Tear film dynamics with evaporation, wetting, and time-dependent flux boundary condition on an eye-shaped domain

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We study tear film dynamics with evaporation on a wettable eye-shaped ocular surface using a lubrication model. The mathematical model has a time-dependent flux boundary condition that models the cycles of tear fluid supply and drainage; it mimics blinks on a stationary eye-shaped domain. We generate computational grids and solve the nonlinear governing equations using the OVERTURE computational framework. *In vivo* experimental results using fluorescent imaging are used to visualize the influx and redistribution of tears for an open eye. Results from the numerical simulations are compared with the experiment. The model captures the flow around the meniscus and other dynamic features of human tear film observed *in vivo*. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4871714>]

I. INTRODUCTION

The tear film is essential for clear vision and eye health. It helps to protect the ocular surface with moisture, to transport waste away, and to provide a smooth optical surface.¹ A properly functioning tear film maintains a critical balance between tear secretion and loss within each blink cycle. The malfunction or deficiency of the tear film causes a collection of problems that are believed to cause dry eye syndrome (DES).² DES symptoms include, but are not limited to, blurred vision, burning, foreign body sensation, and tearing. Studies up to 2007 estimate that there are 4.91×10^6 Americans suffering from DES.³ The ocular surface community is interested in understanding the function of the tear film⁴ as well as the interaction of tear film dynamics and the connection between tear film volume, evaporation, and break up with DES.³

Commonly, the tear film is described as a thin liquid film with multiple layers. At the anterior interface with air is an oily lipid layer that decreases the surface tension and retards evaporation, both of which help retain a smooth well-functioning tear film.⁵ The aqueous layer is posterior to the lipid layer and consists mostly of water.⁶ At the ocular surface, there is a region with transmembrane mucins protruding from the cells in the corneal or conjunctival epithelial. This forest of glycosolated mucins, called the glycocalyx, has been referred to as the mucus layer in the past. It is generally agreed that the presence of the hydrophilic glycocalyx on the healthy ocular surface prevents the tear film from dewetting.^{7–9} The overall thickness of the tear film is a few microns,¹⁰ while the average thickness of the lipid layer is of the order of 50–100 nm^{5,11} and the thickness of the glycocalyx is

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a few tenths of a μm .⁹ This structure is rapidly reformed after each blink in a properly functioning tear film.

The aqueous part of tear fluid is supplied from the lacrimal gland and the excess is drained through the puncta. Previous estimates are available for these quantities. For example, Mishima *et al.*¹² estimated the total tear volume and the rate of influx from the lacrimal gland, as well as the time for the entire volume of tear fluid to be replaced (tear turnover rate). Doane¹³ described the mechanism of tear drainage *in vivo*: tear fluid is drained into the canaliculi through the puncta during the interblink phase (the time between blinks), because of the pressure difference in the meniscus and the canaliculi. The drainage stops when the pressure equalizes in the canaliculi. Zhu and Chauhan¹⁴ proposed a mathematical model to study the tear drainage rate. The model predicts the rate can vary from $0.10\ \mu\text{l}/\text{min}$ to $4.00\ \mu\text{l}/\text{min}$. They verified the results by comparing with literature results. Water lost from the tear film due to evaporation into air is an important process as well.^{15–17}

The supply and drainage of tear fluid affects the distribution and flow of the tear film. A number of methods have been used to visualize and/or measure tear film thickness and flow, including interferometry,^{10,18,19} optical coherence tomography,²⁰ fluorescence imaging,^{21,22} and many others. We mention only a small number here that are relevant for our discussion of tear fluid flow over the exposed ocular surface. Maurice²³ inserted lamp black into the tear film and watched their trajectories with a slit lamp. He observed that the particle paths in the upper meniscus near the temporal canthus diverge, with some proceeding toward the nasal canthus via the upper meniscus and others going around the outer canthus before proceeding toward the nasal canthus via the lower meniscus. (To our knowledge, no images from this experiment exist.) We use the term “hydraulic connectivity” as shorthand for this splitting of flow connecting the menisci. A similar pattern of the tear film was observed by Harrison *et al.*²¹ using fluorescein to visualize the tear film thickness. In this experiment, concentrated sodium fluorescein is instilled in the eye. Shining blue light on the eye causes the fluorescein to glow green; the fluorescence allows one to visualize the tear film.²⁴ The concentration is such that, if evaporation occurs, the concentration of fluorescein increases and the intensity of the fluorescence decreases; if fresh tear fluid enters the tear film, the concentration decreases and fluorescent intensity increases.^{25–27} Harrison *et al.*²¹ visualized the entry of fresh tear fluid into the meniscus near the outer canthus, where the tear film was observed to brighten. Subsequently, this bright region split and fluid moved toward the nasal canthus along both the upper and lower lids via the menisci. The fluorescence could thus visualize hydraulic connectivity in the flow of tears. In this paper, a less invasive visualization using concentrated fluorescein was recorded,²² and we will compare images from this experiment with our model for tear film flow. Both the experiment and the model will exhibit hydraulic connectivity in a very similar way.

The tear film is also redistributed near the eyelid margins by surface tension. McDonald and Brubaker²⁸ studied the importance of surface tension on the meniscus-induced thinning of tear films in their experiments with milk and paper clips. The wetting of a the paper clip was a model for the wetting of the eyelid margin; the curvature generated by the meniscus draws in fluid from surrounding areas, creating locally thin regions near the meniscus. When fluorescein is used to visualize tear film thickness, this locally thin region near the lids is dark, and had been called the “black line.” The black line is typically thought to be a barrier between the meniscus at the lid margins and the rest of the tear film.^{28,29} Finally, though the healthy ocular surface is wettable, the tear film may still rupture; the term break up is used in the ocular science community for this phenomenon. The effects listed above are what we aim to include in this study.

A variety of mathematical models have incorporated various important effects of tear film dynamics as recently reviewed by Braun.³⁰ The most common assumptions for these models are a Newtonian tear fluid and a flat cornea, following Berger and Corrsin.³¹ Braun *et al.*³² investigated thin film dynamics on a prolate spheroid and justified that corneal curvature has a negligible effect on tear film dynamics. Tear film models are typically formulated on a one dimensional domain oriented vertically through the center of the cornea with stationary ends corresponding to the eyelid margins.^{33,34} We refer to models on this kind of domain as 1D models. Surface tension, viscosity, gravity, and evaporation are often incorporated into 1D models. Wong *et al.*³³ and Sharma *et al.*³⁵ studied the relaxation of related 1D models and found that the meniscus caused localized thinning like the black line observed *in vivo*. Miller *et al.*²⁹ incorporated gravity and surface tension forces to model

the dynamic black line formation under the lubrication approximation at low Reynolds number. Braun and Fitt³⁴ developed a fluid dynamic model to explore the drainage of the aqueous layer; their model incorporated evaporation and gravity. Winter *et al.*³⁶ improved previous evaporation models by including a conjoining pressure via a van der Waals type term that approximated the wettable corneal surface. The van der Waals term mimicked the wetting action of the glycocalyx mentioned previously by preventing the thickness from going below the height of the glycocalyx. This term could also be used to set the rate of spread of break up regions. Li and Braun³⁷ resolved a discrepancy of the tear film surface temperature between predictions of existing evaporation models and *in vivo* measurements by allowing for cooling in a realistically thick substrate that includes the cornea and part of the aqueous humor.

