Tortuosity and the Averaging of Microvelocity Fields in Poroelasticity

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The relationship between the macro- and microvelocity fields in a poroelastic representative volume element (RVE) has not been fully investigated. This relationship is considered to be a function of the tortuosity: a quantitative measure of the effect of the deviation of the pore fluid streamlines from straight (not tortuous) paths in fluid-saturated porous media. There are different expressions for tortuosity based on the deviation from straight pores, harmonic wave excitation, or from a kinetic energy loss analysis. The objective of the work presented is to determine the best expression for tortuosity of a multiply interconnected open pore architecture in an anisotropic porous media. The procedures for averaging the pore microvelocity over the RVE of poroelastic media by Coussy and by Biot were reviewed as part of this study, and the significant connection between these two procedures was established. Success was achieved in identifying the Coussy kinetic energy loss in the pore fluid approach as the most attractive expression for the tortuosity of porous media based on pore fluid viscosity, porosity, and the pore architecture. The fabric tensor, a 3D measure of the architecture of pore structure, was introduced in the expression of the tortuosity tensor for anisotropic porous media. Practical considerations for the measurement of the key parameters in the models of Coussy and Biot are discussed. In this study, we used cancellous bone as an example of interconnected pores and as a motivator for this study, but the results achieved are much more general and have a far broader application than just to cancellous bone. [DOI: 10.1115/1.4007923]

Keywords: poroelasticity, cancellous bone, RVE, tortuosity, fabric

1 Introduction

Tortuosity is somewhat of a misnomer, because it generally turns out to be a measure of the loss of energy due to fluid flow in a porous medium, not only to the tortuosity of the fluid streamlines, but also the additional viscous dissipation that occurs due to the fluid particles traveling a longer path than the one that has the least dissipation of energy. There are many different expressions for the tortuosity of a fluid-saturated porous medium. These expressions depend upon the assumptions made about the porous architecture of the fluid-filled medium and the degree of excitation imposed upon the pore fluid. We are principally interested in the definition of tortuosity suggested by Coussy [1], because it appears to be appropriate for the open pore structure of cancellous bone tissue. The open pore structure of cancellous bone tissue is described in Sec. 2, and a point is made about the low Reynolds number associated with marrow flow in this open pore structure. The RVE structure and the sub-RVE microvelocity of the poroelastic continua are described in Sec. 3. Two proposals for averaging the microvelocity over the RVE are considered in Sec. 4. The proposals are due to Coussy [1] and Biot [2]. Expressions for closed tube models of the tortuosity and the dynamic tortuosity model associated with harmonic wave propagation are described in Sec. 5, while all of Sec. 6 is dedicated to formulation of Coussy's [1] expression for tortuosity. At the end of Sec. 6, a relationship is established between the Coussy [1] and Biot [2] methods for averaging the microvelocity over the RVE. Practical considerations for the measurement of the key parameters in the models of Coussy and Biot are discussed in Sec. 7. In Sec. 8, the relationship between tortuosity and fabric is reviewed and extended to the Coussy [1] expression for tortuosity.

2 Cancellous Bone

Cancellous bone generally exists only within the confines of the cortical bone coverings. Cancellous bone is also called trabecular bone, because it is composed of short struts of bone material called trabeculae (from the Latin for "little beam"). The connected trabeculae give cancellous bone a spongy appearance, and it is often called spongy bone (Fig. 1). There are seldom blood vessels within the trabeculae, but there are vessels immediately adjacent to the tissue, and they weave in and out of the pores between the individual trabeculae. Cancellous bone has a vast surface area, as would be suggested by its spongy appearance. This is illustrated by the human pelvis, which has an average volume of 40 cm^3 and an average periosteal (external) surface area of 80 cm^2 , but the average surface area of its trabecular bone is 1600 cm^2 . The human cancellous bone pore size varies from 300 to $2200 \mu\text{m}$ for the porosity of 52% to 96% [3–7].

The pores of cancellous bone contain bone marrow, a rather viscous fluid. The viscosity of marrow is 37.5 cP , about 42 times the viscosity of water; the viscosity of water is 0.894 cP . If we assume that the density of marrow is approximately the same as that of water (1000 kg/m^3), that the characteristic length of the pore size is 300 to $2200 \mu\text{m}$, and if we employ the viscosity of the marrow, it follows that the Reynolds number is $8v$ to $58.7v$, where v is the velocity in m/s . Thus, in order to maintain a Reynolds number of less than 10, the velocity of the marrow must be less than 0.17 m/s (1.25 m/s for the pore size $300 \mu\text{m}$ and 0.17 m/s for $2200 \mu\text{m}$). If the Reynolds number is less than 10, the flow will remain laminar and turbulence will not occur in a porous medium, like cancellous bone (Bear [7]). We assume that this is the situation we are dealing with in cancellous bone.

