=f(x_1, x_2), \mathbf{x}=(x_1, x_2) \in R^2, \quad (1)$$

while its projection is defined by²⁰

$$p(\varphi, r)=\int_R f(r\boldsymbol{\varphi}+t\boldsymbol{\varphi}^\perp)dt, r \in R, \varphi \in [0, \pi), \quad (2)$$

where $\boldsymbol{\varphi}=(\cos\varphi, \sin\varphi)$ and $\boldsymbol{\varphi}^\perp=(-\sin\varphi, \cos\varphi)$ are unit vectors perpendicular to each other. A CT algorithm is to reconstruct the object function $f(\mathbf{x})$ from its projection $p(\varphi, r)$.

The Hilbert filtering is a linear operator that, in the 1D real space R , can be defined as²⁰

$$Hg(r)=\int_R \frac{g(r')}{\pi(r-r')}dr'=\int_R -i\text{sgn}(\rho)G(\rho)e^{i2\pi r\rho}d\rho, \quad (3)$$

where $\text{sgn}(\rho)$ is the sign function of variable ρ ; $G(\rho)$ is the Fourier transform of $g(r)$, i.e.,

$$G(\rho)=\int_R g(r)e^{-i2\pi r\rho}dr. \quad (4)$$

In order to utilize the *a priori* knowledge in a small round area for interior tomography, it is desired to derive the Hilbert filtering in a local polar coordinate system. Letting the object function be represented as

$$\tilde{f}_{\mathbf{x}^0}(s, \theta) = f(s\boldsymbol{\theta} + \mathbf{x}^0), \quad s \in R, \quad \theta \in [0, \pi), \quad \mathbf{x}^0 = (x_1^0, x_2^0) \in R^2, \quad (5)$$

where s and θ denote the radial distance from origin $\mathbf{x}^0$ of the local polar coordinate system and the azimuthal angle from x_1 -axis, respectively, we may define the Hilbert filtering of an object function as

$$H_\theta \tilde{f}_{\mathbf{x}^0}(s) = \int_R \frac{\tilde{f}_{\mathbf{x}^0}(s', \theta)}{\pi(s-s')} ds' = \int_R \frac{f(s'\boldsymbol{\theta} + \mathbf{x}^0)}{\pi(s-s')} ds', \quad (6)$$

Using the Fourier series of object function $f(\mathbf{x})$

$$f(\mathbf{x}) = \int_{R^2} F(\mathbf{v}) e^{i2\pi \mathbf{v} \cdot \mathbf{x}} d\mathbf{v}, \quad (7)$$

where

$$F(\mathbf{v}) = F(v_1, v_2) = \int_{R^2} f(\mathbf{x}) e^{-i2\pi \mathbf{v} \cdot \mathbf{x}} d\mathbf{x}, \quad (8)$$

eq. (6) can be converted into

$$H_\theta \tilde{f}_{\mathbf{x}^0}(s) = \int_{R^2} F(\mathbf{v}) e^{i2\pi \mathbf{v} \cdot \mathbf{x}^0} \int_R \frac{1}{\pi(s-s')} e^{i2\pi s' \mathbf{v} \cdot \boldsymbol{\theta}} ds' d\mathbf{v}. \quad (9)$$

Changing the integral variable as

$$\int_R \frac{1}{\pi(s-s')} e^{i2\pi s' \mathbf{v} \cdot \boldsymbol{\theta}} ds' = \int_R \frac{1}{\pi s'} e^{i2\pi(s-s') \mathbf{v} \cdot \boldsymbol{\theta}} ds', \quad (10)$$