Recently, studies of 1D models with a moving end representing the upper lid have appeared. Wong *et al.*³³ formulated a coating flow model that predicted deposited tear film thickness from a moving meniscus as a function of tear viscosity, surface tension and meniscus radius. Their tear film formation model was separate from the relaxation model for the interblink described above. Jones *et al.*^{38,39} developed models for tear film formation and relaxation that were unified; one end of the domain moved to model the upper lid motion during the opening phase of the blink, then remained stationary for the subsequent relaxation during the interblink. Braun and King-Smith⁴⁰ modeled eyelid motion for blink cycles by moving one end of the domain sinusoidally and they computed solutions for multiple complete blink cycles. Heryudono *et al.*⁴¹ followed their study with a more realistic lid motion and specified a flux boundary condition. Good agreement on tear film thickness between experiments and simulations was found by Heryudono *et al.*⁴¹ Zubkov *et al.*⁴² presented a model describing the spatial distribution of tear film osmolarity that incorporates fluid and solute dynamics, blink and vertical saccadic eyelid motion. They found both osmolarity and polar lipid profiles are heterogeneous across the ocular surface and that measurements of the solute concentrations within the lower meniscus need not reflect those elsewhere in the tear film. Deng *et al.* (in revision) extended the model of Li and Braun³⁷ to include upper lid motion (blink) and explained theoretically that the minimum temperature of the cornea is slightly inferior to the geometric center of the cornea (GCC) as measured by Efron *et al.*⁴³

Because tear film dynamics vary significantly over the entire ocular surface, 1D models are unable to provide a complete picture of the tear film. It is a natural and important step forward to study tear film dynamics on an eye-shaped domain. To our knowledge, Maki *et al.*^{44,45} were the first to extend the tear film model to a geometry that approximated the exposed ocular surface. They formulated a relaxation model on a stationary 2D eye-shaped domain approximated from a digital photo of an eye. They specified tear film thickness and pressure boundary conditions⁴⁴ or flux boundary conditions.⁴⁵ Their simulations recovered features seen in 1D models such as formation of black line, and captured some experimental observations of the tear film dynamics around the lid margins. Maki *et al.*⁴⁵ simplified the *in vivo* mechanisms and imposed a flux boundary condition having only space dependence (specifying the location of the lacrimal gland and the puncta holes) in their tear film relaxation model. Under some conditions, they were able to recover hydraulic connectivity as seen experimentally and described above.

In this paper, we formulate a tear film dynamics model on a 2D eye-shaped domain that incorporates the effects of evaporation and ocular surface wettability. In an effort to mimic some effects of blinks on a stationary boundary, we specify the normal component of the flux at the boundary with both time and space dependence (flux cycle). The time-dependent flux boundary condition is formulated according to Doane's mechanism for tear drainage¹³ and the tear drainage model of Zhu and Chauhan,¹⁴ but with simplification regarding blinking: there is no lid motion. Here we investigate how much of the observed *in vivo* tear film dynamics can be recovered using only a flux cycle and by compensating for evaporative water losses with extra influx but no actual blinks. The numerically challenging treatment of a blinking eye-shaped domain is currently under development and will be treated in a future paper.

We begin by describing the experiment in Sec. II A and formulating the model in Sec. II B. A brief description of the numerical methods used for the simulation is discussed in Sec. III, followed by detailed results of simulations and comparison with experiments in Sec. IV. Conclusions and further directions are pointed out in Sec. V.

II. METHODS

A. Experiment

We begin by briefly describing the fluorescence imaging method used for visualizing tear film dynamics.^{22,26,27} This established method was used to produce original images in this work. Video recordings were made from subjects, including normal and dry eyes, for a 60 s period after instillation of 1 μ l of 5% (sodium) fluorescein. Subjects were instructed to blink about 1 s after the start of the recording and try to hold their eyes open for the remainder of the recording. The subjects' eyes were illuminated with blue light and a blocking interference filter was used to reduce the response to reflected illumination light. This allowed a better detection of the fluorescence of the tear film. The horizontal illumination width was 15 mm, thus including the cornea and part of the conjunctiva. The research protocol was approved by an Institutional Review Board in accordance with the Declaration of Helsinki. Informed consent was obtained from each subject at study enrollment.

Tear film thickness was studied by using the self-quenching of fluorescein, i.e., the reduction of fluorescent efficiency with increasing fluorescein concentration.²⁴ When diluted by about 7 μ l of tears, fluorescein concentration in the tears will be about 0.625%. As the tear film thins from evaporation, the fluorescent intensity is reduced inversely proportional to the square of concentration and is therefore proportional to the square of tear thickness. Thus, tear thickness is proportional to the square root of fluorescent intensity.²⁶ The method estimates relative changes in thickness given an initial concentration, and does not directly visualize flow as with, say, particle image velocimetry. However, the lubrication model that we use computes the film thickness directly, so it is appropriate to use this method to compare the theory with this type of experiment.

In Figure 1, we show some images from the video recording of one subject. The first frame shows the tear film at 5.5 s after a blink; this image is representative of the first 9 s after the blink. There is a region of slightly brighter tear film around the lid margins; this is the meniscus. At 5.5 s after the blink, within the meniscus and just above the outer canthus (right side), part of the meniscus is slightly brighter; it is labeled “lacrimal gland” to indicate the location of input of tear fluid to the tear film from the lacrimal gland. This part of the tear fluid brightens because fresh tear, comprising primarily water, is entering the exposed tear film due to reflex tearing. At 9.5 s (right panel, top row), the bright region has grown, showing where fresh tears have penetrated the tear film. The next two panels (10.17 s and 10.83 s) show the bright region growing around the outer canthus and along both the upper and lower menisci. The bright region is indicating where fresh tear fluid has entered

FIG. 1. Sequence of tear film images of the experiment after a blink. The first image is 5.5 s after a blink; the next three images are, respectively, 9.5 s, 10.17 s, and 10.83 s after that blink. The final image is after a blink that began less than a second after the sequence shown. Gravity is oriented downward in the panels. (Multimedia view) [URL: <http://dx.doi.org/10.1063/1.4871714.1>]

the tear film and lowered the concentration of fluorescein.^{25–27} In the final panel, the subject had blinked in less than 1 s after the last panel, and fresh tear fluid has mixed with the existing tear fluid, diluted it, and made the entire exposed tear film glow. This supply of fluid without blinking is often called reflex tearing (e.g., King-Smith *et al.*⁴⁶ and Maki *et al.*⁴⁷). A movie (cf. video corresponding to Figure 1 (Multimedia view)) shows the complete observation.

There is some variation in the experimental results. For example, in Harrison *et al.*,²¹ tear fluid traveled faster along the lower eyelid while our experiment shows the opposite. Maurice²³ did not report the fluid velocity in the menisci. The relative amount of flow along the upper or lower meniscus under general conditions remains an open question.

In Sec. III, we will compare our thin film model with the visualization of tear fluid entering the tear film. We now turn to formulating the tear film model.

B. Model formulation

The model presented here is an extension of the tear film relaxation model^{44,45} that incorporates evaporation, ocular surface wettability, and time-dependent boundary conditions. The aqueous tear film is assumed to be an incompressible Newtonian fluid with constant density ρ , viscosity μ , specific heat c_p , and thermal conductivity k . Dynamics of the lipid layer and mucus layer are not studied explicitly in this paper; their influence is incorporated only through the choice of boundary conditions and thinning rates as discussed below. The ocular surface is flat in the model since the characteristic tear film thickness is much less than the radius of curvature of the ocular surface.^{31,32}

Maki *et al.*⁴⁴ constructed the eye-shaped geometry, shown in Figure 2. They approximated the boundary from a digital photo, and developed equations to specify the domain.^{44,45} The coordinate directions (x' , y' , z') and corresponding velocity components (u' , v' , w') are shown in Figure 2. Gravity is in the negative y' direction. The tear film-air interface is at $z' = h(x', y', t')$ with t' being the time. $\mathbf{n}'_b$ and $\mathbf{t}'_b$ are the unit normal and tangential vectors of the boundary, respectively. Primed variables are dimensional.

