The Use of Generalized Laguerre Polynomials in Spectral Methods for Solving Fractional Delay Differential Equations

M. M. Khader¹

Department of Mathematics and Statistics,
College of Science,
Al-Imam Mohammed Ibn Saud,
Islamic University (IMSIU),
Riyadh 11566, Saudi Arabia
e-mail: mohamedmbd@yahoo.com

In this paper, an efficient numerical method for solving the fractional delay differential equations (FDDEs) is considered. The fractional derivative is described in the Caputo sense. The proposed method is based on the derived approximate formula of the Laguerre polynomials. The properties of Laguerre polynomials are utilized to reduce FDDEs to a linear or nonlinear system of algebraic equations. Special attention is given to study the error and the convergence analysis of the proposed method. Several numerical examples are provided to confirm that the proposed method is in excellent agreement with the exact solution. [DOI: 10.1115/1.4024852]

Keywords: fractional delay differential equations, Caputo fractional derivatives, Laguerre polynomials, spectral collocation method, error analysis

1 Introduction

It is well known that fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearance in various applications such as in fluid mechanics, viscoelasticity, biology, physics, and engineering applications; for more details see, for example, Refs. [1,2]. Consequently, considerable attention has been given to the efficient numerical solutions of FDEs of physical interest because it is difficult to find the exact solution for it. Different numerical methods have been proposed in the literature for solving FDEs [3–13].

The representation of a function in terms of a series expansion using orthogonal polynomials is a fundamental concept in approximation theory and forms the basis of spectral methods of solution of differential equations [14–16]. Spectral collocation methods are efficient and highly accurate techniques for the numerical solution of FDEs. The basic idea of the spectral collocation method is to assume that the unknown solution $u(x)$ can be approximated by a linear combination of some basis functions, called the trial functions, such as orthogonal polynomials. The orthogonal polynomials can be chosen according to their special properties, which make them particularly suitable for the problem under consideration.

The main goal in this article is concerned with the application of the Laguerre pseudospectral method to obtain the numerical solution of FDDEs of the form

$$D^\alpha u(x) = f(x; u(x), u(g(x))), \quad a \leq x \leq b, \quad 1 < \alpha \leq 2 \quad (1)$$

with the following boundary conditions

$$u(a) = \gamma_0, \quad u(b) = \gamma_1, \quad u(x) = \psi(x), \quad x \in [a^*, a] \quad (2)$$

where α refers to the Caputo fractional order of spatial derivatives. The function g is called the delay function and it is assumed to be continuous on the interval $[a, b]$ and satisfies the inequality

$a^* \leq g(x) \leq x, x \in [a, b]$ and $\psi \in C[a^*, a]$. In our examples in Sec. 4, the interval $[0, 1]$ is given. Our idea in this work is to apply the Laguerre collocation method to discretize Eq. (1) to obtain a linear or nonlinear system of algebraic equations, thus greatly simplifying the problem, and solve the resulting system.

Note that $\alpha = 2$ and Eq. (1) is the classical second-order delay differential equation

$$u''(x) = f(x, u(x), u(g(x))), \quad a \leq x \leq b \quad (3)$$

In Ref. [17], the author studied the first order delay differential equation, using spline functions and studied the stability and the error analysis and in Ref. [9], the authors studied the system of first order delay differential equations, using spline functions, and studied the stability and the error analysis. In addition, in Ref. [18], the authors studied the numerical solution of the FDDE using the Legendre collocation method.

The structure of this paper is arranged in the following way: in Sec. 2, we introduce some basic definitions about the Caputo fractional derivatives and the properties of the generalized Laguerre polynomials. In Sec. 3, we introduce the fundamental theorems for the approximated formula of the fractional derivatives of the generalized Laguerre polynomials and its convergence analysis. In Sec. 4, we give the procedure of the solution for the FDDE (1). In Sec. 5, we present three examples in order to clarify the advantages of the proposed method. We end the paper with a conclusion in Sec. 6.

2 Preliminaries and Notations

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory that will be required in the present paper.

2.1 The Fractional Derivative in the Caputo Sense

Definition 1. The Caputo fractional derivative operator D^α of order α is defined in the following form

$$D^\alpha f(x) = \frac{1}{\Gamma(m - \alpha)} \int_0^x \frac{f^{(m)}(t)}{(x - t)^{\alpha - m + 1}} dt, \quad \alpha > 0, \quad x > 0$$

where $m - 1 < \alpha \leq m, m \in \mathbb{N}$.