we can get

$$\begin{aligned} H_\theta \tilde{f}_{\mathbf{x}^0}(s) &= \int_{R^2} F(\mathbf{v}) e^{i2\pi \mathbf{v} \cdot \mathbf{x}^0} \int_R \frac{1}{\pi s'} e^{i2\pi(s-s') \mathbf{v} \cdot \boldsymbol{\theta}} ds' d\mathbf{v} \\ &= \int_0^\pi \int_R F(\rho \boldsymbol{\varphi}) e^{i2\pi \rho \mathbf{x}^0 \cdot \boldsymbol{\varphi}} \int_R \frac{1}{\pi s'} e^{i2\pi(s-s') \rho \boldsymbol{\theta} \cdot \boldsymbol{\varphi}} ds' |\rho| d\rho d\varphi \\ &= \int_0^\pi \int_R \text{sgn}(\rho) \rho P(\varphi, \rho) e^{i2\pi \rho(\mathbf{x}^0 + s\boldsymbol{\theta}) \cdot \boldsymbol{\varphi}} \int_R \frac{1}{\pi s'} e^{-i2\pi s' \rho \boldsymbol{\theta} \cdot \boldsymbol{\varphi}} ds' d\rho d\varphi \quad (11) \\ &= -i \int_0^\pi \int_R \text{sgn}(\rho) \rho P(\varphi, \rho) e^{i2\pi \rho(\mathbf{x}^0 + s\boldsymbol{\theta}) \cdot \boldsymbol{\varphi}} \text{sgn}(\rho \boldsymbol{\theta} \cdot \boldsymbol{\varphi}) d\rho d\varphi \\ &= - \int_0^\pi \int_R i \rho P(\varphi, \rho) e^{i2\pi \rho(\mathbf{x}^0 + s\boldsymbol{\theta}) \cdot \boldsymbol{\varphi}} d\rho \text{sgn}(\boldsymbol{\theta} \cdot \boldsymbol{\varphi}) d\varphi, \end{aligned}$$

where

$$\int_R i \rho P(\varphi, \rho) e^{i2\pi \rho(\mathbf{x}^0 + s\boldsymbol{\theta}) \cdot \boldsymbol{\varphi}} d\rho = \frac{1}{2\pi} p'(\varphi, (\mathbf{x}^0 + s\boldsymbol{\theta}) \cdot \boldsymbol{\varphi}). \quad (12)$$

Then, the reconstruction formula in the DBP fashion with radial Hilbert filtering in the parallel-beam geometry is attained as

$$H_{\theta} \tilde{f}_{x^0}(s) = -\frac{1}{2\pi} \int_0^{\pi} p'(\varphi, (x^0 + s\theta) \cdot \varphi) \operatorname{sgn}(\theta \cdot \varphi) d\varphi. \quad (13)$$

In practice, the counterpart of eq. (13) in the fan-beam geometry is preferable, and the case corresponding to a full scan (360°) is

$$H_{\theta} \tilde{f}_{x^0}(s) = -\frac{1}{4\pi} \int_0^{2\pi} p'(\varphi, (x^0 + s\theta) \cdot \varphi) \operatorname{sgn}(\theta \cdot \varphi) d\varphi. \quad (14)$$

Then, we get

$$H_{\theta} \tilde{f}_{x^0}(s) = -\frac{1}{4\pi} \int_0^{2\pi} \int_R p(\varphi, r') \delta'((x^0 + s\theta) \cdot \varphi - r') dr' \operatorname{sgn}(\theta \cdot \varphi) d\varphi, \quad (15)$$

according to the following relationships between the parallel-beam and fan-beam geometries,

$$\varphi = \xi + \gamma, \quad r = R_0 \sin \gamma, \quad (16)$$

$$\xi = \varphi - \gamma, \quad \gamma = \arcsin(r/R_0), \quad (17)$$

$$p(\varphi, r) = \widetilde{p}(\xi, \gamma). \quad (18)$$

where R_0 is the distance from the x-ray source to the rotation axis. Similar to the strategy exercised by Noo *et al*²¹, the DBP formula in the equi-angular fan-beam geometry can be finally obtained as

$$H_{\theta} \tilde{f}_{x^0}(s) = -\frac{1}{4\pi} \int_0^{2\pi} \frac{R_0 \cos^2 \gamma^*}{T^2} \operatorname{sgn}(\cos(\xi + \gamma^* - \theta)) \frac{d}{d\gamma} (\cos \gamma \times \widetilde{p}(\xi, \gamma)) \Big|_{\gamma=\gamma^*} d\xi + \frac{1}{2\pi} \sum_{i=1}^2 (-1)^i \frac{\widetilde{p}(\xi_i, \gamma_i^*)}{\|x - v_i\|}, \quad (19)$$

where

$$x = s\theta + x^0, \quad T = R_0 - x \cdot \xi^\perp, \quad \gamma^* = \arctan(x \cdot \xi/T), \quad (20)$$

$$\xi_1 = \theta - \frac{\pi}{2} + \arcsin(x^0 \cdot \theta^\perp / R_0), \quad \xi_2 = 2\theta - \xi_1, \quad (21)$$

$$v_i = R_0 \xi_i^\perp, \quad T_i = R_0 - x \cdot \xi_i^\perp, \quad \gamma_i^* = \arctan(x \cdot \xi_i / T_i), \quad i=1, 2. \quad (22)$$