3 The Poroelastic Continuum Model

The conceptual structure of the poroelastic continuum model is illustrated in Fig. 2. The representative volume element (RVE)

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Fig. 1 An image of a cancellous bone section; the gray color areas represent the solid matrix portion and the black regions the pores

Fig. 2 A cartoon of an enlarged RVE for a continuum model of a porous medium

associated with each point continuum of the poroelastic continuum is shown at an exaggerated scale. The thick dark lines/curves represent the pore structure, a structure that is not of uniform diameter, as this graphic suggests. The modeling approach illustrated by the RVE in Fig. 2 is a schematic version of the viewpoint described in Biot [8] in 1941. He wrote, “Consider a small cubic element of soil, its sides being parallel with the coordinate axes. This element is taken to be large enough compared to the size of the pores so that it may be treated as homogeneous, and at the same time small enough, compared to the scale of the macroscopic phenomena in which we are interested, so that it may be considered as infinitesimal in the mathematical treatment.” This prose written by Biot appears to be the first statement of what later came to be called the RVE concept. In Biot’s proposal, a small but finite volume of the porous medium is used as a model for a continuum point in the development of constitutive equations for the fluid-infiltrated porous solid. These constitutive equations are then assumed to be valid at a point in the continuum. The length or size of the RVE is assumed to be many times larger than the length scale of the microstructure of the material, say the size of a pore. The length of the RVE is the length of the material structure over which the material microstructure is averaged or “homogenized” in the process of forming a continuum model. The homogenization approach is illustrated in Fig. 1 by the dashed lines from the four corners of the RVE to the continuum point. The porosity of the matrix material is denoted by ϕ .

In 1956, Biot published two papers on wave propagation in porous media [9,10]; in 1962, he published another two [2,11]. In his 1962 paper on wave propagation, Biot [2] let $\mathbf{u}$ represent the displacement vector of the solid matrix phase and $\mathbf{U}$ represent the displacement vector of the fluid phase. In Biot [2], the displacement vector of the fluid phase $\mathbf{U}$ was replaced by the product of the porosity and the relative displacement vector $\mathbf{w}$ of the fluid relative to the solid multiplied by the porosity, where

$$\mathbf{w} = \mathbf{U} - \mathbf{u} \quad (1)$$

Thus, Biot used $\phi\mathbf{w}$ where we will use $\mathbf{w}$. Aside from that small difference, the present development follows Biot [2], and the two basic kinematic fields are considered to be the displacement vectors $\mathbf{u}$ and $\mathbf{w}$. The relative velocity of the fluid and solid components is, from [1],

$$\dot{\mathbf{w}} = \dot{\mathbf{U}} - \dot{\mathbf{u}} \quad (2)$$

The three variables $\mathbf{U}$, $\mathbf{u}$, and $\mathbf{w}$ are each at the level of the RVE. The relative microdisplacement in the pores, the displacement of a fluid particle in a pore at the sub-RVE with respect to the solid matrix (skeleton), is denoted by $\mathbf{w}^{\text{micro}}$.

The usual continuum mechanics material $\mathbf{X}$ and spatial $\mathbf{x}$ coordinate systems are employed for the poroelastic continuum; it is in the material $\mathbf{X}$ coordinate system that the RVE is just a point. A sub-RVE coordinate system denoted by ξ is used to represent the pore volume with the RVE. In Sec. 4, the problem of determining $\mathbf{w}^{\text{micro}}$ from $\mathbf{w}$ is discussed.

4 Averaging the Microvelocity Over the RVE

Biot [2] equation (8.3) related the components of the relative microvelocity field $\dot{\mathbf{w}}^{\text{micro}}$ in the sub-RVE pores to the RVE level vector $\dot{\mathbf{w}}$ by a linear transformation or second order tensor $\mathbf{a}$. We define the second order tensor $\mathbf{J}$ with components J_{ij} as equal to ϕa_{ij} , where a_{ij} are coefficients employed by Biot,

$$\dot{\mathbf{w}}^{\text{micro}} = \mathbf{J} \cdot \dot{\mathbf{w}} \quad (3)$$

Biot noted that the coefficients a_{ij} depend on the coordinates in the pores and the pore geometry, and thus $J_{ij} = \phi a_{ij}$, since the porosity ϕ is constant. But Biot wrote nothing more specific about these coefficients that we could locate. One of the contributions of this work will be to relate $\mathbf{J}$ to the tortuosity tensor as defined by Coussy [1] and thus remove part of the uncertainty about this hypothesis of Biot [2].