¹Permanent address: Department of Mathematics, Faculty of Science, Benha University, Benha, 13518 Egypt.

Contributed by Design Engineering Division of ASME for publication in the JOURNAL OF COMPUTATIONAL AND NONLINEAR DYNAMICS. Manuscript received August 23, 2012; final manuscript received April 2, 2013; published online July 18, 2013. Assoc. Editor: J. A. Tenreiro Machado.

Similar to the integer-order differentiation Caputo fractional derivative operator is a linear operation $D^\alpha(\lambda f(x) + \mu g(x)) = \lambda D^\alpha f(x) + \mu D^\alpha g(x)$, where λ and μ are constants.

For Caputo's derivative we have

$$D^\alpha C = 0, \quad C \text{ is a constant} \quad (4)$$

$$D^\alpha x^n = \begin{cases} 0, & \text{for } n \in \mathbb{N}_0 \text{ and } n < [\alpha] \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & \text{for } n \in \mathbb{N}_0 \text{ and } n \geq [\alpha] \end{cases} \quad (5)$$

We use the ceiling function $[\alpha]$ to denote the smallest integer greater than or equal to α and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.

For more details on fractional derivatives definitions and its properties see Refs. [10,19].

2.2 The Definition and Properties of the Generalized Laguerre Polynomials. The generalized Laguerre polynomials constitute a complete orthogonal set of functions on the semi-infinite interval $[0, \infty)$ [20,21]. The convolution structures of the Laguerre polynomials were presented in Ref. [22].

The generalized Laguerre polynomials $[L_n^{(\nu)}(x)]_{n=0}^\infty$, $\nu > -1$ are defined on $[0, \infty)$ and can be determined with the aid of the following recurrence formula

$$(n+1)L_{n+1}^{(\nu)}(x) + (x-2n-\nu-1)L_n^{(\nu)}(x) + (n+\nu)L_{n-1}^{(\nu)}(x) = 0, \quad n = 1, 2, \dots \quad (6)$$

where $L_0^{(\nu)}(x) = 1$ and $L_1^{(\nu)}(x) = \nu + 1 - x$.

$$L_n^{(\nu)}(x) = \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n+\nu}{n-k} x^k = \binom{n+\nu}{n} \sum_{k=0}^n \frac{(-n)_k}{(\nu+1)_k} \frac{x^k}{k!} \quad (7)$$

with $(a)_0 := 1$ and $(a)_k := a(a+1)(a+2)\dots(a+k-1)$, $k = 1, 2, 3, \dots$ and

$$L_n^{(\nu)}(0) = \binom{n+\nu}{n}$$

These polynomials are orthogonal on the interval $[0, \infty)$ with respect to the weight function $w(x) = (1/\Gamma(1+\nu))x^\nu e^{-x}$. The orthogonality relation is

$$\frac{1}{\Gamma(1+\nu)} \int_0^\infty x^\nu e^{-x} L_m^{(\nu)}(x) L_n^{(\nu)}(x) dx = \binom{n+\nu}{n} \delta_{mn} \quad (8)$$

Thus, they satisfy the differentiation formula

$$D^k L_n^{(\nu)}(x) = (-1)^k L_{n-k}^{(\nu+k)}(x), \quad k = 0, 1, \dots, n \quad (9)$$

Any function $u(x)$ belongs to $L_w^2[0, \infty)$, the space of all square integrable functions on $[0, \infty)$ with weight function $w(x)$, can be expanded in the following Laguerre series

$$u(x) = \sum_{i=0}^\infty c_i L_i^{(\nu)}(x) \quad (10)$$

where the coefficients c_i are given by

$$c_i = \frac{\Gamma(i+1)}{\Gamma(i+\nu+1)} \int_0^\infty x^\nu e^{-x} L_i^{(\nu)}(x) u(x) dx, \quad i=0, 1, 2, \dots \quad (11)$$

Consider only the first $(m+1)$ terms of the generalized Laguerre polynomials, thus we can write

$$u_m(x) = \sum_{i=0}^m c_i L_i^{(\nu)}(x) \quad (12)$$

3 The Fractional Derivatives of $L_n^{(\nu)}(x)$ and Its Convergence Analysis

The main goal of this section is to introduce the following theorems for the fractional derivatives of the generalized Laguerre polynomials and the resulting truncating error.