We formulate the model in the following way: inside the tear film, the fluid dynamics are governed by the incompressible Navier-Stokes equations together with conservation of mass and energy. At the tear film free surface, we impose a mass balance, energy balance, tangential immobility, as well as normal stress balance. The nonequilibrium condition (cf. Eq. (A7) in Appendix A) gives a constitutive law that relates the interfacial temperature to the mass flux and pressure jump at the

FIG. 2. Coordinate system.

TABLE I. Dimensional parameters.

Parameter	Description	Value	Reference
μ	Viscosity	$1.3 \times 10^{-3} \text{ Pa s}$	Tiffany ⁵⁰
σ	Surface tension	0.045 N m^{-1}	Nagyová and Tiffany ⁵¹
k	Tear film thermal conductivity	$0.68 \text{ W m}^{-1} \text{ K}^{-1}$	Water
ρ	Density	10^3 kg m^{-3}	Water
$\mathcal{L}_m$	Latent heat of vaporization	$2.3 \times 10^6 \text{ J kg}^{-1}$	Water
T'_s	Saturation temperature	27°C	Estimated
T'_B	Body temperature	37°C	Estimated
g	Gravitational acceleration	9.81 m s^{-2}	Estimated
A^*	Hamaker constant	$3.5 \times 10^{-19} \text{ Pa m}^3$	Winter <i>et al.</i> ³⁶
α	Pressure coefficient for evaporation	$3.6 \times 10^{-2} \text{ K Pa}^{-1}$	Winter <i>et al.</i> ³⁶
K	Non-equilibrium coefficient	$1.5 \times 10^5 \text{ K m}^2 \text{ s kg}^{-1}$	Estimated
d'	Characteristic thickness	$5 \times 10^{-6} \text{ m}$	King-Smith <i>et al.</i> ¹⁰
L'	Half-width of palpebral fissure	$5 \times 10^{-3} \text{ m}$	Estimated
U_0	Characteristic speed	$5 \times 10^{-3} \text{ m/s}$	References 19, 39, and 49

interface, as studied by Ajaev and Homsy.⁴⁸ In addition, at the interface between the tear film and ocular surface, we impose no-slip and impenetrability conditions, as well as body temperature.

The scales used to non-dimensionalize the equations are

$$x' = L'x, y' = L'y, z' = d'z, h' = d'h, u' = U_0u, v' = U_0v, t' = \frac{L'}{U_0}t, \quad (1)$$

$$w' = \frac{d'U_0}{L'}w, p' = \frac{\mu U_0}{L'\epsilon^2}p, T = \frac{T' - T'_s}{T'_B - T'_s}, J' = \frac{k}{d'\mathcal{L}_m}(T'_B - T'_s)J. \quad (2)$$

Here $L' = 5 \times 10^{-3} \text{ m}$ is the half width of an open eye, $d' = 5 \times 10^{-6} \text{ m}$ is the characteristic tear film thickness, $U_0 = 5 \times 10^{-3} \text{ m/s}$ is the velocity scale, $T'_B = 37^\circ\text{C}$ is the body temperature and $T'_s = 27^\circ\text{C}$ is the estimated saturation temperature. The choice of $U_0 = 5 \times 10^{-3} \text{ m/s}$ is based on experimental observations.^{19,49} One other possible choice would be making the capillary number unity, however this choice would, for example, introduce very large domain sizes. In addition, one could also base the scalings on the thinning rate, which is another experimental observation. Although our choice of velocity scale looks simplistic, it is an effective choice for computation. The small parameter $\epsilon = d'/L' = 1 \times 10^{-3}$ indicates the separation of scales in the thin tear film. Values of viscosity μ , thermal conductivity k , the latent heat of vaporization $\mathcal{L}_m$ are given in Table I.

1. Thin film PDE

After non-dimensionalization, we apply lubrication theory to simplify the equations; details of the leading order equations are given in Appendix A. Solving for the velocity and temperature fields, integrating mass conservation and using the kinematic condition yields a single partial differential equation (PDE) for $h(x, y, t)$:

$$\partial_t h + EJ + \nabla \cdot \mathbf{Q} = 0, \quad (3)$$

with the evaporative mass flux J being

$$J = \frac{1 - \delta (S\nabla^2 h + Ah^{-3})}{\bar{K} + h}, \quad (4)$$

and the fluid flux across any cross-section of the film, $\mathbf{Q}$, being

$$\mathbf{Q} = \frac{h^3}{12} \nabla (S\nabla^2 h + Ah^{-3} - Gy). \quad (5)$$

TABLE II. Dimensionless parameters.

Parameter	Expression	Value
ϵ	$\frac{d'}{L'}$	1×10^{-3}
E	$\frac{k(T'_B - T'_s)}{d' \mathcal{L}_m \epsilon \rho U_0}$	118.3
S	$\frac{\sigma \epsilon^3}{\mu U_0}$	6.92×10^{-6}
$\bar{K}$	$\frac{kK}{d' \mathcal{L}_m}$	8.9×10^3
G	$\frac{\rho g (d')^2}{\mu U_0}$	0.05
δ	$\frac{\alpha \mu U_0}{\epsilon^2 L' (T'_B - T'_s)}$	4.66
A	$\frac{A^*}{L' d \mu U_0}$	2.14×10^{-6}

E characterizes the evaporative contribution to the surface motion, δ measures the pressure influence to evaporation, S is the ratio of surface tension to viscous forces, A is the Hamaker constant in nondimensional form related to the unretarded van der Waals force, and G is the ratio of gravity to the viscous force. When specifying the van der Waals force in the model, we only consider the perfectly wetting case due to the fact that the presence of hydrophilic glycocalyx on the underlying ocular surface prevents the tear film from completely dewetting the ocular surface. When the van der Waals term and evaporation balance, thinning stops, and we interpret this equilibrium value, $(\delta A)^{1/3}$, as tear film break up. We set this thickness to be the same as the height of the glycocalyx. Definitions of these nondimensional parameters can be found in Table II.

The Eqs. (3)–(5) are the extension of the 1D model³⁷ to the 2D eye-shaped domain. Li and Braun³⁷ developed a tear film dynamics model that incorporated heat diffusion inside the cornea and part of the anterior chamber. They determined parameters that best simulated experimental results for the temperature of the ocular surface; however, we do not consider the heat diffusion underneath the tear film in this paper in order to focus on the tear fluid motion on the exposed ocular surface. We may also reduce our model (3)–(5) to the tear film relaxation model of Maki *et al.*^{44,45} by further making $E = 0$ and $A = 0$, i.e., neglecting effects of evaporation and ocular surface wettability.