III. IMPLEMENTATION

The object function $f(x)$ inside an ROI can be inversely solved from eqs (19–22) using the DBP-POCS method^{15,16}. The implementation of DBP-POCS method should be carried out on each Hilbert line associated with the local coordinate system^{15–19,28,29}, and thus the object function $f(x)$ may be viewed as a 1D function $f(s)$ with a definition domain of normalized interval $(-1,1)$. On each Hilbert line, Λ_K , Λ_H , Λ and X represent the intervals with the *a priori* knowledge $f^{(p)}$, the DBP values that can be calculated from eqs (19–22), the compact support of the object function $f(s)$, and the whole interval wherein the DBP-POCS method is implemented^{15,16}. For the situation considered in this paper, there is a relationship among these intervals: $\Lambda_H \subseteq X$, $\Lambda_K \subseteq \Lambda \subseteq X = (-1, 1)$. $L_W^2(X)$ represents a weighted L^2 space with $W(s)=(1-s^2)^{0.5}$, and the associated inner product and norm are defined by

$$(f_1, f_2)_W = \int_{-1}^1 ds f_1(s) f_2(s) W(s), \quad \|f\|_W^2 = (f, f)_W. \quad (23)$$

To implement the DBP-POCS method, five convex sets are defined^{15,16}:

$$\begin{aligned} C_1 &= \{f \in L_W^2(X) | (H_X f)(s) = g(s) \text{ for } s \in \Lambda_H\}, \\ &\quad (H_X f)(s) = \frac{1}{\pi} p.v. \int_{-1}^1 ds \frac{1}{s-s'} f(s'); \\ C_2 &= \{f \in L_W^2(X) | f(s) = 0 \text{ for } s \notin \Lambda\}; \\ C_3 &= \{f \in L_W^2(X) | f(s) = f^{(p)}(s) \text{ for } s \in \Lambda_K\}; \\ C_4 &= \{f \in L_W^2(X) | \int_{\Lambda \setminus \Lambda_K} ds f(s) = C_\Lambda^{(p)}\}, \quad C_\Lambda^{(p)} = \pi C_\Lambda - \int_\Lambda ds f^{(p)}(s); \\ C_5 &= \{f \in L_W^2(X) | f(s) \geq 0 \text{ for } s \in X\}, \end{aligned} \quad (24)$$

where πC_Λ is equal to the ideal projection data acquired along the Hilbert line passing through the origin of local polar coordinate system. In the equi-angular fan-beam geometry, we have

$$C_\Lambda = \frac{1}{\pi} P_{x^0}(\theta) = \frac{1}{\pi} \widetilde{p}(\xi_i, \gamma_i^0), \quad \gamma_i^0 = \arctan\left(\frac{x^0 \cdot \xi_i}{R_0 - x^0 \cdot \xi_i^\perp}\right), \quad i=1 \text{ or } 2. \quad (25)$$

Projection operator P_i corresponding to each convex set C_i is required for numerically implementing the DBP-POCS method^{15,16}. Projectors P_2 , P_3 and P_5 are trivial, and P_1 and P_4 are specified as^{15,16},

$$\begin{aligned} (P_1 f)(s) &= \left(\frac{1}{\pi} \frac{1}{W(s)} \int_{-1}^1 ds' f(s')\right) + (H_X^+ M H_X f)(s), \\ (H_X^+ h)(s) &= \frac{1}{\pi} \frac{1}{W(s)} p.v. \int_{-1}^1 ds' \frac{1}{s-s'} h(s') W(s'), \\ (Mh)(s) &= \begin{cases} g(s) & (s \in \Lambda_H), \\ h(s) & (s \notin \Lambda_H); \end{cases} \\ (P_4 f)(s) &= \begin{cases} f(s) + \frac{\left(C_\Lambda^{(p)} - \int_{\Lambda \setminus \Lambda_K} ds' f(s')\right)/W(s)}{\int_{\Lambda \setminus \Lambda_K} ds'/W(s')} & (s \in \Lambda \setminus \Lambda_K), \\ f(s) & (s \notin \Lambda \setminus \Lambda_K). \end{cases} \end{aligned} \quad (26)$$