The main approximate formula of the fractional derivative of $u(x)$ is given in the following theorem.

THEOREM 1. Let $u(x)$ be approximated by the generalized Laguerre polynomials as Eq. (12) and suppose $\alpha > 0$ then, that its Caputo fractional derivative can be written in the following form

$$D^\alpha(u_m(x)) = \sum_{i=[\alpha]}^m \sum_{k=[\alpha]}^i c_i w_{i,k}^{(\alpha)} x^{k-\alpha} \quad (13)$$

where $w_{i,k}^{(\alpha)}$ is given by

$$w_{i,k}^{(\alpha)} = \frac{(-1)^k}{\Gamma(k+1-\alpha)} \binom{i+\nu}{i-k} \quad (14)$$

Proof. Since Caputo's fractional differentiation is a linear operation we have

$$D^\alpha(u_m(x)) = \sum_{i=0}^m c_i D^\alpha(L_i^{(\nu)}(x)) \quad (15)$$

Employing Eqs. (4) and (5) in the formula (7) we obtain

$$D^\alpha L_i^{(\nu)}(x) = 0, \quad i = 0, 1, \dots, [\alpha] - 1, \quad \alpha > 0 \quad (16)$$

In addition, for $i = [\alpha], [\alpha] + 1, \dots, m$ and by using Eqs. (4) and (5) in the formula (7) we obtain

$$\begin{aligned} D^\alpha L_i^{(\nu)}(x) &= \sum_{k=0}^i \frac{(-1)^k}{k!} \binom{i+\nu}{i-k} D^\alpha x^k \\ &= \sum_{k=[\alpha]}^i \frac{(-1)^k}{\Gamma(k+1-\alpha)} \binom{i+\nu}{i-k} x^{k-\alpha} \end{aligned} \quad (17)$$

A combination of Eqs. (14)–(17) leads to the desired result in Eq. (13). $\square$

THEOREM 2. The error in approximating $D^\alpha u(x)$ by $D^\alpha u_m(x)$ is bounded by

$$\begin{aligned} |E_T(m)| &\leq \sum_{i=m+1}^\infty c_i \Pi_\alpha(i, j) \frac{(\nu+1)_j}{j!} e^{x/2}, \\ \nu &\geq 0, \quad x \geq 0, \quad j=0, 1, 2, \dots \end{aligned} \quad (18)$$

$$\begin{aligned} |E_T(m)| &\leq \sum_{i=m+1}^\infty c_i \Pi_\alpha(i, j) \left(2 - \frac{(\nu+1)_j}{j!}\right) e^{x/2}, \\ -1 &< \nu \leq 0, \quad x \geq 0, \quad j=0, 1, 2, \dots \end{aligned} \quad (19)$$

where

$$\Pi_\alpha(i, j) = \sum_{k=[\alpha]}^i \sum_{j=0}^{k-\alpha} \Omega_{ijk} \text{ and } |E_T(m)| = |D^\alpha u(x) - D^\alpha u_m(x)|$$

Proof. A combination of Eqs. (10), (12), and Theorem 2 from Ref. [23] leads to

$$|E_T(m)| = |D^\alpha u(x) - D^\alpha u_m(x)| \leq \sum_{i=m+1}^{\infty} c_i \Pi_\alpha(i, j) |L_j^{(\nu)}(x)| \quad (20)$$

using formulas S_1 and S_2 in Ref. [24] and subtracting the truncated series from the infinite series, bounding each term in the difference, and summing the bounds completes the proof of the theorem. $\square$

4 Procedure of Solution for Fractional Delay Differential Equation

Consider the FDDE of the type given in Eq. (1). In order to use the Laguerre collocation method, let $w(x) = (1/\Gamma(1+\nu))x^\nu e^{-x}$ be a positive weight function on the interval $I = [0, \infty)$ and $L_w^2(I)$ is the weighted L^2 space with the inner product

$$(u, v)_w = \int_0^\infty w(x) u(x) v(x) dx$$

and the associated norm $\|u\|_w = \sqrt{(u, u)_w}$. It is well known that $[L_n^{(\nu)}(x) : n \geq 0]$ forms a complete orthogonal system in $L_w^2(I)$, therefore if we define

$$S_m(I) = \text{Span}[L_0^{(\nu)}(x), L_1^{(\nu)}(x), \dots, L_m^{(\nu)}(x)]$$

then the Laguerre spectral solution to Eq. (1) is to find $u_m \in S_m(I)$ such that