For numerical purposes, we write the PDE (3)–(5) as a system of nonlinear PDEs, in order to lower the order of spatial derivatives by introducing a new dependent variable p (the pressure):

$$\partial_t h + E \frac{1 + \delta p}{\bar{K} + h} + \nabla \cdot \left[-\frac{h^3}{12} \nabla (p + Gy) \right] = 0, \quad (6)$$

$$p + S \nabla^2 h + A h^{-3} = 0. \quad (7)$$

2. Boundary conditions

We parameterize the dimensionless boundary curve with its arclength s , and specify the tear film thickness and the normal component of the flux along the boundary. The tear film thickness is fixed all over the boundary:

$$h|_{\partial\Omega} = h_0. \quad (8)$$

Here we set $h_0 = 13$, following Maki *et al.*^{44,45} This choice of h_0 is in the range of experimental measurement (48–66 μm or 9.6–13.2 nondimensionally) from Golding *et al.*⁵² We note that the specified thickness results in a nonzero contact angle at the stationary contact line. This surface along the lid margin has different biology and properties as discussed elsewhere; for summaries, see Bron *et al.*⁷ and Braun.³⁰ No van der Waals term are present on the eyelid margin as opposed to the ocular surface.

FIG. 3. Time and space dependent functions for the flux boundary condition. Black curves indicate the lacrimal gland contribution; lighter (red) curves indicate the punctal contribution. (a) Space dependent functions are such that $x < 0$ is the nasal side and $x > 0$ is the temporal side. (b) Time dependent functions. (Multimedia view) [\[URL: http://dx.doi.org/10.1063/1.4871714.2\]](http://dx.doi.org/10.1063/1.4871714.2)

To approximate some of the effects of blinks, we specify the normal component of the flux on the boundary with both space and time dependence:

$$\mathbf{Q} \cdot \mathbf{n}_b = Q_{lg}(s, t) + Q_p(s, t). \quad (9)$$

We assume the lacrimal gland influx $Q_{lg}(s, t)$ and the punctal drainage $Q_p(s, t)$ can be written as the product of the space dependent function $\hat{Q}_m(s)$ and the time dependent function $f_m(t)$, where $m = lg$ or p , representing the lacrimal gland and the puncta, respectively. That is,

$$Q_{lg}(s, t) = f_{lg}(t)\hat{Q}_{lg}(s), \text{ and } Q_p(s, t) = f_p(t)\hat{Q}_p(s).$$

The functions in space, which define the locations of the lacrimal gland, the puncta and the maximum amount of the flux at those locations, are plotted in Figure 3(a). We assume the lacrimal gland supply starts at the beginning of a flux cycle with the punctal drainage starting one time unit later. Both the supply and drainage start to shut off at $t = 5$. The duration of a complete flux cycle in the model is $\Delta t_{bc} = 10$. Figure 3(b) shows the time dependent functions; detailed formulas are listed in Appendix B (cf. video corresponding to Figure 3 for the time-dependent flux boundary condition). We set the amount of fluid supplied by lacrimal gland and that of puncta drainage to be equal as a start, and then modify the lacrimal gland supply so as to compensate for the evaporation loss during a flux cycle. Results of both specifications are included in Sec. IV.

3. Initial condition

The initial condition (IC) is based on a smoothed version of

$$h(x, y, 0) = 1 + (h_0 - 1)e^{-\min(\text{dist}((x, y), \partial\Omega))/x_0}, \quad (10)$$

where $x_0 = 0.06$ and $\text{dist}(\mathbf{x}, \partial\Omega)$ is the distance between a point with position vector $\mathbf{x}$ and a point on the boundary $\partial\Omega$.^{44,45} The IC specifies a dimensional initial volume of about $1.805 \mu\text{l}$. This value is well within the experimental measurements by Mathers and Daley;⁵³ they found the volume of exposed tear fluid to be $2.23 \pm 2.5 \mu\text{l}$. The initial pressure, $p(x, y, 0)$, is calculated from $h(x, y, 0)$ according to Eq. (7).

FIG. 4. Smoothed initial condition. The dark band (maroon) around the boundary (maroon) shows thickness greater than or equal to 3 for the $h(x, y, 0)$ plot.

The pressure calculated from Eq. (10) is often not smooth enough. We numerically integrate the system for a short time to $t = 0.01$ in order to smooth out the pressure derived from Eq. (10). Although our numerical methods would smooth out the pressure for each simulation, for efficiency, we decide to only do the smoothing process once, and then we use the smoothed data for both h and p as the IC for all of the computations. The physical effects, such as evaporation, time-dependent flux BC, and gravity, hardly begin to affect the system after such a short period of time ($t = 0.01$). Therefore, it is legitimate to use a universal smoothed data as the IC for all the various cases. The smoothed IC speeds up the computations dramatically. Figure 4 shows the actual IC used from this approach. Note that the dark band (maroon) indicates $h \geq 3$ in the tear film thickness contour plot, which corresponds to the meniscus at the eyelid margin.

III. NUMERICAL METHODS

The domain is discretized using a composite overlapping grid, which is a collection of logically rectangular curvilinear component grids that overlap where they meet, and cover the whole domain. We use four boundary-fitting components grids to approximate the boundary, and a background Cartesian grid to cover the rest of the domain. Solutions on the different grids are coupled by interpolation. The domain and the associated grids^{44,45} were created with the grid generation capabilities of the OVERTURE computational framework.^{54,55} (<http://www.overtureframework.org>. Primary developer and contact: W. D. Henshaw, henshw@rpi.edu.)

We solve the system (6) and (7) by the method of lines. The spatial derivatives are approximated using the second-order accurate finite difference methods for curvilinear and Cartesian grids from OVERTURE, which leads to a system of index-1 differential algebraic equations (DAE). The DAE system is advanced using a variable step size backward differentiation formula (BDF) with fixed leading coefficient that was added to OVERTURE.^{44,45,56} Finally, we solve the resulting nonlinear system using Newton's method.

The method achieves good accuracy on test problems with exact solutions using this domain and gridding approach.^{45,57} We provide additional numerical evidence in Sec. IV verifying that the method also performs well on the current problem.

FIG. 5. Contours of tear film thickness with $E = 0$ and $G = 0$. The dark band (from the right end of the color bar; maroon) around the boundary shows thickness greater than or equal to 3.

IV. RESULTS

A. Tear film dynamics

In this section, we present and compare computed results for various parameter settings: (i) $E = 0$ and $G = 0$, (ii) $E \neq 0$ and $G = 0$, and (iii) $E \neq 0$ and $G \neq 0$. Recall that E helps characterize the evaporation rate and G characterizes the effect of gravity. We switch these effects on and off in order to study the individual effects of time-dependent flux boundary condition, evaporation, and their combined effects on tear film dynamics. We shall later modify the lacrimal gland supply function $Q_{lg}(s, t)$ so that the net amount of fluid provided by the flux boundary condition compensates for the evaporative loss during a flux cycle.

