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Parameter estimation approach to banding artifact reduction in balanced steady-state free precession

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Abstract

Purpose—The balanced steady-state free precession (bSSFP) pulse sequence has shown to be of great interest due to its high signal-to-noise ratio efficiency. However, bSSFP images often suffer from banding artifacts due to off-resonance effects, which we aim to minimize in this paper.

Methods—We present a general and fast two-step algorithm for 1) estimating the unknowns in the bSSFP signal model from multiple phase-cycled acquisitions, and 2) reconstructing band-free images. The first step, Linearization for Off-resonance Estimation (LORE), solves the nonlinear problem approximately by a robust linear approach. The second step applies a Gauss-Newton algorithm, initialized by LORE, to minimize the nonlinear least squares criterion. We name the full algorithm LORE-GN.

Results—We derive the Cramér-Rao bound (CRB), a theoretical lower bound of the variance for any unbiased estimator, and show that LORE-GN is statistically efficient. Furthermore, we show that simultaneous estimation of T_1 and T_2 from phase-cycled bSSFP is difficult, since the CRB is high at common SNR. Using simulated, phantom, and in vivo data, we illustrate the band-reduction capabilities of LORE-GN compared to other techniques, such as sum-of-squares.

Conclusion—Using LORE-GN we can successfully minimize banding artifacts in bSSFP.

Keywords

Parameter estimation; bSSFP; off-resonance; banding artifacts; LORE

Introduction

One of the challenges of magnetic resonance imaging (MRI) is to acquire an image with high signal-to-noise ratio (SNR) in a short scan time, and for this the balanced steady-state free precession (bSSFP) sequence has proven to be of great interest. The main drawback of

bSSFP is due to off-resonance effects, typically manifesting as banding artifacts (1, 2). These artifacts are of major concern, especially at high field strengths. Off-resonance effects can lead to signal losses in parts of the image, and techniques for improving image quality are necessary.

When several acquisitions are made with different phase increments of the radio frequency (RF) excitation, the resulting images can be combined to minimize these off-resonance artifacts. Commonly, two or four phase-cycled acquisitions are used as a compromise between performance and scan time. Several image-based techniques have been previously proposed, such as *sum-of-squares*, where the square root of the sum of the squared magnitude of the images is used; or *maximum-intensity*, where the maximum magnitude over all images is combined into one image (1). These methods can in some cases give insufficient banding suppression. For example, when using small flip angles the passband in the bSSFP signal profile is not flat, and the resulting image will not have a uniform intensity (1). Additionally, these techniques do not provide estimates of the model parameters, which can be of interest in quantitative MRI.

Recent works have applied parameter estimation techniques to reduce banding artifacts in bSSFP. The principle is to use a signal model and estimate a parameter that is independent of the off-resonance. This estimate is then used as the band-free image. In (3), the authors treat the special case occurring when setting the echo time, TE , to zero and acquiring data with a specific choice of phase increments. Then, the off-resonance effects can be removed using an analytical solution named the *cross-solution*. The resulting image will, however, have a different contrast compared to the original images. This is due to the fact that the parameter estimated relates to the original images through a function depending on both T_1 and T_2 . The approach is also sub-optimal in the least squares sense, since it is derived with the assumption of no noise. This causes problems when the SNR is low, leading to poor estimates. Furthermore, the method does not provide estimates of all unknown parameters in the model equation and cannot be directly generalized to more than four phase-cycled images.

The approach suggested in (4) is to identify some of the unknown model parameters while assuming the others to be constant. Keeping the relaxation parameters constant makes the estimates less reliable, since in practice, the true values can vary significantly over an image. The approach is based on a manually initialized Levenberg-Marquardt (LM) nonlinear minimization algorithm applied to magnitude data. The use of magnitude data makes the estimation less tractable from a mathematical viewpoint due to the non-differentiability of the absolute value. Furthermore, it changes the noise properties from a Gaussian distribution to a Rician distribution, making the nonlinear least squares (NLS) criterion sub-optimal (biased).

Another characteristic of bSSFP is the T_2/T_1 -weighted image contrast and the subsequent difficulty to sensitize the signal to T_1 or T_2 alone. Quantification of T_1 and T_2 gives valuable tissue information used in a wide range of applications (5). There are various techniques to estimate T_2 from bSSFP data (6, 7). One popular technique is the DESPOT2, which was introduced in (6), and later improved to account for off-resonance effects (8). The

DESPOT2 method has the drawback that it needs a T_1 estimate obtained prior to the T_2 estimation, which requires the acquisition of an additional dataset. In (9) an outline of a method for simultaneous estimation of T_1 and T_2 from bSSFP data was proposed. The method is evaluated using 12 images, which is more than what is needed for DESPOT2. To the best of our knowledge, no accuracy analysis has so far been presented for this technique. Both aforementioned methods (8, 9) utilize a variable flip angle in combination with phase cycling. There are also methods for T_1 estimation as well as simultaneous estimation of T_1 and T_2 , using inversion recovery bSSFP (10, 11). However, neither of these methods take off-resonance effects into account. Furthermore, it is shown that in the presence of off-resonances, the method in (11) can suffer from significant bias.

In this paper, we first describe a parameter estimation algorithm for the phase-cycled bSSFP signal model with the aim of reducing banding artifacts. A scenario with a different model has been investigated in (12). We use complex-valued data to estimate all unknown parameters in a model derived from (13). From the parameter estimates we can reconstruct band-free images with bSSFP-like contrast. As a first step, we derive a fast and robust linear method based on least squares to approximately solve the estimation problem. This eliminates the need for user-defined parameters, such as manual initialization. In the second step, we fine tune the estimates using a nonlinear iterative minimization algorithm. The obtained estimates can then be used to reconstruct band-free images. The proposed algorithm can be applied to datasets regardless of the used echo time (TE) and repetition time (TR); does not rely on any prior assumption on the flip angle (α); and can be used with any number of phase-cycled images larger than or equal to three. Here, we will focus on four-image datasets, since they enable the parameter estimation approach, and generally provide better banding suppression compared to using two images. We then proceed to generalize the algorithm to simultaneously estimate T_1 and T_2 and the equilibrium magnetization including coil sensitivity (KM_0) from phase-cycled data. We derive the Cramér-Rao bound (CRB) for the bSSFP model, a theoretical lower bound on the estimated parameter variances independent of the estimation algorithm. Using the CRB we can determine the statistical efficiency of the proposed algorithm, as well as the maximum theoretical accuracy we can expect when estimating T_1 and T_2 simultaneously, using phase-cycled bSSFP.