$$u_m(x) \cong \sum_{i=0}^m c_i L_i^{(\nu)}(x) \quad (21)$$

From Eqs. (1), (21), and Theorem 1 we have

$$\sum_{i=\lceil \alpha \rceil}^m \sum_{k=\lceil \alpha \rceil}^i c_i w_{i,k}^{(\alpha)} x_p^{k-\alpha} = f\left(x; \sum_{i=0}^m c_i L_i^{(\nu)}(x), \sum_{i=0}^m c_i L_i^{(\nu)}(g(x))\right) \quad (22)$$

We now collocate Eq. (22) at $(m+1-\lceil \alpha \rceil)$ points x_p , $p = 0, 1, \dots, m-\lceil \alpha \rceil$ as

$$\sum_{i=\lceil \alpha \rceil}^m \sum_{k=\lceil \alpha \rceil}^i c_i w_{i,k}^{(\alpha)} x_p^{k-\alpha} = f\left(x_p; \sum_{i=0}^m c_i L_i^{(\nu)}(x_p), \sum_{i=0}^m c_i L_i^{(\nu)}(g(x_p))\right) \quad (23)$$

For suitable collocation points we use the roots of the generalized Laguerre polynomial $L_{m+1-\lceil \alpha \rceil}^{(\nu)}(x)$.

By substituting Eq. (21) into the boundary conditions (2) and using the property

$$L_i^{(\nu)}(0) = \binom{\nu+i}{i},$$

we can obtain $\lceil \alpha \rceil$ equations

$$\sum_{i=0}^m c_i \binom{\nu+i}{i} = \gamma_0, \quad \sum_{i=0}^m c_i L_i^{(\nu)}(1) = \gamma_1 \quad (24)$$

Equation (23), together with the $\lceil \alpha \rceil$ equations of boundary conditions (24), yield $(m+1)$ linear or nonlinear algebraic equations, which can be solved for the unknowns c_n , $n = 0, 1, \dots, m$. Consequently, $u(x)$ given in Eq. (1) can be calculated.

5 Numerical Results

In this section, we implement the proposed method in order to solve linear and nonlinear examples of fractional delay differential equations. These examples [17,18] can be used as a basis to illustrate the applicability of the proposed method.

Example 1. Consider the following linear fractional delay differential equation

$$D^{1.5} u(x) = -u(x) + u(x/2) + \frac{7}{8} x^3 + \frac{6}{\Gamma(2.5)} x^{1.5} \quad (25)$$

with the boundary conditions

$$u(0) = 0, \quad u(1) = 1 \quad (26)$$

We implement the suggested method with $m=3$ and we approximate the solution as

$$u_3(x) = \sum_{i=0}^3 c_i L_i^{(\nu)}(x) \quad (27)$$

Using Eq. (23) we obtain

$$\begin{aligned} \sum_{i=2}^3 \sum_{k=2}^i c_i w_{i,k}^{(\alpha)} x_p^{k-\alpha} &= -\sum_{i=0}^3 c_i L_i^{(\nu)}(x_p) + \sum_{i=0}^3 c_i L_i^{(\nu)}\left(\frac{x_p}{2}\right) \\ &+ \frac{7}{8} x_p^3 + \frac{6}{\Gamma(2.5)} x_p^{1.5} \end{aligned} \quad (28)$$

with $p=0, 1$, where x_p are roots of the Laguerre polynomial $L_2^{(\nu)}(x)$ and their values are (at $\nu = -0.5$)

$$x_0 = 2 + \nu - \sqrt{2 + \nu} = 0.27526,$$

$$x_1 = 2 + \nu + \sqrt{2 + \nu} = 2.72470$$

By using Eqs. (24) and (26) we obtain

$$k_0 c_0 + k_1 c_1 + k_2 c_2 + k_3 c_3 = 0 \quad (29)$$

$$h_0 c_0 + h_1 c_1 + h_2 c_2 + h_3 c_3 = 1 \quad (30)$$

where $k_i = L_i^{(\nu)}(0)$ and $h_i = L_i^{(\nu)}(1)$, $i = 0, 1, 2, 3$.