To determine the object function $f(s)$ belonging to the intersection of five convex sets defined in eq. (24), the following iterative formula is carried out,

$$f^{(k+1)} = P_5 P_4 P_3 P_2 P_1 f^{(k)}, \quad k=0, 1, \dots, K-1, K, \quad (27)$$

where $f^{(0)}$ is an initial image with each pixel set to zero. In this work, the maximal iteration number, K , is empirically set as 100, since no improvement is noticeable after 100 iterations. To be distinguishable from the DBP-POCS method derived and implemented in references^{15,16}, we call the DBP-POCS method with *a priori* knowledge in a small round area the radial DBP-POCS method, even though their iterative formula is the same as that in eq. (27).

IV. PERFORMANCE EVALUATION

The performance of the radial DBP-POCS method with *a priori* knowledge in a small round area is evaluated using projection data simulated by computer and acquired by a CT scanner.

3.1 Evaluation with the projection data simulated by computer

The standard²⁶ and modified²⁷ Shepp-Logan phantoms are used to simulate the projection data in the equi-angular fan-beam geometry. The contrast in internal structure of the modified Shepp-logan is deliberately increased in comparison to the standard one. As such, the performance of the proposed DBP-POCS method can be thoroughly evaluated and verified. In the simulation of projection data, the distance from x-ray source to the rotation axis is 541 mm. The projection data are uniformly acquired in a full scan (360°) at 1160 angular positions, with a detector array consisting of 888 cells spanning a 52.1° full fan-angle. The interior tomography is simulated by setting the radius of the ROI to be scanned as 40.5 mm (small size) and 59.25 mm (moderate size), while the projection data are truncated appropriately with lateral collimation. According to the interior structure of these two phantoms, the origin x^0 of the local polar coordinate system is deliberately moved from (0, 0) to (-20, 0) mm. In such a way, the *a priori* knowledge can be assumed as the LAC of the left ventricle. Transaxial tomographic images are reconstructed in a 512×512 matrix with pixel size $0.5 \times 0.5 \text{ mm}^2$. The dimension of the sampling lattice in the local polar coordinate system is 2048×2048 , i.e., 2048 sampling points along the azimuthal direction (360°) and radial direction (a distance of 128 mm from the origin x^0), respectively. With these sampling parameters, a bilinear interpolation can convert the reconstructed image in the local polar coordinate system to the 2D Cartesian coordinate system without significant degradation of spatial resolution. A region with a constant value (centered at the origin x^0 , and with a radius of 10^2 sampling points along radial direction) is chosen as the small round area wherein the *a priori* knowledge is demanded.

Initially, the evaluation is carried out using the simulated projection data without noise to verify the algorithm's accuracy, followed by the cases with noise at typical levels to evaluate the algorithm's robustness. A Gaussian random number generator with the variance dependent on the attenuated transmission is employed to simulate the noise²². It should be

pointed out that the transmitted x-ray photons observe the Poisson distribution in reality. However, we can use Gaussian noise to approximate the Poisson noise in a CT scanner²³.

3.2 Evaluation with the projection data acquired by a CT scanner

The performance of the radial DBP-POCS method with *a priori* knowledge in a small round area is further evaluated and verified using the projection data of a large animal (sheep) acquired by a CT scanner. In data acquisition, the distance from the x-ray source to the CT's rotation axis is 570 mm. The projection data are uniformly acquired in a full scan (360°) at 1160 angular positions. The detector array consists of 672 cells with a 52.1° full fan-angle. Truncated projection data are acquired by setting the radius of the ROI to be scanned as 50 mm (small size), 75 mm (moderate size) and 225 mm (full size). Tomographic images are reconstructed at a 512×512 matrix in the Cartesian coordinate system with pixel size 0.5664×0.5664 mm². In the local polar coordinate system, the sampling matrix at dimension 2048×2048 is also employed, wherein 2048 points are sampled along the azimuthal direction (360°) and radial direction (a distance of 225 mm from origin x^0), respectively. An air-filled cavity (the bronchus with its LAC approximately equal to zero, centered at origin x^0 , and with a radius of 10 sampling points along the radial direction) within the sheep's body is chosen as the small round area wherein the *a priori* knowledge is demanded.