Theory

Signal model

In bSSFP imaging, the complex signal, S , at an arbitrary pixel of the n^{th} phase-cycled image can be modeled as (13, 14)

$$S_n = KM e^{-\frac{TE}{T_2}} e^{i\Omega TE} \frac{1 - ae^{-i(\Omega + \Delta\Omega_n)TR}}{1 - b\cos[(\Omega + \Delta\Omega_n)TR]} + v_n, \quad [1]$$

where we have the following definitions:

$$M = iM_0 \frac{(1 - E_1)\sin\alpha}{1 - E_1\cos\alpha - (E_1 - \cos\alpha)E_2^2}, \quad a = E_2, \quad b = E_2 \frac{1 - E_1 - E_1\cos\alpha + \cos\alpha}{1 - E_1\cos\alpha - (E_1 - \cos\alpha)E_2^2}, \quad E_1 = e^{-TR/T_1}, \quad E_2 = e^{-TR/T_2}. \quad [2]$$

The joint variable KM_0 is defined as the complex-valued coil sensitivity K multiplied by the equilibrium magnetization M_0 ; T_1 and T_2 are the longitudinal and transverse relaxation times, respectively; α is the flip angle, $\Omega_n TR$ the user-controlled phase increment, TE the echo time, and TR the repetition time. The off resonance corresponds to $\Omega = 2\pi\gamma B_0 = 2\pi f_{OR}$, where γ is the gyromagnetic ratio, B_0 is the effective deviation from the ideal static magnetic field strength, including both tissue susceptibility and inhomogeneities, and f_{OR} is the corresponding off-resonance frequency. Finally, v_n denotes the noise, which is assumed to be independent and complex Gaussian distributed. For a derivation of Eqs. 1 and 2 see the Appendix.

There are five real-valued unknown parameters in Eqs. 1 and 2 that can be estimated: Ω $\text{Re}(KM_0)$, $\text{Im}(KM_0)$, T_1 , and T_2 , where $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ denote the real and imaginary part, respectively. The parameters here assumed known are: Ω_n , α , TE , and TR , however, as will be shown next, α does not have to be known for reconstructing band-free images. An estimate of α is only needed to get explicit estimates of KM_0 and T_1 .

We introduce the following variables:

$$S_0 = KM_0 e^{-TE/T_2}, \quad \theta = \Omega TR, \quad \Delta\theta_n = \Delta\Omega_n TR, \quad \theta_n = \theta + \Delta\theta_n. \quad [3]$$

This enables us to rewrite the pixel-wise signal model in Eq. 1 as

$$S_n = S_0 e^{i\theta TE/TR} \frac{1 - ae^{-i\theta_n}}{1 - b\cos\theta_n} + v_n = g_n(\mathbf{u}) + v_n, \quad [4]$$

where $g_n(\mathbf{u})$ is the noise-free data model of the n^{th} phase-cycled image, and the vector of new unknown model parameters is denoted by $\mathbf{u} = [\theta, \text{Re}(S_0), \text{Im}(S_0), a, b]^T$, where $\mathbf{w}^T$ is the transpose of the vector $\mathbf{w}$. Acquiring images with different phase increments θ_n allows us to estimate the unknown model parameters of Eq. 4. Using these parameters, we can reconstruct a band-free image from the model. Here, α will be estimated implicitly, since it does not explicitly occur in Eq. 4, and hence, no prior information regarding the flip angle is needed when using Eq. 4 for band reduction.

Even though the phase increments can be arbitrary, using four images ($N = 4$) with phase increments $\mathbf{\theta} = [0, \pi/2, \pi, 3\pi/2]^T$ is common practice and will therefore be considered here as well. It is possible to optimize the phase increments in some sense, especially if some prior information is available. However, a preliminary study (15) showed that the gain from doing so is small, assuming no prior knowledge, and this possibility is therefore not pursued further here.

The Cramér-Rao bound

The Cramér-Rao bound (CRB) gives a lower bound on the variance of the parameter estimates for any unbiased estimator (16). The performance of the estimation algorithm, in terms of parameter variance, can be compared to the optimal performance as stated by the CRB. If an estimation algorithm achieves the CRB, it is statistically efficient.

Under the assumption of circularly Gaussian-distributed, zero-mean, white complex noise of variance σ^2 , the Fisher Information Matrix, $\mathbf{I}(\mathbf{u})$, is given by the Slepian-Bangs formula (16):

$$\mathbf{I}(\mathbf{u}) = \frac{2}{\sigma^2} \sum_{n=1}^N \text{Re} \left\{ \left(\frac{\partial g_n(\mathbf{u})}{\partial \mathbf{u}} \right)^* \left(\frac{\partial g_n(\mathbf{u})}{\partial \mathbf{u}} \right) \right\}, \quad [5]$$

where $*$ denotes the conjugate transpose and $g_n(\mathbf{u})$ is the gradient of the scalar function $g_n(\mathbf{u})$, which is by convention defined to be a row vector. For the model in Eq. 4 the gradient in Eq. 5 is given by:

$$\frac{\partial g_n(\mathbf{u})}{\partial \mathbf{u}} = e^{i\theta \frac{TE}{TR}} \frac{1 - ae^{-i\theta_n}}{1 - b\cos(\theta_n)} \begin{bmatrix} S_0 \left(i \frac{ae^{-i\theta_n}}{1 - ae^{-i\theta_n}} - \frac{b\sin(\theta_n)}{1 - b\cos(\theta_n)} + i \frac{TE}{TR} \right) \\ 1 \\ i \\ S_0 \frac{e^{-i\theta_n}}{1 - ae^{-i\theta_n}} \\ S_0 \frac{\cos(\theta_n)}{(1 - b\cos(\theta_n))} \end{bmatrix}^T. \quad [6]$$

The CRB matrix is computed by taking the inverse of Eq. 5:

$$\mathbf{P}_{\text{CRB}} = \mathbf{I}^{-1}(\mathbf{u}). \quad [7]$$

Finally, the diagonal elements of Eq. 7 give lower bounds on the variance of the corresponding parameter estimates, that is $\sigma_{u_j}^2 \geq [\mathbf{P}_{\text{CRB}}]_{jj}$.