We first show results for $E = G = 0$. This case was studied by Maki *et al.*⁴⁵ for a *time-independent* flux BC and with no van der Waals forces. Here the flux BCs are time dependent and the substrate under the film is wettable. Computed thickness contours are shown in Figure 5. As observed previously in many tear film papers, the relatively thick film at the boundary induces positive curvature, which lowers the pressure and sucks fluid into the meniscus; near the meniscus a local minimum forms that is called the “black line” in the eye literature. The name comes from the dark band that appears near the bright meniscus when using fluorescence imaging. (In our figures, it is colored dark from the low end of the color bar, or blue.) The mechanism was illustrated by McDonald and Brubaker²⁸ using milk and a paper clip, and has been recovered by all of the papers that have the ends modeling the lids. In the two-dimensional domain, the additional features that appear are the canthi, which induce a second direction of curvature, creating an even lower pressure that attracts fluid toward themselves. The especially thin black line creates a boundary that separates the interior of the tear film from the meniscus; this occurs because h is small and the flux $\mathbf{Q}$ is proportional to h^3 . The interior is sometimes referred to a “perched” tear film.²⁹ The redistribution of fluid due to surface tension also causes a small ridge to form on the interior side of the black line. The thickness of the film in the interior is roughly constant. In the meniscus, the tear film thickens near the region of influx from lacrimal gland (right side, above temporal canthus), and narrows in the region of efflux by the puncta (left side, nasal canthus). The new fluid is unable to penetrate into the interior under these conditions. More extreme influx conditions,¹⁴ such as the reflex tearing that arises when cutting an onion or from crying, may overcome the barrier created by the black line and cause fluid to reach the interior without blinking.⁴⁷

Contour plots for the pressure corresponding to $E = G = 0$ are shown in Figure 6 at two different times. The highest pressures occur in the interior, particularly under the ridge near the black line, and

FIG. 6. Pressure distribution for $E = G = 0$ with time-dependent flux boundary conditions and a wettable substrate.

near the input from the lacrimal gland. The meniscus pressure is always lower than the interior in this computation, and this lower pressure pulls fluid into the meniscus. But within the meniscus, there is a pressure gradient that drives flow away from the input region of the lacrimal gland and ultimately toward the lowest pressure region by the puncta and nasal canthus. The plots in Figure 6 illustrate the mechanisms for the flow that forms the black line, and for the flow around the meniscus that we refer to as hydraulic connectivity. We show the pressure at $t = 4$ to illustrate how the time-dependent flux BC influences the pressure distribution when both lacrimal gland influx and puncta efflux are fully functioning; the latter contributes to the very low pressure seen in the neighborhood of the nasal canthus. Then we plot p at the end of the first flux cycle ($t = 10$) when the flux BC is zero all around the boundary. The pressures near the temporal and nasal canthi are less extreme at this time because of the surface tension driven relaxation that occurs in the meniscus when the boundary fluxes are zero. We can also plot the direction and magnitude of the volumetric flux inside the film; representation results are shown in the next case. We discuss the flux in general in the cases with evaporation, which we turn to next.

Figure 7 shows the evolution of the tear film thickness with evaporation. The first column shows results for the case $E \neq 0$ and $G = 0$, and the second column for $E \neq 0$ and $G \neq 0$. We first focus on $G = 0$. The presence of the meniscus again creates a pressure difference between the interior and the meniscus, which drives the tear fluid from the interior towards the boundary, resulting in the black line separating the two. The lower pressure in the meniscus can be clearly seen in the pressure distribution plots; the plots for this case are quite similar to those in Figure 6, but the specific plots for these parameters are shown in Figure 15 in Appendix C. We find the global minimum in the thickness occurring near the nasal canthus, in a manner similar to the last case. With evaporation, however, the tear film thins in the interior throughout the computation which is visualized by the continual darkening of the interior. As seen in Figure 7, the minimum thickness is smaller than that in Figure 5. Our model is consistent with the expected contribution of evaporation to tear film thinning of the interior region between blinks and with the experimental studies of Kimball *et al.*,¹⁷ Nichols *et al.*,²⁶ and others. The capillary-driven tangential flow only explains the thinning of tear film near the meniscus, i.e., the formation of black line; it has little to do with the thinning of tear film in the interior region.^{29,58}

The flux boundary condition (9) imposed on our model enforces time-dependent influx from the lacrimal gland and efflux through the two punctal holes on the nasal side. The influx pumps fluid into the meniscus above the temporal canthus, while the efflux drains fluid out near the nasal canthus. We plot the direction field of the flux $\mathbf{Q}$ over contours of its magnitude in Figure 8. The lengths of the arrows are normalized to unity, thus showing the directions only, and we use the shading to indicate the magnitude of the flux vector: the darker the background, the smaller the flux. In particular white indicates a flux greater than 10^{-2} ; dark gray is less than 10^{-3} . Finally, there are far fewer arrows than the computational grids for clarity. The arrows in Figure 8 illustrate the effect of the boundary flux at $t = 4$. The computed flux is consistent with the maroon band becoming wider near the lacrimal gland with increasing time and narrower near the nasal canthus in Figure 7. Additionally, the flux

FIG. 7. Contour plots of the tear film thickness without gravity (left column) and with gravity (right column). The dark band (maroon) around the boundary shows thickness greater than or equal to 3.

FIG. 8. The flux direction field plotted over the contours of the norm of the flux at $t = 4$ with $E \neq 0$ and $G = 0$. (Far fewer arrows than the computational grid points are shown for clarity. All the arrows in this plot start at different locations.)

$t=10$

FIG. 9. The flux direction field at the end of first flux cycle. $E \neq 0$ and $G = 0$. (Far fewer arrows than the computational grids are shown for clarity. All the arrows in this plot start at different locations.)

boundary condition changes the curvature of the meniscus causing the pressure gradient that drives the fluid flow in the meniscus (Figure 15 in Appendix C).

The second plot of Figure 6 is representing the pressure at the end of the first flux cycle (see Figure 15 of Appendix C for the exact plot). For most of the interior region, the pressure gradient is approximately zero, and there is little motion there; correspondingly, Figure 9 for the flux at $t = 10$ is basically dark in the interior, i.e., with $\|\mathbf{Q}\| < 10^{-2}$. Relatively fast ($\|\mathbf{Q}\| \geq 10^{-2}$) fluid movement occurs near the boundary in the meniscus, splitting at the lacrimal gland input, and traveling around the menisci toward the puncta and nasal canthus. Thus, Figure 9 illustrates the hydraulic connectivity observed by Maurice,²³ Harrison *et al.*,²¹ and our experiment described in Sec. II A, and as computed in some cases by Maki *et al.*⁴⁵

When the flux through the boundary is off (no flux anywhere), hydraulic connectivity makes the fluid move towards the nasal canthus, makes the meniscus wider, and lowers the pressure difference between the canthi. However, when the punctal drainage is on for the next flux cycle, the meniscus near the nasal canthus gets narrower again. In the model, we assume a flux cycle to be 10 time units. The time-dependent flux boundary condition (9) is specified as non-zero for the first 5 time units of each flux cycle and zero for $5 < t < 10$. The on-off transition time is very short (cf. Table IV). Once the boundary flux turns off, tear fluid starts to collect at the nasal canthus due to the low pressure caused by the relatively large curvature of the film surface there. Thus, we see the maroon band near the nasal canthus is wider at the end of each flux cycle in Figure 7 ($t = 10$ and $t = 20$). The pressure at the nasal canthus starts to increase and this prevents a steep pressure gradient that would eventually cause the simulation with time independent, nonzero boundary fluxes to stop.⁴⁵ In general, as Maki *et al.*⁴⁵ speculated in their paper, the time-dependent boundary condition helps avoid large interior pressure gradients by having the punctal drainage active for a short time.

We now turn to the case with gravity active, that is, $E \neq 0$ and $G \neq 0$. Gravity redistributes the tear film from the top to the bottom of the domain and may be another important effect on tear film dynamics, e.g., if the tear film is thick enough.^{34,47} The right column of Figure 7 illustrates the tear film thickness with $G = 0.05$. Similar to prior results, the black line develops rapidly and is persistent. However, due to the gravitational effect, the dark band (maroon) representing the lower meniscus widens significantly in this computation; this means more tear fluid is collected at the bottom. Tracking the same time sequence of plots with or without gravity, we see that a bulge of

FIG. 10. The flux direction field at the end of first flux cycle with $E \neq 0$ and $G = 0.05$. (Far fewer arrows than the computational grids are shown for clarity. All the arrows in this plot start at different locations.)

new tear fluid supplied by the lacrimal gland is being driven downwards. It penetrates through the black line and then exits the interior to the lower meniscus, but only near the temporal canthus. The direction of the fluid motion at the end of first cycle ($t = 10$) can be seen in Figure 10. All the arrows in the inner dark region are pointing toward the lower eyelid for that slow flow. The light area intruding inside the dark region shows the relatively fast movement of the bulge of fluid.