Now, solving Eqs. (28)–(30) we obtain

$$c_0 = \frac{15}{8}, \quad c_1 = -\frac{45}{4}, \quad c_2 = 15, \quad c_3 = -6$$

Thus, using Eq. (27) we get

$$u(x) \simeq \frac{15}{8} L_0^{(\nu)}(x) - \frac{45}{4} L_1^{(\nu)}(x) + 15 L_2^{(\nu)}(x) - 6 L_3^{(\nu)}(x) = x^3$$

which is the exact solution of Eq. (25).

In order to compare with the classical Laguerre polynomials $L_n(x)$, we solved the system of Eqs. (28)–(30) at $\nu = 0$ and we obtained $c_0 = 6$, $c_1 = -18$, $c_2 = 18$, $c_3 = -6$.

Thus using Eq. (27) we obtain

$$u(x) \simeq 6 L_0^{(0)}(x) - 18 L_1^{(0)}(x) + 18 L_2^{(0)}(x) - 6 L_3^{(0)}(x) = x^3$$

which is the exact solution of Eq. (25).

Example 2. Consider the following nonlinear fractional delay differential equation

$$D^{1.5} u(x) = u(x-0.5) + u^3(x) + f(x) \quad (31)$$

where $f(x) = (2/\Gamma(1.5))x^{0.5} - (x - 0.5)^2 - x^6$, with the boundary conditions

$$u(0) = 0, \quad u(1) = 1 \quad (32)$$

We implement the suggested method with $m=3$ and approximate the solution as Eq. (27). Using Eq. (23) we have

$$\sum_{i=|\alpha|}^m \sum_{k=|\alpha|}^i c_i w_{i,k}^{(\alpha)} x_p^{k-\alpha} = \sum_{i=0}^3 c_i L_i^{(\nu)}(x_p - 0.5) + \left(\sum_{i=0}^3 c_i L_i^{(\nu)}(x_p) \right)^3 + f(x_p) \quad (33)$$

with $p=0, 1$, where x_p are roots of the Laguerre polynomial $L_2^{(\nu)}(x)$ and their values are (at $\nu = -0.5$)

$$x_0 = 2 + \nu - \sqrt{2 + \nu} = 0.27526, \\ x_1 = 2 + \nu + \sqrt{2 + \nu} = 2.72470$$

By using Eqs. (24) and (33) we obtain the following nonlinear system of algebraic equations

$$c_2 w_{2,2}^{(1.5)} x_0^{0.5} + c_3 \left(w_{3,2}^{(1.5)} x_0^{1.5} + w_{3,3}^{(1.5)} x_0^{1.5} \right) - \sum_{i=0}^3 r_i c_i - \left(\sum_{i=0}^3 s_i c_i \right)^3 - f(x_0) = 0 \quad (34)$$

$$c_2 w_{2,2}^{(1.5)} x_1^{0.5} + c_3 \left(w_{3,2}^{(1.5)} x_1^{1.5} + w_{3,3}^{(1.5)} x_1^{1.5} \right) - \sum_{i=0}^3 u_i c_i - \left(\sum_{i=0}^3 v_i c_i \right)^3 - f(x_1) = 0 \quad (35)$$

$$k_0 c_0 + k_1 c_1 + k_2 c_2 + k_3 c_3 = 0 \quad (36)$$

$$h_0 c_0 + h_1 c_1 + h_2 c_2 + h_3 c_3 = 1 \quad (37)$$

where

$$r_i = L_i^{(\nu)}(x_0 - 0.5), \quad s_i = L_i^{(\nu)}(x_0),$$

$$u_i = L_i^{(\nu)}(x_1 - 0.5), \quad v_i = L_i^{(\nu)}(x_1)$$

$$k_i = L_i^{(\nu)}(0), \quad h_i = L_i^{(\nu)}(1), \quad i = 0, 1, 2, 3$$

Now, Eqs. (34)–(37) give a nonlinear system of equations by solving it using the Newton iteration method at $\nu = -0.5$ we obtain

$$c_0 = \frac{3}{4}, \quad c_1 = -3, \quad c_2 = 2, \quad c_3 = 0$$

Thus, using Eq. (27) we obtain

$$u(x) \simeq \frac{3}{4} L_0^{(\nu)}(x) - 3 L_1^{(\nu)}(x) + 2 L_2^{(\nu)}(x) + 0 L_3^{(\nu)}(x) = x^2$$

which is the exact solution of Eq. (31).