Additionally, we derived the CRB with respect to $\mathbf{u}_0 = [\theta \text{Re}(KM_0), \text{Im}(KM_0), T_1, T_2]^T$, using MATLAB's Symbolic Math Toolbox. This was done to draw conclusions about the initial estimation problem based on the model in Eq. 1.

The LORE-GN algorithm

To estimate the unknown parameters and remove the off-resonance artifacts, we propose a two-step algorithm. The first step is named *Linearization for Off-resonance Estimation* (LORE). The second step is a Gauss-Newton nonlinear search (GN), hence we name the full algorithm LORE-GN. In the first step we rewrite the model in Eq. 4 so that it becomes linear in the unknown parameters, by making use of an over-parameterization. This enables the application of ordinary linear least squares (LS), which is both fast and robust. However, the resulting estimates will be biased in general. In the following step, the final estimates are obtained using GN, initialized with the LORE estimates. This removes any bias and makes

the estimates NLS optimal, which under the assumption of identically distributed Gaussian noise is the maximum likelihood (ML) estimate.

From the estimates of $S_0 \in \mathbb{C}$ and $a, b, \theta \in \mathbb{R}$ it is possible to recover the original parameters $\mathbf{u}_0$ by successively inverting the equations for a and b in Eq. 2, and substituting the results. We have:

$$\hat{E}_1 = \frac{\hat{a}(1 + \cos\alpha - \hat{a}\hat{b}\cos\alpha) - \hat{b}}{\hat{a}(1 + \cos\alpha - \hat{a}\hat{b}) - \hat{b}\cos\alpha}, \quad \hat{E}_2 = \hat{a}, \quad [8]$$

which in turn can be used to compute the estimates:

$$\hat{T}_1 = -TR/\log(\hat{E}_1), \quad \hat{T}_2 = -TR/\log(\hat{E}_2), \quad K\hat{M}_0 = \hat{S}_0 \frac{1 - \hat{E}_1 \cos\alpha - (\hat{E}_1 - \cos\alpha)\hat{E}_2^2}{ie^{-TE/\hat{T}_2}(1 - \hat{E}_1)\sin\alpha} \quad [9]$$

Step 1: Parameter estimation using LORE—For the LORE algorithm, we introduce the following complex parameters:

$$\eta = S_0 e^{i\theta TE/TR}, \quad \beta = S_0 a e^{i\theta(T/TE-1)}, \quad \zeta = b e^{i\theta}. \quad [10]$$

Note the slight over-parameterization with six real-valued parameters as opposed to five in Eq. 4. This enables us to rewrite the noise-free part of Eq. 4 as

$$S_n = \frac{\eta - \beta e^{-i\Delta\theta_n}}{1 - \text{Re}(\zeta e^{i\Delta\theta_n})}. \quad [11]$$

To simplify the notation we introduce the subscripts r and i to denote the real and imaginary part, respectively. Multiplying both sides by the denominator we can now express Eq. 11 in linear form:

$$S_n [1 - \zeta_r \cos\Delta\theta_n + \zeta_i \sin\Delta\theta_n] = \eta - \beta e^{-i\Delta\theta_n}. \quad [12]$$

Furthermore, $|1 - \zeta_r \cos\theta_n + \zeta_i \sin\theta_n| \geq 2$ since it can be shown that $0 \leq b \leq 1$, which implies that the noise is amplified by at most a factor of two in this operation. Moving the unknown variables to the right hand side and gathering the real and imaginary parts of S_n separately in a vector $\mathbf{y}_n = [S_{r,n} \ S_{i,n}]^T$, we can write Eq. 12 in matrix form:

$$\mathbf{y}_n = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\cos(\Delta\theta_n) & \sin(\Delta\theta_n) \\ -\sin(\Delta\theta_n) & -\cos(\Delta\theta_n) \\ S_{r,n}\cos(\Delta\theta_n) & S_{i,n}\cos(\Delta\theta_n) \\ -S_{r,n}\sin(\Delta\theta_n) & -S_{i,n}\sin(\Delta\theta_n) \end{bmatrix}}_{\mathbf{A}_n} \underbrace{\begin{bmatrix} \eta_r \\ \eta_i \\ \beta_r \\ \beta_i \\ \zeta_r \\ \zeta_i \end{bmatrix}}_{\mathbf{x}} \quad [13]$$

By stacking the measurements in a vector $\mathbf{y} = [\mathbf{y}_1 \dots \mathbf{y}_N]^T$ and a matrix $\mathbf{A} = [\mathbf{A}_1^T \dots \mathbf{A}_N^T]^T$, where N is the number of phase-cycled images, we obtain $\mathbf{y} = \mathbf{A}\mathbf{x}$, from which the LS estimate of $\mathbf{x}$ is readily found as:

$$\hat{\mathbf{x}} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{y}. \quad [14]$$

These estimates are biased, since the noise enters the regressor matrix $\mathbf{A}$ through the measured data S_n . The estimates of the sought parameters in Eq. 4 can then be obtained as

$$\hat{\theta} = -\angle(\hat{\beta}/\hat{\eta}), \hat{a} = |\hat{\beta}/\hat{\eta}|, \hat{b} = |\hat{\zeta}|, \hat{S}_0 = \hat{\eta} e^{-i\hat{\theta}TE/TR}, \quad [15]$$

where $\angle(\cdot)$ denotes the phase of a complex number. The information in ζ regarding the off-resonance is not used since ζ can be small in magnitude, leading to unreliable estimates of θ . While LORE can give accurate estimates on its own, the algorithm is sub-optimal in the NLS sense. To tackle this, the LORE estimates can be used as an initial guess for the next step.

Step 2: Fine tuning using Gauss-Newton—We propose to use a Gauss-Newton iterative method to truly minimize the NLS and further improve the results. GN is chosen since it is simple, computationally efficient, and has fast convergence (17). However, this minimization method is unconstrained, so any physical constraints on the parameters cannot be taken into account. Given a good initial estimate, here provided by LORE, GN converges to the correct global optimum with high probability. This is what distinguishes LORE-GN from other general nonlinear methods.