The pressure plots at $t = 4$ and 10 are shown as Figure 16 in Appendix C. Other than a slight pressure gradient from top to bottom in these figures, the plots are similar to the other pressure plots shown.

The existence of black line once formed is persistent for all the cases in Figures 5 and 7. Even though, with the presence of gravity, the influx bulge breaks through the black line near the lacrimal gland and temporal canthus, it does not move into the interior and does not help to ameliorate the thinning process of the interior region.

B. Volume conservation

Now that we have visualized some results for the model, we now turn to some integrated (global) quantities to illustrate that the method is working consistently. The accuracy of the numerical method has been tested by Maki *et al.*⁴⁵ by formulating a test problem with known exact solution on a domain comprising a rectangle with a circular hole cut out. Refinement of the grids yielded convincing numerical evidence of a second order convergence rate in the space variables, consistent with theoretical expectation. In addition, they conducted tests on the eye-shaped domain and achieved good accuracy as well. In both cases, mass conservation was a reasonably good indicator of the error. To give an indication that the numerics also work well for this problem, we verify conservation of mass, or in our case volume since the density is constant. The volume of the tear film at all time equals the initial volume minus the amount of evaporation loss and the amount of the net flux from the boundary condition:

$$V(t) = V(0) - e(t) - F(t), \quad (11)$$

where $e(t)$ and $F(t)$ are the volume of tear film that evaporates away and the net flux through the boundary, respectively. Expressions for these can be found by integrating (3) with respect to space

TABLE III. No flux boundary condition and no evaporation.

Time	$Err(t)$	Percentage of $V(0)$ (%)
1	0.0181	0.1253
5	0.0353	0.2442
10	0.0456	0.3159
20	0.0582	0.4030

and time:

$$V(t) - V(0) + \underbrace{E \int_0^t \int_{\Omega} J dA d\tau}_{e(t)} + \underbrace{\int_0^t \int_{\partial\Omega} \mathbf{Q} \cdot \mathbf{n}_b ds d\tau}_{F(t)} = 0. \quad (12)$$

Here Ω is the (eye-shaped) domain of the computation, and the integration in space is over this domain. To verify if the numerical results conserve the fluid volume as stated in Eq. (11), we plot $V(t)$ and $V(0) - e(t) - F(t)$ for various cases.

First, we neglect evaporation and use a no-flux boundary condition, i.e., $e(t) = F(t) = 0$. In this case, the difference between $V(t)$ and $V(0)$ increase with time, but remains small. The error in volume conservation is defined as $Err(t) = |V(0) - V(t)|$ and is listed in Table III. The error in volume conservation remains well below 1% for all the times considered.

Second, we check volume conservation for those cases for which we imposed the time-dependent flux boundary condition (9) with or without evaporation. The results are plotted in Figure 11. The error in the volume conservation at the end of the first flux cycle ($t = 10$) is 0.0585 for $E = 0$, and 0.0574 for $E \neq 0$. These represent an error in volume conservation of less than 1%. Overall, the method appears to work well for the model with evaporation and time-dependent flux boundary condition.

FIG. 11. Results for the time-dependent flux boundary condition. (a) Without evaporation and (b) with evaporation.

FIG. 12. Evaporation rate $\frac{d}{dt}e(t)$ of various cases.

C. Recovering the thinning rate

We obtain the non-dimensional evaporation rate theoretically by differentiating $e(t)$, the volume of tear film that evaporates. Equation (12) yields

$$\frac{d}{dt}e(t) = E \int_{\Omega} J(x, y, t) dA.$$

Figure 12 shows the evaporation rates vs. time for various cases. The evaporation rate is nearly constant for the first 20 time units. We denote the constant as $Q_e \approx 0.078$ (notice that the scale of the coordinate of Figure 12 ranges from only 0.077 to 0.0784). Dimensionally, our model predicts the evaporation rate to be $0.58 \mu\text{l}/\text{min}$.

The model recovers the tear film thinning rate that we used to set the evaporation parameters, providing a consistency check on the computations. Nichols *et al.*⁵⁹ have found that the mean rate of thinning of the pre-corneal tear film is $3.79 \pm 4.20 \mu\text{m}/\text{min}$. In our model, the area of the eye-shaped domain is about 5.934, i.e., 148.34 mm^2 dimensionally. If the calculated evaporation rate is divided by area of the domain, we obtain an average thinning rate of the tear film of $3.91 \mu\text{m}/\text{min}$. That matches our assumed average thinning rate, $4 \mu\text{m}/\text{min}$.

D. Compensation of evaporation loss

For the previous results, the influx and efflux from the time-dependent flux boundary condition (9) are balanced over a flux cycle. However, this may not be what happens in the eye. New tear fluid is supplied from the lacrimal gland to offset tear film loss, and then tear film is recovered to its initial state from evaporation and any redistribution due to surface tension and gravity after each blink. Hence, we attempt to modify the flux boundary condition to compensate for the evaporation loss. To the original balanced flux boundary condition, we add $\tilde{Q}_{lg}(s)$ to provide additional time-independent tear supply through the lacrimal gland at a rate of Q_e , the same as the model predicted evaporation rate, as formulated in Eq. (13):

$$\mathbf{Q} \cdot \mathbf{n}_b = Q_{lg}(s, t) + Q_p(s, t) + \tilde{Q}_{lg}(s). \quad (13)$$

Equation (B5) in Appendix B is used for $\tilde{Q}_{lg}(s)$.

The simulation with flux boundary condition specified as Eq. (13) is useful for understanding tear film dynamics. We make an attempt to understand the natural tear film dynamics by adjusting the conditions in the computations. In Figure 13 (cf. Video 3 (Multimedia view)), similar patterns to the cases with a balanced flux boundary condition are observed in the tear film thickness. A black line emerges early in time near the meniscus and persists throughout the flux cycles. The tear film in the interior keeps thinning as a result of evaporation. More fluid is collected in the meniscus, especially near the lacrimal gland, because of the influx. However, the larger amount of fluid there

FIG. 13. Contours of tear film thickness with flux compensating the evaporation loss and $G = 0$. (Multimedia view) [\[URL: http://dx.doi.org/10.1063/1.4871714.3\]](http://dx.doi.org/10.1063/1.4871714.3)

still cannot penetrate past the black line into the interior.²⁹ We cannot fully restore the tear film to close to its initial uniformly thick distribution in the interior by merely providing more fluid through the lacrimal gland. The blink, with its attendant lid motion, is indispensable to evenly spread the new tear fluid collected in the meniscus into the inner region separated by the black line.