In order to compare with the classical Laguerre polynomials $L_n(x)$, we solved the system of Eqs. (34)–(37) at $\nu = 0$ we obtain $c_0 = 2, c_1 = -4, c_2 = 2, c_3 = 0$.

Thus, using Eq. (27) we obtain

$$u(x) \simeq 2 L_0^{(0)}(x) - 4 L_1^{(0)}(x) + 2 L_2^{(0)}(x) + 0 L_3^{(0)}(x) = x^2$$

which is the exact solution of Eq. (31).

Example 3. Consider the following nonlinear fractional delay differential equation

$$D^{1.8} u(x) = 2u(x - 0.2) + u^2(x) + f(x) \quad (38)$$

where $f(x) = (120/\Gamma(4.2))x^{3.2} - (x - 0.2)^2 - x^{10}$, with the boundary conditions $u(0) = 0, u(1) = 1$. We implement the suggested method with $m=5$ and approximate the solution as

$$u_5(x) = \sum_{i=0}^5 c_i L_i^{(\nu)}(x) \quad (39)$$

Using Eq. (23) we have

$$\sum_{i=2}^m \sum_{k=2}^i c_i w_{i,k}^{(1.8)} x_p^{k-1.8} = 2 \sum_{i=0}^5 c_i L_i^{(\nu)}(x_p - 0.2) + \left(\sum_{i=0}^5 c_i L_i^{(\nu)}(x_p) \right)^2 + f(x_p) \quad (40)$$

with $p = 0, 1, 2, 3$ where x_p are roots of the Laguerre polynomial $L_4^{(\nu)}(x)$ and their values are (at $\nu = -0.5$)

$$x_0 = 0.145304, \quad x_1 = 1.3391, \quad x_2 = 3.92696, \quad x_3 = 8.58864$$

By using Eq. (24) and the boundary conditions we obtain

$$k_0 c_0 + k_1 c_1 + k_2 c_2 + k_3 c_3 + k_4 c_4 + k_5 c_5 = 0 \quad (41)$$

$$h_0 c_0 + h_1 c_1 + h_2 c_2 + h_3 c_3 + h_4 c_4 + h_5 c_5 = 1 \quad (42)$$

where $k_i = L_i^{(\nu)}(0), h_i = L_i^{(\nu)}(1), i = 0, 1, 2, 3, 4, 5$.

Now, Eqs. (40)–(42) give a nonlinear system of equations by solving it using the Newton iteration method we obtain

$$c_0 = 29.5312, \quad c_1 = -295.312, \quad c_2 = 787.5,$$

$$c_3 = -945, \quad c_4 = 540, \quad c_5 = -120$$

Thus, using Eq. (39) we obtain

$$u(x) \simeq 29.531 L_0^{(\nu)}(x) - 295.31 L_1^{(\nu)}(x) + 787.5 L_2^{(\nu)}(x) - 945 L_3^{(\nu)}(x) + 540 L_4^{(\nu)}(x) - 120 L_5^{(\nu)}(x) = x^5$$

which is the exact solution of Eq. (38).

Fig. 1 The behavior of the exact solution and the approximate solution at $m=3$ and $m=5$

The behavior of the approximate solution at different values of m ($m = 3, 5$) is presented in Fig. 1. From this figure, we can conclude that the proposed approximate solution is in excellent agreement with the exact solution and the accuracy is increased with increasing the value of m . In light of the numerical results of all of the proposed examples, we can confirm the advantages from the introduced technique since we obtained the solutions with high accuracy and with a small amount of time compared with other techniques [18,23].

6 Conclusion

The properties of the generalized Laguerre polynomials are used to derive an approximate formula for the Caputo fractional derivative and we used it to reduce the fractional delay differential equations to the solution of the system of linear or nonlinear algebraic equations. Special attention is given to estimate an upper error bound of the approximated fractional derivative. From the solutions obtained by using the suggested method, we can conclude that these solutions are in excellent agreement with the exact solution. Additionally, it is evident that the overall errors can be made smaller by adding new terms from the series (12). This shows that the proposed method is an efficient method. In the presented examples, we have the exact solution. All of the numerical results are obtained using MATLAB 7.1.