V. RESULTS

The images reconstructed from the projection data described above are used to evaluate the performance of the radial DBP-POCS method. The images reconstructed by the filtered back-projection (FBP) algorithm are employed as a reference to illustrate the merit of the radial DBP-POCS method to solve the interior tomography problem. It should be indicated that, in the reconstruction with the FBP algorithm, a 0th-order extrapolation – repeating the outmost data of the truncated projection to the dimension corresponding to the object's support – is employed, though other more sophisticated extrapolation approaches have been reported in the literature³⁰.

4.1 Performance evaluation – phantom study

Figs. 1 – 5 show the images of the standard Shepp-Logan phantom reconstructed at different noise levels (noise-free, moderate and severe noise). Presented in Fig. 1 are the transaxial images reconstructed by the FBP algorithm with the origin x^0 at (0, 0) (left column), and the radial DBP-POCS method with the origin x^0 at (0, 0) (mid-column) and (−20, 0) mm (right column), respectively, from projection data without lateral truncation. It is clearly observed that, in the case with complete projection data, all reconstructed images are in good quality without any artifact. Presented in Figs. 2 – 5 are the transaxial images reconstructed by the FBP algorithm and the radial DBP-POCS method with lateral data truncation in the projection, i.e., the results of interior tomography. Specifically, Figs. 2 and 3 display the images reconstructed with origin x^0 at (0, 0) and ROIs of small (radius 40.5 mm) and moderate (radius 59.25 mm) size, respectively, while Figs. 4 and 5 correspond to the origin x^0 at (−20, 0) mm and ROIs of small (radius 40.5 mm) and (radius 59.25 mm) size. It is observed that severe direct current (DC) shift exists in the images reconstructed by the FBP algorithm, showing that interior tomography is a real challenge to CT imaging with the

conventional image reconstruction algorithm, even though the strategy of data extrapolation is exercised. However, it is observed that there is no DC shift in the images reconstructed by the radial DBP-POCS method. Meanwhile, it is also observed that the performance of DBP-POCS method is influenced by noise. For example, minor artifact exists in the images in the right column of Figs. 2 – 5, especially near boundaries of the ROIs.

The transaxial images of the modified Shepp-logan phantom are in turn displayed in Figs. 6 – 10. Presented in Fig. 6 are the transaxial images reconstructed by the FBP algorithm with the origin $\mathbf{x}^0$ at (0, 0) (left column), and the radial DBP-POCS method with the origin $\mathbf{x}^0$ at (0, 0) (mid-column) and (–20, 0) mm (right column), respectively, from projection data without lateral truncation. Again, with complete projection data, all reconstructed images are in good quality. Presented in Figs. 7 – 10 are the transaxial images reconstructed by the FBP algorithm and the radial DBP-POCS method with lateral data truncation in the projection. Specifically, Figs. 7 and 8 display the images reconstructed with origin $\mathbf{x}^0$ at (0, 0) and ROIs of small (radius 40.5 mm) and moderate (radius 59.25 mm) size, respectively, while Figs. 9 and 10 correspond to the origin $\mathbf{x}^0$ at (–20, 0) mm and ROIs of small (radius 40.5 mm) and (radius 59.25 mm) size. Similar to what we have observed in Figs. 2 – 5, there is no DC shift in the images reconstructed by the radial DBP-POCS method, while severe DC shift occurs in the images reconstructed by the FBP algorithm. Moreover, the performance of DBP-POCS method varies over noise, as evidenced by the artifacts with increasing noise levels.

In addition to visual evaluation, we compare the performance of FBP and DBP-POCS methods by quantitatively gauging their spatial resolution (full width at half maximum – FWHM) and noise (standard deviation – SD). The location and orientation for FWHM measurement and the ROI for SD measurement are illustrated in Fig. 1(a) and Fig. 6(a). Itemized in Table I are the results corresponding to the FBP and proposed radial DBP-POCS methods measured in Figs. 8 and 10, which shows that the DBP-POCS method is of better noise property than the FBP method, while their spatial resolution is comparable, regardless of where the origin $\mathbf{x}^0$ of the local coordinate system is located.