The NLS criterion is

$$L(\mathbf{u}) = \sum_{n=1}^N |S_n - g_n(\mathbf{u})|^2. \quad [16]$$

Letting $\mathbf{r}$ denote the residual vectorized over the measurements, according to

$$\mathbf{r} = \begin{bmatrix} \text{Re}(\mathbf{S}) \\ \text{Im}(\mathbf{S}) \end{bmatrix} - \begin{bmatrix} \text{Re}(\mathbf{g}(\mathbf{u})) \\ \text{Im}(\mathbf{g}(\mathbf{u})) \end{bmatrix}, \quad [17]$$

the update formula for GN with the search direction $\mathbf{p}_k$ is

$$\mathbf{u}_{k+1} = \mathbf{u}_k + c\mathbf{p}_k = \mathbf{u}_k + c(\mathbf{J}_k^T \mathbf{J}_k)^{-1} \mathbf{J}_k^T \mathbf{r}_k, \quad [18]$$

where $\mathbf{J}_k = \mathbf{J}(\mathbf{u})|_{\mathbf{u}=\mathbf{u}_k}$ is the Jacobian matrix evaluated at the current point in the parameter space $\mathbf{u}_k$. In the same manner, we derived the GN algorithm with respect to the original model parameters $\mathbf{u}_0 = [\theta, \text{Re}(KM_0), \text{Im}(KM_0), T_1, T_2]^T$.

The step length c is chosen by back-tracking so that the *Armijo condition* is fulfilled, that is, $c = 2^{-m}$, where m is the smallest non-negative integer that fulfills

$$L(\mathbf{u}_{k+1}) \leq L(\mathbf{u}_k) - \mu c \mathbf{r}_k^T \mathbf{J}_k (\mathbf{J}_k^T \mathbf{J}_k)^{-1} \mathbf{J}_k^T \mathbf{r}_k, \quad [19]$$

and $\mu \in [0, 1]$ is a constant (18). A stopping condition based on the norm of the gradient $\|\mathbf{J}_k^T \mathbf{r}_k\|$ was used. In the following, μ was set to 0.5 and the stopping condition to $\|\mathbf{J}_k^T \mathbf{r}_k\| < 10^{-8}$.

The obtained estimates can then be used to reconstruct band-free images with bSSFP contrast by using the model in Eq. 4, setting $\theta = 0$, and letting θ be any constant value. When considering the explicit parameter estimates, however, phase wrapping and nonphysical optima can cause ambiguities. These problems are treated in the next section.

The MATLAB code for the algorithm is available for general use at: <http://www-mrsrl.stanford.edu/~jbarral/orm.html>

Post-processing

By analyzing the model, a few interesting properties can be seen. The following relation holds:

$$S_0 e^{i\theta TE/TR} \frac{1 - ae^{-i\theta_n}}{1 - b\cos\theta_n} = S_0 e^{\mp i\pi TE/TR} e^{i(\theta \pm \pi) TE/TR} \frac{1 + ae^{-i(\theta_n \pm \pi)}}{1 + b\cos(\theta_n \pm \pi)}. \quad [20]$$

This means that for a set of optimal parameters a , b , θ and S_0 , the NLS criterion will have another global optimum at $\tilde{a} = -a$, $\tilde{b} = -b$, $\tilde{\theta} = \theta \pm \pi$ and $\tilde{S}_0 = S_0 e^{\mp i\pi TE/TR}$. It can be shown that a and b are positive, and hence, we can remove the resulting non-physical minima. Furthermore, we have

$$S_0 e^{i(\theta + 2\pi k) TE/TR} \frac{1 - ae^{-i(\theta_n + 2\pi k)}}{1 - b\cos(\theta_n + 2\pi k)} = S_0 e^{i2\pi k TE/TR} e^{i\theta TE/TR} \frac{1 - ae^{-i\theta_n}}{1 - b\cos\theta_n}, \quad \forall k \in \mathbb{Z}, \quad [21]$$

that is, a shift of θ by $2\pi k$ is equivalent to a phase shift of S_0 by $2\pi k TE/TR$. The estimate of θ is confined in the interval $[-\pi, \pi]$ (wrapped phase), meaning that if the true θ is outside of our estimation interval, we will obtain the wrong phase of S_0 . It is important to realize that the magnitude signal is not affected by Eq. 20 and Eq. 21, and hence, the post-processing step is not needed for band reduction. The problems only arise when estimating the absolute off-resonance and a complex-valued S_0 , in which case phase unwrapping is needed to get consistent estimates. By assuming that θ is close to zero in the center of the image, which can be obtained through proper shimming, we can unwrap the estimated phase to obtain k in Eq. 21, and then compensate our S_0 estimates according to

$$\hat{S}_{0u} = \hat{S}_0 e^{-i2\pi k TE/TR}. \quad [22]$$

Given that proper shimming has indeed been ensured, we obtain the true estimate of S_0 . Phase unwrapping in two or three dimensions is a common problem in MRI, and several methods in the literature tackle it, see for example (19) for a review. Here, a MATLAB

implementation of the quality-guided 2D phase-unwrapping algorithm was used (20). A detailed description is beyond the scope of this paper.

The two correction steps, that is Eq. 20 and phase unwrapping together with Eq. 22, constitute the post-processing step, which can be used to avoid some local minima of the criterion function in Eq. 16.

Methods

Simulations and the CRB

Simulations were performed with the following parameters:

$$T_1=675\text{ms}, T_2=75\text{ms}, K M_0=1, \alpha=30^\circ, TR=5\text{ms}, TE=2.5\text{ms}. \quad [23]$$

These parameters were chosen as a representative case targeting brain white matter at 1.5 T, and they were the basis of all simulations unless stated otherwise.

The simulated data was generated by adding complex Gaussian noise of appropriate variance σ^2 to the model $g_n(\mathbf{u})$ in Eq. 4. The variance was chosen to achieve a certain SNR as defined by

$$SNR = \frac{\sum_{n=1}^N |g_n(\mathbf{u})|}{N\sigma}, \quad [24]$$

which is the common definition of SNR in the MRI community (21).

The root mean square error (rMSE) of the parameter estimates is defined as

$$rMSE(\hat{z}) = \sqrt{\frac{1}{M} \sum_{m=1}^M |\hat{z}_m - z|^2}, \quad [25]$$

where $\hat{z}_m$ is the parameter estimate in simulation m , z is the true parameter value, and M is the number of simulations.