References

- [1] Bagley, R. L., and Torvik, P. J., 1984, "On the Appearance of the Fractional Derivative in the Behavior of Real Materials," *ASME J. Appl. Mech.*, **51**(2), pp. 294–298.
- [2] Enelund, M., and Josefson, B. L., 1997, "Time-Domain Finite Element Analysis of Viscoelastic Structures With Fractional Derivatives Constitutive Relations," *AIAA J.*, **35**(10), pp. 1630–1637.
- [3] Diethelm, K., 1997, "An Algorithm for the Numerical Solution of Differential Equations of Fractional Order," *Electron. Trans. Numer. Anal.*, **5**, pp. 1–6.
- [4] He, J. H., 2011, "A Short Remark on Fractional Variational Iteration Method," *Phys. Lett. A*, **375**(38), pp. 3362–3364.
- [5] He, J. H., 1999, "Homotopy Perturbation Technique," *Comput. Methods Appl. Mech. Eng.*, **178**(3–4), pp. 257–262.
- [6] Khabibrahmanov, I. Z., and Summers, D., 1998, "The Use of Generalized Laguerre Polynomials in Spectral Methods for Non-Linear Differential Equations," *Comput. Math. Appl.*, **36**, pp. 65–70.
- [7] Khader, M. M., 2011, "On the Numerical Solutions for the Fractional Diffusion Equation," *Commun. Nonlinear Sci. Numer. Simul.*, **16**, pp. 2535–2542.
- [8] Khader, M. M., 2012, "Introducing an Efficient Modification of the VIM by Using Chebyshev Polynomials," *Appl. Appl. Math.*, **7**(1), pp. 283–299.
- [9] Ramadan, M. A., and Sharif, M. N., 2006, "Numerical Solution of System of First Order Delay Differential Equations Using Spline Functions," *Int. J. Comput. Math.*, **83**(12), pp. 925–937.
- [10] Samko, S., Kilbas, A., and Marichev, O., 1993, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, London.
- [11] Sweilam, N. H., Khader, M. M., and Al-Bar, R. F., 2007, "Numerical Studies for a Multi-Order Fractional Differential Equation," *Phys. Lett. A*, **371**, pp. 26–33.
- [12] Sweilam, N. H., Khader, M. M., and Nagy, A. M., 2011, "Numerical Solution of Two Sided Space Fractional Wave Equation Using Finite Difference Method," *Comput. Appl. Math.*, **235**, pp. 2832–2841.
- [13] Sweilam, N. H., Khader, M. M., and Mahdy, A. M. S., 2012, "Numerical Studies for Fractional-Order Logistic Differential Equation With Two Different Delays," *J. Appl. Math.*, **2012**, pp. 1–14.
- [14] Bell, W. W., 1968, *Special Functions for Scientists and Engineers*, Butler and Tanner Ltd, Frome, London.
- [15] Canuto, C., Hussaini, M. Y., Quarteroni, A., and Zang, T. A., 1988, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, New York.
- [16] Funaro, D., 1992, *Polynomial Approximation of Differential Equations*, Springer-Verlag, New York.
- [17] Ramadan, M. A., 2005, "Spline Solution of First Order Delay Differential Equation," *J. Egypt. Math. Soc.*, **1**, pp. 7–18.
- [18] Khader, M. M., and Hendy, A. S., 2012, "The Approximate and Exact Solutions of the Fractional-Order Delay Differential Equations Using Legendre Pseudospectral Method," *Int. J. Pure Appl. Math.*, **74**(3), pp. 287–297.
- [19] Podlubny, I., 1999, *Fractional Differential Equations*, Academic, New York, 1999.
- [20] Michalska, M., and Szynal, J., 2001, "A New Bound for the Laguerre Polynomials," *J. Comput. Appl. Math.*, **133**, pp. 489–493.
- [21] Wang, L., and Guo, B., 2006, "Stair Laguerre Pseudospectral Method for Differential Equations on the Half Line," *Adv. Comput. Math.*, **25**, pp. 305–322.
- [22] Askey, R., and Gasper, G., 1977, "Convolution Structures for Laguerre Polynomials," *J. Anal. Math.*, **31**, pp. 48–68.
- [23] Khader, M. M., El Danaf, T. S., and Hendy, A. S., 2012, "Efficient Spectral Collocation Method for Solving Multi-Term Fractional Differential Equations Based on the Generalized Laguerre Polynomials," *Fractional Calculus Appl.*, **3**(13), pp. 1–14.
- [24] Lewandowski, Z., and Szynal, J., 1998, "An Upper Bound for the Laguerre Polynomials," *J. Comput. Appl. Math.*, **99**, pp. 529–533.