4.2 Performance evaluation – animal study

According to the anatomic structure of the sheep, the origin $\mathbf{x}^0$ of the local polar coordinate system is deliberately moved to (–13, 9) mm. Consequently, the *a priori* knowledge within a small round area, i.e., the air-filled lumen of primary and secondary bronchus, can be readily available. Presented in Fig 11(a), (b) and (c) are the images reconstructed by the conventional FBP algorithm from the truncated and complete projection data, respectively, while the corresponding images reconstructed by the radial DBP-POCS method are in Fig 11(a'), (b') and (c'). To reinforce our visual observation, the intensity profiles along the central horizontal and vertical lines of the images are plotted in Fig. 12. Again, it is observed that the radial DBP-POCS method outperforms the conventional FBP algorithm, especially with respect to the DC shifts. Meanwhile, we notice that there are minor artifacts near boundaries of the ROIs.

VI. DISCUSSIONS AND CONCLUSIONS

In this work, we propose and implement the radial DBP-POCS method for practical interior tomography, in which the required *a priori* knowledge is confined within a small round area. We conducted an experimental evaluation and verification of its performance with the data simulated by computer and acquired by a CT scanner. Based on the preliminary results obtained, the following observations and discussions are in order.

The required *a priori* knowledge in a small round area is readily available in reality. For example, there exists a variety of *in vivo* air-filled sinus or fluid-filled ventricles among other cavities in human or animal body and their corresponding LACs are almost exactly known. This means that the requirement of the *a priori* knowledge does not bring about a challenge that cannot be met in practice. Thus, in comparison to the implementation of the DBP-POCS method demanding the *a priori* knowledge within a horizontal or vertical strip, which may be across a large area in human or animal's body, the radial DBP-POCS method is substantially more feasible in terms of algorithmic and system implementation.

In the practical DBP-POCS method to solve the interior tomography problem, the location of the local polar coordinate (green in Fig. 13) system's origin $\mathbf{x}^0$ needs to coincide with the centroid of the small round area (blue in Fig. 13) wherein the *a priori* knowledge is required. On the other hand, the coincided local polar coordinate system's origin $\mathbf{x}^0$ and centroid of the small round area do not have to coincide with the center of the ROI (red in Fig. 13) to be imaged. In fact, the origin $\mathbf{x}^0$ can be anywhere within the ROI, as long as it coincides with the centroid of the small round area, e.g., sinuses or ventricles among other cavities in human or animal body. Such a decoupling between the ROI's center and the polar coordinate system's origin $\mathbf{x}^0$ may lead to convenience and flexibility in practice for the selection of the small round area with the demanded *a priori* knowledge. As such, an optimal zoning of the organs or tissues to be imaged can be warranted in the light of radiation dose reduction or making use of a CT detector with limited dimension as efficiently as possible.

It should be pointed out that, an estimation approach similar to that in reference¹⁶ has been exercised to enforce the object support constraint in the implementation of DBP-POCS method (see constraint C_2 in reference¹⁶). This approach needs certain *a priori* knowledge about the dimension or size of the object to be imaged, e.g., the habitus of a patient or animal. However, as indicated in the literature and confirmed in this work, the DBP-POCS method to solve the interior tomography problem is quite tolerable to inaccuracy in the estimation of object's dimension or size to enforce the object support constraint. Therefore, despite the possible inaccurate knowledge of object's dimension or size estimation, the implemented radial DBP-POCS method with *a priori* knowledge in a small round area should still be robust for clinical and preclinical applications.

It is worth noting that, in the DBP-POCS method presented in reference¹⁶, examples with the implementation along the so-called polar Hilbert lines were given as special cases. The radial DBP-POCS method is systematically re-derived, implemented and evaluated in this paper. Hence, the work reported in this paper may be viewed as an extension of the work in

reference¹⁶. In reference²⁴, the image reconstructed by an FBP algorithm is used as the starting point for the TV minimization based reconstruction, followed by a DBP-POCS reconstruction carried out in the Cartesian coordinate system. In fact, the DBP-POCS reconstruction is carried out twice along x_1 -, x_2 - and then x_2 -, x_1 -directions, respectively, followed by summing the results with adequate weighting schemes. In reference²⁵, the DBP-POCS method is used to generate the image as the initial point for a statistical reconstruction using the TV minimization criterion to reduce noise and accordingly radiation dose. Clearly, all these recent progresses^{15–19, 24, 25, 28–31} are advancing the research in interior tomography substantially.