The simulations and calculations were performed in MATLAB on an HP desktop computer with a 2.8 GHz Intel Core i7 860 quad-core processor and 16GB RAM. All computation times were measured when running a single thread.

Monte Carlo simulations provided the rMSE of the parameter estimates. The performance of the proposed algorithm was compared to 1) a Levenberg-Marquardt (LM) algorithm suggested by Santini and Scheffler (4), 2) LM with our suggested post-processing step (LMpost), and 3) the optimum performance given by the CRB. The standard MATLAB LM implementation in the function “lsqnonlin” was utilized. The estimates obtained with LORE were also included in the comparison to illustrate the accuracy of this linear algorithm and the initial estimates used by GN. The cross-solution by Xiang and Hoff (3) is not included here, since it does not estimate the model parameters.

To illustrate the CRB of T_1 and T_2 , the minimum SNR needed to obtain a 5% relative standard deviation (RSD) in the estimates was calculated. The RSD was defined as the CRB standard deviation of the parameter estimate, relative to the true parameter value, that is for parameter j in Eq. 7, $\text{RSD}_j = \sqrt{[P_{\text{CRB}}]_{jj}}/u_j$. Numerical simulations were performed for true parameter values in the ranges $T_1 = 100 - 3000$ ms and $T_2 = 5 - 200$ ms.

Phantom and in vivo data

A phantom and an in vivo brain dataset was acquired using a 1.5 T scanner (GE Healthcare, Milwaukee, WI). Each dataset consisted of four complex-valued 3D bSSFP images with linear phase increments of $0, \pi/2, \pi$, and $3\pi/2$. The scan parameters were: field-of-view = $24 \times 24 \times 16$ cm³, matrix size = $128 \times 128 \times 32$, $TR = 5$ ms, $TE = 2.5$ ms, $\alpha = 30^\circ$. The phantom dataset was included since the banding artifacts remaining after the processing are easily visualized when the ideal intensity is uniform. For demonstration purposes, in order to induce more significant banding artifacts in the phantom and the in vivo datasets, the automatic shimming was disabled at 1.5 T. The estimated average SNR was 170 and 33 for the phantom and in vivo data, respectively.

Similarly, an in vivo brain dataset was acquired using a 7 T system (GE Healthcare). The scan parameters were as follows: field-of-view = $20 \times 20 \times 16$ cm³, matrix size = $200 \times 200 \times 160$, $TR = 10$ ms, $TE = 5$ ms, $\alpha = 10^\circ$. High-order shimming was used to achieve best-possible field homogeneity. The estimated average SNR was 11. This dataset was included to show that the proposed method can be applied to higher field strengths, where banding artifacts are typically more significant. Also, the low flip angles used at high field strengths due to specific absorption rate (SAR) constraints give a non-flat passband in the bSSFP profile, which is problematic for many of the competing approaches. The longer TR for the 7 T dataset is motivated by the application in (22), where phase-cycled bSSFP at 7 T is used for high resolution imaging of the hippocampus.

Before running the LORE-GN algorithm, the data was masked to remove the background, thereby reducing the computation time. This was done by thresholding the sum-of-squares image and masking pixels with intensity below a certain percentage of the maximum value, in this case 15% for 1.5 T and 6% for 7 T. The resulting number of computed pixels was 9467, 22712 and 22844 for the 1.5 T phantom, 1.5 T in vivo, and 7 T in vivo data, respectively.

Here we reconstructed the images at $\theta = \pi/2$ to be able to compare directly with one of the original phase-cycled images. This corresponds to the image closest to maximum SNR reconstruction for both white and grey matter. Since the computed reconstructions, as well as the the collected phase-cycled images, are complex-valued, the corresponding magnitudes were used when displaying the images.

Results

Simulations and the CRB

The results of the Monte Carlo simulations, based on 10000 noise realizations, are shown in Fig. 1. As can be seen, the LORE-GN algorithm is efficient when estimating the parameters of Eq. 1 given an SNR above 5, since it achieves the CRB, and the performance of LORE alone is comparable in this case. SNRs below 5 have been excluded since the variance is bound to be high, which would generally not result in useful estimates.

Indirect estimates of T_1 , T_2 and KM_0 can be obtained from Eq. 9, and the corresponding performance is shown in Fig. 2. In these figures, outliers have been removed at the lower SNR values. This was done by omitting estimates with an absolute distance larger than 20 CRB standard deviations from the true value. The reason for removing outliers is that single large values, caused by a noise realization leading to singularity, can have a great impact on the rMSE values, but these cases are easily detected and removed. The estimates in Fig. 2 have a rather high variance in general, and an SNR above 50 is needed to achieve the CRB for T_1 and T_2 .

Figures 3a and 3b show how the needed SNR varies for T_1 and T_2 estimation, respectively, for different true values of T_1 and T_2 . It can be seen that the minimum required SNR is generally quite high. For example, an SNR of roughly 72 is needed to estimate the relaxation parameters of white matter and grey matter in the brain with 5% RSD.

Phantom example

Four phase-cycled images from a central slice of the 1.5 T 3D phantom dataset, all showing some degree of banding, are shown in Fig. 4a. The LORE-GN reconstructed image at $\theta = \pi/2$ is shown in Fig. 4b, together with the sum-of-squares, maximum-intensity, and cross-solution images in Figs. 4c, 4d, and 4e, respectively. The proposed method results in an image showing no bands, while in the sum-of-squares and maximum-intensity images, some artifacts still remain. The cross-solution gives a uniform image, but has an intensity corresponding to the parameter S_0 . The off-resonance frequency estimated by LORE-GN is shown in Fig. 4f. For comparison, the off-resonance was also estimated using two gradient echo images with $TE = 4$ and 5 ms, respectively, and computing the field map from the phase difference of these images as is described in (23). The result is shown in Fig. 4g. As can be seen, LORE-GN provides a smooth low noise estimate that corresponds well with the gradient echo based technique.