E. Comparison with experiment

1. Evaporation

It is important to put our computed evaporation rate in context with measured values over the last few decades for comparison. A number of researchers, as reviewed by Tomlinson *et al.*,¹⁶ have measured the evaporation rate with both direct fluid-capture techniques (that measures the fluid loss from the ocular surface) and indirect interferometric technique (that measures the tear film thinning). Results of the direct mass measurements were reported in different units, such as $\text{g}/\text{cm}^2 \text{ s}$ and $\mu\text{l}/\text{min}$. In order to compare all these results, Tomlinson *et al.*¹⁶ converted the different units by assuming the density of tear fluid to be that of water ($10^3 \text{ kg}/\text{m}^3$) and the area of the ocular surface to be 167 mm^2 . The evaporation rates reported by Nichols *et al.*⁵⁹ and Tomlinson *et al.*¹⁶ have a discrepancy when they are converted to the same units. Kimball *et al.*¹⁷ speculate that the origin of the difference is that most of the measurements reviewed by Tomlinson *et al.*¹⁶ were conducted using preocular chambers. The air flow over the tear film surface is restricted by the chambers, which retards evaporation. The evaporative parameter of the model is estimated based on the assumption of room temperature and free air conditions. The evaporation rate ($0.58 \mu\text{l}/\text{min}$) predicted by our model agrees well with that measured by Nichols *et al.*⁵⁹ using interferometric technique in a free air condition, and close to the results of Liu *et al.*⁶⁰ ($0.40 \pm 0.14 \mu\text{l}/\text{min}$), which was measured using the ventilated-chamber method, even though it is about 4 times larger than the average evaporation rate of those reported from direct measures of fluid mass loss.¹⁶

2. Hydraulic connectivity

In Figure 1, we showed experimental results from an interblink where reflex tearing caused new tear fluid to enter the exposed tear film and flow along the menisci. From that same sequence we show a different time, and compare it with our computed results. Figure 14 (cf. videos corresponding

FIG. 14. Comparison with experiment. (a) An image from the experiment at 10.6 s post blink. (b) The flux field at $t = 6$ with flux boundary condition compensating the evaporation loss and $G = 0$. (Far fewer arrows than the computational grids are shown for clarity.) (Multimedia view) [URL: <http://dx.doi.org/10.1063/1.4871714.4>] [URL: <http://dx.doi.org/10.1063/1.4871714.5>]

to Figure 14) shows this comparison at one time, and reveals that our model captures some of the details of the development of hydraulic connectivity correctly. In Figure 14(b), the arrows indicate the direction of the fluid flow and shading represents the magnitude of the flux as before. Along the upper eyelid, all the arrows are pointing towards the nasal side, and the fluid moves relatively fast. Along the lower eyelid, near the lacrimal gland, the arrows in the light background push the dark region towards the nasal side, which means the new tear provided by the boundary condition is pushing its way towards the nasal canthus. These trends agree with the frame shown in Figure 14(a). More detailed comparison can be problematic, however; the volume of the exposed tear film and the volume flux of fluid into it are unknown and difficult to measure.

Our model provides a global prediction about the fluid motion in the eye-shaped domain: all the fluid in the meniscus is traveling towards the nasal canthus. The motion of the dimmed part of the fluid in the experimental image is yet to be measured by any experiments to our knowledge.

V. CONCLUSION

In this paper, we present a tear film dynamics model on an eye-shaped domain that includes capillarity, gravity, evaporation, ocular surface wettability, and time-dependent flux boundary conditions. We significantly extended existing tear film models, providing new insights about tear film

dynamics. The inclusion of ocular surface wettability in the model prevents the tear film from reaching zero thickness in the computations, but includes break up as reaching a small nonzero equilibrium thickness, which enables us to conduct numerical simulations for longer times. Our model also captures new details about tear flows in the meniscus, which is beyond the reach of one-dimensional models. We also described experiments using fluorescein that visualized tear film thickness changes, and indirectly, the supply of fresh tear fluid and where the more diluted fluid moves.

The time-dependent flux boundary condition that captures some effects of blinks is formulated based on eye researchers' descriptions¹³ and other studies about the tear supply and drainage mechanisms.^{21,23} We began by balancing the influx and efflux through the lacrimal gland and puncta in the boundary condition. The model captures some key physics of the tear film dynamics observed by ocular scientists, such as the emergence of black line, evaporation, and hydraulic connectivity. Then, we modified the lacrimal supply to compensate the evaporation loss in an attempt to restore the tear film structure to its initial thick state in the interior. In the model, tear fluid supplied from the lacrimal gland is unable to relieve the evaporation thinning in the interior region with the presence of black line. A blink is well necessary to evenly spread the new tear fluid.

Evaporation is well captured by our model for comparison with well-controlled laboratory experiments. The result yielded by the model is comparable to the measurements conducted with the ventilated-chamber method or in the free air,^{17,60} though that rate is larger than the average evaporation rate measured using preocular chambers.¹⁶

Our results also reveal that the hydraulic connectivity is largely controlled by the pressure gradient created by the flux through the lacrimal gland and puncta. The hydraulic connectivity is also aided by the shape of the lid margin, which causes low pressure regions to form and draw fluid toward the canthi. The model correctly captures the overall trend that tear fluid flows towards the nasal canthus along the upper and lower eyelids from the temporal canthus as observed *in vivo* by various researchers. However, due to the lack of experimental information about the flux through the lacrimal gland and flow in the meniscus, we are unable to make quantitative comparisons about the hydraulic connectivity with experimental results.

In the future, incorporating blinking and osmolarity dynamics into the existing tear film model is of interest. Osmolarity is essentially the salt ion concentration in the tear film. It is hypothesized to cause the dry eye symptoms such as irritation and redness.³ However, the measurement of the osmolarity is nearly always limited to the temporal canthus (e.g., Lemp *et al.*⁶¹). Measurements of the osmolarity in the meniscus provide little direct information for the interior regions, though the variation measured near the outer canthus is thought to be helpful in diagnosing dry eye. Therefore, theoretical studies of the osmolarity in an eye-shaped domain may provide insight about the osmolarity distribution over the ocular surface for the eye community; such computations are underway.

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APPENDIX A: LEADING ORDER APPROXIMATIONS

We use standard thin-layer theory to model tear film dynamics on the 2D eye-shaped domain. We first non-dimensionalize all the governing equations and boundary conditions. We then estimate the size of the non-dimensional parameters:

$$\epsilon = \frac{d'}{L'} = 1 \times 10^{-3}, \quad \text{Re} = \frac{U_0 L'}{\mu/\rho} \approx 19.23, \quad \text{Pr} = \frac{c_p \mu}{k} \approx 8.01,$$

where Re is the Reynolds number and Pr is the Prandtl number. Terms involving the following parameters are regarded as small:

$$\epsilon^2 = 1 \times 10^{-6}, \quad \epsilon^2 Re \approx 1.92 \times 10^{-5}, \quad \epsilon^2 Re Pr \approx 1.54 \times 10^{-4}.$$

Applying lubrication theory, we neglect the small terms, and have the following leading order approximations. Inside the tear film ($0 < z < h$), the equations governing the conservation of mass and energy are

$$\partial_x u + \partial_y v + \partial_z w = 0, \quad 0 = \partial_z^2 T. \quad (A1)$$

Here (u, v, w) are the velocity components in the coordinate directions (x, y, z) . Equations for the conservation of momentum in the x, y, z directions are, respectively,

$$0 = -\partial_x p + \partial_z^2 u, \quad 0 = -\partial_y p + \partial_z^2 v - G, \quad 0 = -\partial_z p. \quad (A2)$$

At the intersection of the tear film and ocular surface ($z = 0$), we have the no-slip and no penetration conditions, and we prescribe body temperature:

$$u = v = w = 0, \quad T = 1. \quad (A3)$$

At the free surface of the tear film ($z = h$), we have the equations to balance mass and energy:

$$EJ = w - u\partial_x h - v\partial_y h - \partial_t h, \quad J + \partial_z T = 0. \quad (A4)$$