As demonstrated in the evaluation section, the conventional FBP algorithm cannot solve the interior tomography problem with data extrapolation, as evidenced by the severe DC shift artifacts shown in the images (a), (b) and (c) in Figs. 2 – 5. This phenomenon is basically owing to the fact that the conventional FBP algorithm with data extrapolation is in principle a heuristic approach to cope with the interior tomography problem. On the other hand, as demonstrated by the images (a'), (b') and (c') in Figs. 2 – 5, there is almost no DC shift artifacts in the ROIs reconstructed by the radial DBP-POCS method with *a priori* knowledge in a small round area, because it is in principle an exact approach to solve the interior tomography problem. A close inspection of the results shows that there exist minor artifacts in the images reconstructed by the proposed radial DBP-POCS method. Based on the facts (i) there is almost no artifacts in the images without noise and (ii) the artifacts worsen with increasing noise level, we believe that these artifacts are more likely due to noise's influence on the algorithm's performance, although it has been pointed out in the literature¹⁷ that instability of the DBP-based reconstruction algorithm may also be a reason. Further effort will be invested by us in the future to improve the robustness of the proposed radial DBP-POCS method for interior tomography applications.

We would like to conclude this paper by indicating that, in the radial DBP-POCS method for interior tomography, the small area wherein the *a priori* knowledge is demanded can be round, rather than a horizontal or vertical strip, which is readily available in reality. In addition, the proposed radial DBP-POCS method in principle can be readily extended for interior tomography under cone beam geometry. Thus, the radial DBP-POCS method is of practical significance and may find its extensive application in clinical human imaging and preclinical animal imaging, although several related issues in its implementation are still waiting for their final resolutions.

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Figure 1.

The images of the standard Shepp-Logan phantom reconstructed from complete projection data by the FBP algorithm with $\mathbf{x}^0 = (0, 0)$ mm (**left**), the radial DBP-POCS method with $\mathbf{x}^0 = (0, 0)$ mm (**middle**) and with $\mathbf{x}^0 = (-20, 0)$ mm (**right**); (**top**) without noise, (**middle**) with moderate noise, and (**bottom**) with severe noise. (Display window: [0.9 1.1]; Black lines in (a) show the location and orientation for FWHM measurement, while the white circle defines the ROI for noise measurement.).

Figure 2.

The images of the standard Shepp-Logan phantom reconstructed from truncated projection data by the FBP algorithm (**left**) (Display window: [0.75 0.95]) and the radial DBP-POCS method (**right**) (Display window: [0.9 1.1]); (**top**) without noise, (**middle**) with moderate noise, and (**bottom**) with severe noise. ($\mathbf{x}^0 = (0, 0)$ mm; the white dash circle represents the ROI at radius 40.5 mm).

Figure 3.

The images of the standard Shepp-Logan phantom reconstructed from truncated projection data by the FBP algorithm (**left**) (Display window: [0.8 1.0]) and the radial DBP-POCS method (**right**) (Display window: [0.9 1.1]); (**top**) without noise, (**middle**) with moderate noise, and (**bottom**) with severe noise. ($\mathbf{x}^0 = (0, 0)$ mm; the white dash circle represents the ROI at radius 59.25 mm).

Figure 4.

The images of the standard Shepp-Logan phantom reconstructed from truncated projection data by the FBP algorithm (**left**) (Display window: [0.75 0.95]) and the radial DBP-POCS method (**right**) (Display window: [0.9 1.1]); (**top**) without noise, (**middle**) with moderate noise, and (**bottom**) with severe noise. ($\mathbf{x}^0 = (-20, 0)$ mm; the white dash circle represents the ROI at radius 40.5 mm).

Figure 5.