Coussy [1] introduces an averaging process for the microvelocity field $\dot{\mathbf{w}}^{\text{micro}}$ that appears to have the same function as the proposal of Biot discussed above. Coussy calls this averaging process the barycentric average. The barycentric volume average of the quantity $*$ between the pointed brackets $\langle * \rangle$ for the fluid microscopic particles contained in the elementary pore volume Ω_p of a RVE is represented by

$$\dot{\mathbf{w}}(\mathbf{x}, t) = \langle \dot{\mathbf{w}}^{\text{micro}}(\xi, \mathbf{x}, t) \rangle \quad (4)$$

where the ξ represent a coordinate system in the sub-RVE pores. Coussy [1] notes two reasons for the following inequality to be satisfied:

$$\langle \dot{\mathbf{w}}^{\text{micro}} \cdot \dot{\mathbf{w}}^{\text{micro}} \rangle \geq \langle \dot{\mathbf{w}}^{\text{micro}} \rangle^2 = (\dot{\mathbf{w}})^2 \quad (5)$$

The first reason is that, due to the tortuosity of the porous network, the direction of the microscopic relative velocity field $\dot{\mathbf{w}}^{\text{micro}}$ does not remain constant within pore volume V_p . The second reason is that the absolute value of $\dot{\mathbf{w}}^{\text{micro}}$ does not remain constant across a pore, due to the effect of fluid viscosity being greater at the pore wall than at the pore center. This viscosity implies that the

absolute value of $\dot{\mathbf{w}}^{\text{micro}}$ decreases from its maximal value reached at the greatest distance from the solid internal walls of the skeleton, which delimit the connected porous network through which the fluid flow actually occurs, to zero onto these internal walls that form the matrix-fluid interface.

While it is difficult to see exactly how one could determine the $\mathbf{J}$ introduced by Biot, it is not difficult to see many possibilities to make the barycentric average of Coussy precise by drawing upon the many homogenization studies of recent years. For example, we consider the average of the microvelocity over the RVE as a very simple example,

$$\dot{\mathbf{w}}(\mathbf{x}, t) = \langle \dot{\mathbf{w}}^{\text{micro}}(\xi, \mathbf{x}, t) \rangle = \frac{1}{\Omega_p} \int_{\Omega_p} \dot{\mathbf{w}}^{\text{micro}}(\mathbf{x}, \xi, t) dv(\xi) \quad (6)$$

where Ω_p represents the total pore volume of fluid.

5 Tortuosity

As noted in the introduction, tortuosity is a measure of the loss of energy due, not only to the tortuosity of the fluid streamlines, but also the additional viscous dissipation that occurs due to the fluid particles traveling a longer path than the one that has the least dissipation of energy. The literature on tortuosity and many other related topics up to the early 1970s is summarized in the book of Bear [7]. Bear notes that Carmen [12] proposed a tortuosity factor as an effect on the velocity and as an effect on the driving force, and he defined tortuosity as dependent upon (L_e/L) [2], where L is the straight-line length (end to end) of a tortuous total of length L_e . The concept of a closed tortuous tube as a unit model for the fluid passage pore structure in a porous medium was subsequently widely used in many formulas' tortuosity (Bear [7] and Norris [13]). $\mathbf{T}^{\text{CT}}$ denotes the formulas obtained for the tortuosity tensor based on the closed tube models. From Fig. 1 and the discussion above concerning the structure of cancellous bone, one can see that the closed tortuous tube model is inappropriate for this type of porous structure. In the cancellous bone type of pore structure, the tortuous tubes that form to transmit the fluid through the porous matrix change as a function of Reynolds number (Costa et al. [14]). They wander less from a straight line as the Reynolds number increases.

Two other approaches to tortuosity are of interest. The approach of Coussy will be discussed in Sec. 6, as it is the preferred one for structures of the type that characterize cancellous bone. The other approach to tortuosity is restricted to harmonic wave propagation and is described as dynamic tortuosity. Following Johnson et al. [15] and Perrot et al. [16], the dynamic tortuosity tensor $\mathbf{T}^D(\omega)$ is a function of frequency ω , and it is introduced in the linear momentum equation

$$\rho^f \mathbf{T}^D(\omega) \cdot \frac{\partial \dot{\mathbf{w}}}{\partial t} = -\nabla p, \quad \left(\rho^f T_{ij}^D(\omega) \frac{\partial \dot{w}_j}{\partial t} = -\frac{\partial p}{\partial x_i} \right) \quad (7)$$