In vivo examples

The phase-cycled images from a central slice of the 1.5 T 3D in vivo dataset are shown in Fig. 5a. LORE-GN was applied to estimate the model parameters. The reconstructed image at $\theta = \pi/2$ is shown in Fig. 5b, together with the sum-of-squares, maximum-intensity and cross-solution images in Figs. 5c, 5d and 5e, respectively. As expected, banding artifacts are more subtle than in the phantom experiments. By scaling and subtracting the sum-of-squares, maximum-intensity and cross-solution images from the LORE-GN estimate, the differences can be visualized more clearly. These images are shown in Figs. 5f, 5g and 5h,

respectively. Sum-of-squares shows some non-uniformity compared to the proposed estimate, while the maximum-intensity and cross-solution approaches give a similar level of uniformity. Using any constant initialization of LM seemed to give estimation errors in some parts of the image, leading to defects in the reconstruction. An example of this is shown in Fig. 6, where the upper right corner of the image has been zoomed in to visualize the problem more clearly. The image was reconstructed at $\theta = \pi/2$, similarly to Fig. 5b. In Fig. 7, additional LORE-GN reconstructions at different θ for the 1.5 T in vivo data of 5a are shown. As can be seen, the SNR and contrast vary depending on the reconstruction.

Four phase-cycled images from a central slice of the 7 T 3D in vivo dataset are shown in Fig. 8a. Here LORE was used alone to estimate the parameters, due to the low SNR. The reason for this will be further explained in the discussion. In Fig. 8b the reconstructed image at $\theta = \pi/2$ is shown, together with the sum-of-squares, maximum-intensity and cross-solution images in Figs. 8c, 8d and 8e, respectively. No banding can be seen with LORE, even at such a low flip angle. Sum-of-squares and maximum-intensity do not fully suppress the bands, as indicated by the arrows. The cross-solution method has problems due to the low SNR, and the resulting image is severely degraded.

Run times

The run times for the phantom and the in vivo data are shown in Table 1. The current implementation of LORE-GN is approximately 7 times faster than using LM with a fixed initialization. It can also be seen that LORE accounts for less than 10% of the LORE-GN run time, providing a speedup factor of approximately 80 compared to LM.

Discussion

The fast linear algorithm LORE is the main contribution in this paper. In many cases, LORE provides accurate estimates, but it can also be used to initialize a nonlinear algorithm minimizing the NLS criterion. By adding post-processing steps we can also separate between the several optima that are inherent to the model, which is important for consistent estimation of the parameters. Using the estimated parameters we can reconstruct band-free images by using the model in Eq. 4. As mentioned, under the Gaussian assumption, the reconstructed data will be the ML estimate of the true signal, and hence minimally distorted by noise.

Simulations and the CRB

The case shown in Fig. 1 was chosen to illustrate when LM fails due to wrong initialization of θ . This leads LM into an ambiguous optimum, which is the reason for the poor performance where the rMSE is high and remains constant when increasing the SNR. The suggested post-processing mostly corrects for this, however, noise minima can occur at low SNR, making the initialization increasingly important. This can be seen in Fig. 1 as an increased rMSE for all parameters when using LMpost compared to LORE-GN at low SNR.

LORE and LORE-GN show robustness at low SNR while the other methods sometimes give outliers. This is for example shown in Fig. 1d where the LORE rMSE is even slightly below the CRB at low SNR. This is possible due to the biased estimates provided by LORE.

However, the problem gets quite sensitive at low SNR if only four phase-cycled images are used, which can lead to outliers. For LORE-GN, this is mainly a problem when estimating T_1 , T_2 and KM_0 , which is why outliers were removed when generating Fig. 2. It can be seen in Fig. 2 that the rMSEs at low SNR are typically higher for LORE-GN than LORE. Since GN involves the inversion of a potentially close-to-singular matrix, high noise levels can be a problem. It is therefore advisable to use LORE alone in these cases, or to apply a gradient-based method instead of GN, which does not involve any matrix inversion. However, LORE-GN is advantageous at higher SNR due to its ML formulation.

The difference in signal power and the conditioning of the model are the two major contributors to the variations in Figs. 3a and 3b for different values of T_1 and T_2 . As can be seen, the SNR required to precisely estimate T_1 and T_2 from a set of four phase-cycled bSSFP images is rather high. This means that using purely phase-cycled bSSFP to simultaneously estimate T_1 and T_2 is not a good approach. It is therefore of little use to derive a method that is efficient at lower SNR. Figure 3b indicates that it is easier to estimate T_2 than T_1 in the region of short T_2 values. However, achieving high SNR is harder for short T_2 -species. In practice the noise variance will be fixed and the signal magnitude will vary over the image, in this case the decrease in SNR will contribute to an increased CRB at short T_2 . It should be noted that the difficulty to estimate T_1 and T_2 explicitly does not affect the quality of the image reconstruction, since these estimates are not used in Eq. 4.

Phantom example

For the phantom data, the proposed method, Fig. 4b, has a clear advantage compared to the sum-of-squares and maximum-intensity images, Figs. 4c and 4d. The cross-solution image, Fig. 4e, provides a uniform intensity similar to the proposed approach. However, since the cross-solution assumes $TE = 0$, it is not strictly valid in this case, and therefore provides a biased estimate with a different intensity. It can also be noted that LORE-GN provides an accurate estimate of the off-resonance, Fig. 4f, when comparing to the reference field map in Fig. 4g. Field map estimation is an application in its own, and several methods are described in the literature, see for example (24, 25). Since LORE provides an efficient estimate of the off-resonance, and a rather low variance even at an SNR of 10, it could be a useful method for B_0 field mapping.

In vivo examples

The main difference when using highly structured in vivo images, as opposed to the phantom dataset, is the partial volume effects at tissue borders. As can be seen in Fig. 5b, the reconstructed image using the LORE-GN estimates seem to have a level of detail similar to the sum-of-squares and maximum-intensity images of Figs. 5c and 5d, respectively. The reconstructed image shows no bands as opposed to the sum-of-squares. The banding is further highlighted by the difference image in Fig. 5f. However, no clear advantage in terms of banding artifacts can be seen when compared to maximum-intensity and cross-solution, as can be seen in Figs. 5g and 5h, respectively. Since the contrast, for example between white and grey matter, varies for different phase increments, the maximum-intensity will not provide the largest possible contrast. With the proposed algorithm, it is possible to reconstruct several images with different phase increments θ . There is a trade-off between

contrast and SNR, but both can be kept higher than for maximum-intensity, while getting similar or superior band reduction. The reconstructions in Fig. 7 show that different contrasts and SNRs can be obtained. However, the SNR defines the extent to which these reconstructions can be used. For example, $\theta = 0$ gives a low SNR, while theoretically, this corresponds to the image with the maximum contrast between grey and white matter.