The images of the standard Shepp-Logan phantom reconstructed from truncated projection data by the FBP algorithm (**left**) (Display window: [0.8 1.0]) and the radial DBP-POCS method (**right**) (Display window: [0.9 1.1]); (**top**) without noise, (**middle**) with moderate noise, and (**bottom**) with severe noise. ($\mathbf{x}^0 = (-20, 0)$ mm; the white dash circle represents the ROI at radius 59.25 mm).

Figure 6.

The images of the modified Shepp-Logan phantom reconstructed from complete projection data by the FBP algorithm with $\mathbf{x}^0 = (0, 0)$ mm (**left**), the radial DBP-POCS method with $\mathbf{x}^0 = (0, 0)$ mm (**middle**) and with $\mathbf{x}^0 = (-20, 0)$ mm (**right**); (**top**) without noise, (**middle**) with moderate noise, and (**bottom**) with severe noise. (Display window: [0 0.5]; Black lines in (a) show the location and orientation for FWHM measurement, while the white circle defines the ROI for noise measurement.).

Figure 7.

The images of the modified Shepp-Logan phantom reconstructed from truncated projection data by the FBP algorithm (**left**) and the radial DBP-POCS method (**right**); (**top**) without noise, (**middle**) with moderate noise, and (**bottom**) with severe noise. (Display window: [0 0.5], $x^0 = (0, 0)$ mm; the white dash circle represents the ROI at radius 40.5 mm).

Figure 8.

The images of the modified Shepp-Logan phantom reconstructed from truncated projection data by the FBP algorithm (**left**) and the radial DBP-POCS method (**right**); (**top**) without noise, (**middle**) with moderate noise, and (**bottom**) with severe noise. (Display window: [0 0.5], $x^0 = (0, 0)$ mm; the white dash circle represents the ROI at radius 59.25 mm).

Figure 9.

The images of the modified Shepp-Logan phantom reconstructed from truncated projection data by the FBP algorithm (**left**) and the radial DBP-POCS method (**right**); (**top**) without noise, (**middle**) with moderate noise, and (**bottom**) with severe noise. (Display window: [0 0.5], $x^0 = (-20, 0)$ mm; the white dash circle represents the ROI at radius 40.5 mm).

Figure 10.

The images of the modified Shepp-Logan phantom reconstructed from truncated projection data by the FBP algorithm (**left**) and the radial DBP-POCS method (**right**); (**top**) without noise, (**middle**) with moderate noise, and (**bottom**) with severe noise. (Display window: [0 0.5], $x^0 = (-20, 0)$ mm; the white dash circle represents the ROI at radius 59.25 mm).

Figure 11.

The images of a sheep reconstructed by the conventional FBP algorithm (**left**) and the radial DBP-POCS method (**right**); the radius of ROI is 50 mm (**top**), 75 mm (**middle**), and 225 mm (**bottom**). (Display window: [0 0.03]/mm; $\mathbf{x}^0 = (-13, 9)$ mm; the white dash circle represents the ROI).

Figure 12.

The intensity profiles along the central horizontal (**left**) and vertical (**right**) lines of the images shown in Fig. 11. Fig. 12(a) and (a') correspond to Fig. 11(a) and (a'), while Fig. 12(b) and (b') correspond to Fig. 11(b) and (b'). Note that only the intensity profiles interior to the respective ROIs are plotted.

Figure 13.

The schematic diagram showing the relationship among the local polar coordinate (green) system's origin x^0 , the centroid of the small round area (blue) wherein the *a priori* knowledge are demanded and the center of ROI (red) to be imaged: **(a)** All three centers coincide; **(b)** The local polar coordinate system's origin x^0 coincides the small round area's centroid but away from the ROI's center.

Table I

The measurement of spatial resolution (FWHM) and noise (SD) in the images reconstructed by the FBP and proposed DBP-POCS methods

Shepp-Logan phantom	Reconstruction algorithm	x_0 (mm)	FWHM (mm)		SD	
			Horizontal	Vertical	@ Moderate noise	@ Severe noise
Modified	FBP	(0, 0)	2.4386	2.4768	0.0277	0.0553
	DBP-POCS	(0, 0)	2.3531	2.5396	0.0162	0.0322
	FBP	(-20, 0)	2.4453	2.4770	0.0276	0.0790
	DBP-POCS	(-20, 0)	2.3973	2.5183	0.0159	0.0318