We also assume tangential immobility:³⁴

$$v = u = 0. \quad (A5)$$

To balance normal stress with the conjoining pressure under consideration, we have

$$p - p_v = -S\nabla^2 h - \frac{A}{h^3}. \quad (A6)$$

Finally, we relate the interfacial temperature to the mass flux and pressure jump by the nonequilibrium condition:

$$\bar{K}J = \delta(p - p_v) + T. \quad (A7)$$

APPENDIX B: TIME-DEPENDENT FLUX BOUNDARY CONDITION

We list the space and time dependence functions defined in the flux BC (9) below. Parameters in the formulations are tabulated in Table IV:

$$f_{lg}(t) = \begin{cases} \frac{1}{2} \left[\cos \left(\frac{\pi}{2} \frac{t - t_{lg,on}}{\Delta t_{lg}} - \frac{\pi}{2} \right) + 1 \right], & \text{if } |t - t_{lg,on}| \leq \Delta t_{lg}, \\ 1, & \text{if } t_{lg,on} + \Delta t_{lg} \leq t \leq t_{lg,off} - \Delta t_{lg}, \\ \frac{1}{2} \left[\cos \left(\frac{\pi}{2} \frac{t - t_{lg,off}}{\Delta t_{lg}} + \frac{\pi}{2} \right) + 1 \right], & \text{if } |t - t_{lg,off}| \leq \Delta t_{lg}, \\ 0, & \text{otherwise,} \end{cases} \quad (B1)$$

$$f_p(t) = \begin{cases} \frac{1}{2} \left[\cos \left(\frac{\pi}{2} \frac{t - t_{p,on}}{\Delta t_p} - \frac{\pi}{2} \right) + 1 \right], & \text{if } |t - t_{p,on}| \leq \Delta t_p, \\ 1, & \text{if } t_{p,on} + \Delta t_p \leq t \leq t_{p,off} - \Delta t_p, \\ \frac{1}{2} \left[\cos \left(\frac{\pi}{2} \frac{t - t_{p,off}}{\Delta t_p} + \frac{\pi}{2} \right) + 1 \right], & \text{if } |t - t_{p,off}| \leq \Delta t_p, \\ 0, & \text{otherwise,} \end{cases} \quad (B2)$$

TABLE IV. Parameters appearing in the flux boundary condition.

Parameter	Description	Value
$t_{lg,on}$	On time for lacrimal gland supply	0.2
$t_{lg,off}$	Off time for lacrimal gland supply	5.2
Δt_{lg}	Transition time of lacrimal gland supply	0.2
$t_{p,on}$	On time for punctal drainage	1.05
$t_{p,off}$	Off time for punctal drainage	5.05
Δt_p	Transition time of punctal drainage	0.05
Q_{mI}	Estimated steady supply from lacrimal gland	0.08
$\hat{Q}_{0lg}$	Height of lacrimal gland peak	0.4
$\hat{Q}_{0p}$	Height of punctal drainage peak	4
Δt_{bc}	Flux cycle time	10
$s_{lg,on}$	On-ramp location for lacrimal gland peak	4.2
$s_{lg,off}$	Off-ramp location for lacrimal gland peak	4.6
Δs_{lg}	On-ramp and off-ramp width of lacrimal peak	0.2
p_{out}	Fraction of drainage from upper punctum	0.5
$s_{p,lo}$	Lower punctal drainage peak location	11.16
$s_{p,up}$	Upper punctal drainage peak location	11.76
Δs_p	Punctal drainage peak width	0.05

$$\hat{Q}_{lg}(s) = \begin{cases} 0, & \text{if } s < s_{lg,on} - \Delta s_{lg}, \\ -\frac{1}{2} \hat{Q}_{0lg} \left[\cos \left(\frac{\pi}{2} \frac{s - s_{lg,on}}{\Delta s_{lg}} - \frac{\pi}{2} \right) + 1 \right], & \text{if } |s - s_{lg,on}| \leq \Delta s_{lg}, \\ -\hat{Q}_{0lg}, & \text{if } s_{lg,on} + \Delta s_{lg} \leq s \leq s_{lg,off} - \Delta s_{lg}, \\ -\frac{1}{2} \hat{Q}_{0lg} \left[\cos \left(\frac{\pi}{2} \frac{s - s_{lg,off}}{\Delta s_{lg}} + \frac{\pi}{2} \right) + 1 \right], & \text{if } |s - s_{lg,off}| \leq \Delta s_{lg}, \\ 0, & \text{otherwise,} \end{cases} \quad (B3)$$

$$\hat{Q}_p(s) = \begin{cases} 0, & \text{if } s < s_{p,lo} - \Delta s_p, \\ -\frac{\hat{Q}_{0p}}{2} (1 - p_{out}) \left[\cos \left(\pi \frac{s - s_{p,lo}}{\Delta s_p} - \pi \right) - 1 \right], & \text{if } |s - s_{p,lo}| \leq \Delta s_p, \\ 0, & \text{if } s_{p,lo} + \Delta s_p \leq s \leq s_{p,up} - \Delta s_p, \\ -\frac{\hat{Q}_{0p}}{2} (p_{out}) \left[\cos \left(\pi \frac{s - s_{p,up}}{\Delta s_p} - \pi \right) - 1 \right], & \text{if } |s - s_{p,up}| \leq \Delta s_p, \\ 0, & \text{otherwise.} \end{cases} \quad (B4)$$

Definition of additional tear supply that compensates the evaporation loss as appeared in (13) is

$$\tilde{Q}_{lg}(s) = \begin{cases} 0, & \text{if } s < s_{lg,on} - \Delta s_{lg}, \\ -\frac{1}{2} \tilde{Q}_{0lg} \left[\cos \left(\frac{\pi}{2} \frac{s - s_{lg,on}}{\Delta s_{lg}} - \frac{\pi}{2} \right) + 1 \right], & \text{if } |s - s_{lg,on}| \leq \Delta s_{lg}, \\ -\tilde{Q}_{0lg}, & \text{if } s_{lg,on} + \Delta s_{lg} \leq s \leq s_{lg,off} - \Delta s_{lg}, \\ -\frac{1}{2} \tilde{Q}_{0lg} \left[\cos \left(\frac{\pi}{2} \frac{s - s_{lg,off}}{\Delta s_{lg}} + \frac{\pi}{2} \right) + 1 \right], & \text{if } |s - s_{lg,off}| \leq \Delta s_{lg}, \\ 0, & \text{otherwise,} \end{cases} \quad (B5)$$

with

$$\tilde{Q}_{0lg} = \frac{Q_e}{s_{lg,off} - s_{lg,on}}.$$

FIG. 15. Pressure distribution for $E \neq 0$ and $G = 0$. FIG. 16. Pressure distribution for $E \neq 0$ and $G \neq 0$.

APPENDIX C: SUPPLEMENTARY PRESSURE PLOTS

Pressure plots for the cases $E \neq 0$, $G = 0$, and $E \neq 0$, $G \neq 0$, both with time-dependent flux boundary condition and wettable ocular surface, are collected in this section.

Similar as the pressure plots in Figure 6, the pressure gradient that causes the hydraulic connectivity can be seen in Figure 15 in the meniscus; the pressure is basically constant in the interior and is greater than that of the meniscus, thus driving the formation of the black line. Evaporation has very limited effects on the overall pressure distribution. However, the effects of gravity that create an additional pressure gradient from top to bottom is obviously seen in Figure 16.

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