As can be seen in Fig. 6, LM with fixed initialization does not converge properly, leading to defects in the reconstructed image. These defects are due to local minima caused by noise, and adding the post processing does not correct for this. Using LORE, however, solves the problem, which underlines the importance of a proper initialization.

The 7 T dataset shows that the method is applicable at higher field strengths and low flip angles, as the resulting LORE reconstruction in Fig. 8b gives superior band reduction compared to the sum-of-squares and maximum-intensity in Figs. 8c and 8d, respectively. The low flip angle and the resulting narrow pass band of the bSSFP profile is the main reason for the incomplete band reduction provided by the sum-of-squares and maximum-intensity. Furthermore, this low SNR example gives an example where using LORE alone is favorable, as GN can suffer from singularity problems. As can be seen the LORE reconstruction does not suffer from the low SNR artifacts of the cross-solution, shown in Fig. 8e.

Run times

LORE-GN is fast due to its simplicity and the few GN iterations needed to converge when the initial value is close to the optimum, as was seen in Table 1. The initial value is provided by LORE, which due to its linear formulation is much more computationally efficient than the nonlinear counterparts. The exact computation time will depend on the implementation, and optimizations in this respect are possible. Because of the pixel-wise computations the algorithm can easily be parallelized on multi-core computers to decrease total run time significantly.

Limitations

One limitation of the current method is that magnetization transfer (MT) is not taken into account. Determining the impact of MT is beyond the scope of this paper. The effects could be modeled, but adding MT parameters will increase the complexity of the model (26); and it is unlikely that LORE could be generalized to this case, since it is specific to the model presented here. Furthermore, the number of images would have to be increased, leading to a longer acquisition time.

Imperfections in the slice profile can be the source of poor estimation, especially for 2D datasets. The flip angle is in this paper assumed to be constant across a voxel, but significant variations of the flip angle within a voxel can make the model invalid. Here we have only used 3D acquisitions in which the slice profile is approximately constant for the slice of interest. Finally, the model does not take partial volume effects into account, but no problems were observed here when applying the LORE-GN to the in vivo images.

Conclusion

We have successfully minimized off-resonance effects in bSSFP. The LORE-GN algorithm is designed to be general, and can be applied to any phase-cycled bSSFP datasets with three or more images, regardless of TE and TR . For band removal and off-resonance estimation, the flip angle does not have to be known, and the method provides uniform reconstructed images even at low flip angles, where other techniques often fail. The fast linear estimator LORE is user-parameter free, which in turn makes the method simple to use and robust, and it provides rather accurate estimates. Adding a nonlinear optimization step, we can efficiently minimize the NLS and provide reconstructed images with optimal SNR in the ML sense, under the assumption of Gaussian noise. We have also demonstrated that it is inherently difficult to explicitly estimate T_1 and T_2 from pure phase-cycled bSSFP, as the obtained variance is bound to be high at common SNR.

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Appendix - Derivation of the signal model

The derivation of Eq. 1 was inspired by Freeman and Hill (13), but it is generalized to account for an arbitrary echo time TE and to accommodate the use of phase cycling. Here we assume a right-handed coordinate system with the z-axis along the static magnetic field B_0 . The magnetization vector ($\mathbf{M}$) immediately before the k th RF pulse is related to that after by a simple rotation of $-\alpha$. Representing the rotation by a matrix, we have

$$\mathbf{M}(kTR^+) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \mathbf{M}(kTR^-) = \mathbf{R}_x(-\alpha)\mathbf{M}(kTR^-), \quad [26]$$

assuming that the pulse is along the x-axis. Here we have defined $\mathbf{M}(t^\pm) = \lim_{\tau \rightarrow t^\pm} \mathbf{M}(\tau)$. With linear phase cycling, the phase of the pulse will be incremented by the same phase for each repetition. Introducing a coordinate system, which aligns with the RF phase at each excitation, we can still express the flips as rotations around the x-axis. During each TR, free precession causes the magnetization to rotate about the z-axis, and relaxation causes an exponential recovery towards thermal equilibrium (M_0). When using a discretely rotating frame that rotates with the linear phase increment, an additional rotation about the z-axis (equal and opposite the phase increment) is introduced. Using the definition of a rotation matrix generalized from Eq. 26, we get

$$\mathbf{M}((k+1)TR^-) = \mathbf{R}_z(\Omega TR)\mathbf{R}_z(\Delta\Omega TR)\mathbf{D}(TR)\mathbf{M}(kTR^+) + (1 - E_1)\mathbf{M}_0, \quad [27]$$

where $\mathbf{D}(t)$ is a damping matrix with diagonal elements e^{-t/T_2} , e^{-t/T_2} and e^{-t/T_1} respectively, $E_1 = e^{-TR/T_1}$, $\mathbf{M}_0 = [0, 0, M_0]^T$ is the equilibrium magnetization directed along the z-axis, Ω

$= 2\pi\gamma B_0$ corresponds to the off-resonance frequency in radians per second, and ΩTR is the user-controlled phase increment of the RF pulse.

Substituting Eq. 27 into Eq. 26 and using the fact that $\mathbf{M}((k+1)TR^+) = \mathbf{M}(kTR^+)$ at steady-state, we get a system of equations that can be solved to obtain $\mathbf{M}(TR^+)$, where the arbitrary k has been dropped to simplify the notation. The solution is

$$\mathbf{M}(TR^+) = (\mathbf{I} - \mathbf{R}_x(-\alpha)\mathbf{R}_z(\Omega TR)\mathbf{R}_z(\Delta\Omega TR)\mathbf{D}(TR)^{-1} \times (1 - E_1)\mathbf{R}_x(-\alpha)\mathbf{M}_0. \quad [28]$$

At the echo time TE , the free precession has rotated the magnetization in the transverse plane by an angle $2\pi\gamma B_0 TE = \Omega TE$ under decay from the initial $\mathbf{M}(TR^+)$. This can be expressed as

$$\mathbf{M}(TE) = \mathbf{R}_z(\Omega TE)\mathbf{D}(TE)\mathbf{M}(TR^+). \quad [29]$$

Note that there is no accumulation of phase due to the rotation of the frame, since the frame rotates in discrete steps just before each excitation. We are only interested in the transverse component of Eq. 29. Expressing it as a complex number $S_n = M_x(TE) + iM_y(TE)$ and simplifying, we get

$$S_n = M e^{-\frac{TE}{T_2}} e^{i\Omega TE} \frac{1 - a e^{-i(\Omega + \Delta\Omega_n)TR}}{1 - b \cos[(\Omega + \Delta\Omega_n)TR]}, \quad [30]$$

where we have defined

$$M = iM_0 \frac{(1 - E_1)\sin\alpha}{1 - E_1\cos\alpha - (E_1 - \cos\alpha)E_2^2}, \quad a = E_2, \quad b = \frac{E_2(1 - E_1)(1 + \cos\alpha)}{1 - E_1\cos\alpha - (E_1 - \cos\alpha)E_2^2}, \quad [31]$$

and $E_2 = e^{-TR/T_2}$. Including a coil sensitivity K , the received signal is

$$S_n = K M e^{-\frac{TE}{T_2}} e^{i\Omega TE} \frac{1 - a e^{-i(\Omega + \Delta\Omega_n)TR}}{1 - b \cos[(\Omega + \Delta\Omega_n)TR]}. \quad [32]$$

Equation Eq. 32 is the final signal equation for phase-cycled bSSFP with arbitrary TE .

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(a) $\hat{\theta}$ in degrees. (b) $\hat{S}_0$. (c) $\hat{a}$. (d) $\hat{b}$.**Figure 1.**

rMSE vs. SNR for the parameter estimates, and for the different methods, along with the associated CRB. (a) $\hat{\theta}$ (degrees), (b) $\hat{S}_0$, (c) $\hat{a}$, and (d) $\hat{b}$. The results are based on 10000 Monte Carlo simulations. The true values were $\theta = 0$, $S_0 = 0.1207i$, $a = 0.9355$, and $b = 0.4356$.

(a) $\hat{T}_1$. (b) $\hat{T}_2$. (c) $K\hat{M}_0$.**Figure 2.**

rMSE vs. SNR for the indirect parameter estimates (a) T_1 , (b) T_2 , and (c) KM_0 , and for the different methods, along with the associated CRB. The results are based on 10000 Monte Carlo simulations. The true values were $T_1 = 675$ ms, $T_2 = 75$ ms, and $KM_0 = 1$.

(a) Estimating T_1 . (b) Estimating T_2 .**Figure 3.**

SNR needed to achieve a standard deviation equal to 5% of the true value of the parameters (a) T_1 , and (b) T_2 . The values for $T_2 > T_1$ are excluded (grey) and the SNR range is saturated at a maximum of 400.

(a) Phase-cycled images.

(b) LORE-GN.

(c) Sum-of-squares.

(d) Maximum-intensity.

(e) Cross-solution.

(f) $\hat{f}_{\text{OR}}$. (g) Reference f_{OR} .**Figure 4.**

(a) 1.5 T phantom dataset of four images with phase increments $0, \pi/2, \pi, 3\pi/2$. (b) the reconstructed image with $\theta = \pi/2$, (c) the sum-of-squares image, (d) the maximum-intensity image, (e) the cross-solution estimate, (f) the estimated off-resonance frequency, and (g) the reference off-resonance frequency. LORE-GN and the cross-solution show a uniform intensity, while bands still remain in the sum-of-squares and maximum-intensity images. The estimated average SNR of the data was 170. The automatic shimming was disabled to induce more significant banding artifacts. Each image in (b)–(e) is displayed with the same scale as the original images in (a), and all images have been cropped prior to display.

(a) Phase-cycled images.

(b) LORE-GN.

(c) Sum-of-squares.

(d) Maximum-intensity.

(e) Cross-solution.

(f) Difference: (c)-(b)

(g) Difference: (d)-(b)

(h) Difference: (e)-(b)

Figure 5.

(a) 1.5 T in vivo brain dataset of four images with phase increments $0, \pi/2, \pi, 3\pi/2$; (b) the reconstructed image with $\theta = \pi/2$; (c) the sum-of-squares image; (d) the maximum-intensity image; (e) the estimate obtained with the cross-solution. The relative difference between the proposed estimate and: sum-of-squares (f), maximum-intensity (g), and cross-solution (h); where the difference in average intensity has been removed, shows that some bands remain in the sum-of-squares image. The estimated average SNR was 33. The automatic shimming was disabled to induce more significant banding artifacts. Each image in (b)–(e) is displayed with the same scale as the original images in (a).

Figure 6.

A zoomed-in example of the $\theta = \pi/2$ reconstruction from LM with constant initialization when applied to the 1.5 T in vivo data of Fig. 5a. The reconstruction has defects due to convergence issues of the LM algorithm.

Figure 7.

Reconstructions with different $\Delta\theta$ using the LORE-GN estimates obtained from the 1.5 T in vivo dataset of Fig. 5a. Each image is displayed in a different scale to make the median intensity comparable. As can be seen, the contrast and SNR of the reconstruction depend on $\Delta\theta$.

(a) Phase-cycled images.

(b) LORE.

(c) Sum-of-squares.

(d) Maximum-intensity.

(e) Cross-solution.

(f) LORE
(zoomed-in). (g) Sum-of-squares
(zoomed-in). (h) Maximum-intensity
(zoomed-in). (i) Cross-solution
(zoomed-in).**Figure 8.**

(a) 7 T in vivo brain dataset of four images with phase increments $0, \pi/2, \pi, 3\pi/2$; (b) the LORE reconstructed image with $\theta = \pi/2$; (c) the sum-of-squares image; (d) the maximum-intensity image; (e) the estimate obtained with the cross-solution. (f)–(i) shows zoomed-in versions of (b)–(e), focusing on the top left part of the image. The proposed method shows no banding artifacts while the sum-of-squares and maximum-intensity have some remaining bands as indicated by the arrows. The cross-solution fails due to the low SNR, which was estimated to be 11. A high-order shim was used to achieve the best-possible field

homogeneity. Each image in (b)–(i) is displayed with the same scale as the original images in (a).

Table 1

Average MATLAB run times in seconds for the different methods and the 1.5 T datasets.

	Dataset	
	Phantom	In vivo
LORE	1.5	3.6
LORE-GN	16	40
LM	122	245
LMpost	123	247