and using Darcy's law, relating the fluid mass flow rate, $\mathbf{q}$, to the gradient (∇p) of the pore pressure p (Darcy [17]),

$$\mathbf{q} = \phi \left(\frac{\rho^f}{\rho^s} \right) \dot{\mathbf{w}} = -\frac{\mathbf{K} \cdot \nabla p}{\mu} \quad (8)$$

where $\mathbf{K}$ is the permeability tensor and μ is the viscosity and, if we assume that $\mathbf{w} = \text{constant} \cdot e^{-i\omega t}$, it follows that

$$\mathbf{T}^D(\omega) = \frac{i\mu\phi}{\omega\rho^s} \mathbf{K}^{-1}(\omega) \quad (9)$$

6 Coussy's Formulation of the Tortuosity Tensor

Coussy's [1] definition of the tortuosity tensor, $\mathbf{T}^C$, is based on the energy lost due to the motion of the viscous pore fluid through

the porous matrix. The kinetic energy of the fluid is expressed by the formula

$$K^f = \frac{1}{2} \int_{\Omega_p} \rho^f \dot{\mathbf{U}} \cdot \dot{\mathbf{U}} d\Omega_p \quad (10)$$

If we let $\dot{\mathbf{U}} = \dot{\mathbf{u}} + \dot{\mathbf{w}}^{\text{micro}}$ and perform the integrations that can be done easily, one obtains

$$\begin{aligned} K^f &= \int_{\Omega_p} \frac{1}{2} \rho^f (\dot{\mathbf{u}} + \dot{\mathbf{w}}^{\text{micro}})^2 d\Omega \\ &= \frac{1}{2} \phi \rho^f \dot{\mathbf{u}}^2 V + \rho^f \dot{\mathbf{u}} \cdot \int_{\Omega_p} \dot{\mathbf{w}}^{\text{micro}} d\Omega_p + \frac{1}{2} \rho^f \int_{\Omega_p} (\dot{\mathbf{w}}^{\text{micro}})^2 d\Omega_p \end{aligned} \quad (11)$$

where use has been made of the relation $\dot{\mathbf{u}} = \langle \dot{\mathbf{u}} \rangle$, which follows from the fact that $\dot{\mathbf{u}}$ is independent of the sub-RVE coordinates ξ . Biot [2] introduces the components m_{ij} of a symmetric tensor, and we employ a constant multiple $\mathbf{M}$ of that symmetric tensor defined by

$$M_{ij} = \frac{\phi}{\rho^f} m_{ij} \quad (12)$$

then

$$\frac{1}{\Omega_p} \int_{\Omega_p} (\dot{\mathbf{w}}^{\text{micro}})^2 d\Omega_p \equiv \dot{\mathbf{w}} \cdot \mathbf{M} \cdot \dot{\mathbf{w}} = \dot{w}_i M_{ij} \dot{w}_j \quad (13)$$

where

$$\mathbf{M} = \frac{1}{\Omega_p} \int_{\Omega_p} \mathbf{J}^T \cdot \mathbf{J} d\Omega_p, \quad M_{ij} = \frac{1}{\Omega_p} \int_{\Omega_p} J_{ki} J_{kj} d\Omega_p \quad (14)$$

If the plausible assumption is made that $\mathbf{J}$ is constant for a RVE, then Eq. (14) may be reduced to the following:

$$\mathbf{M} = \mathbf{J}^T \cdot \mathbf{J}, \quad M_{ij} = J_{ki} J_{kj} \quad (15)$$

Substituting Eqs. (5) into (11) and subsequently applying the averaging process $\langle * \rangle$, we obtain

$$\frac{K^f}{\Omega_p} = \frac{1}{2} \rho^f \dot{\mathbf{u}}^2 + \rho^f \dot{\mathbf{u}} \cdot \dot{\mathbf{w}} + \frac{1}{2} \rho^f \dot{\mathbf{w}} \cdot \mathbf{T}^C \cdot \dot{\mathbf{w}} \quad (16)$$

where

$$\dot{\mathbf{w}} \cdot \mathbf{T}^C \cdot \dot{\mathbf{w}} \equiv \frac{1}{\Omega_p} \int_{\Omega_p} (\dot{\mathbf{w}}^{\text{micro}})^2 d\Omega_p = \langle \dot{\mathbf{w}}^{\text{micro}} \cdot \dot{\mathbf{w}}^{\text{micro}} \rangle \quad (17)$$

and where the Coussy tortuosity tensor $\mathbf{T}^C$ is defined by

$$\langle \dot{\mathbf{w}}^{\text{micro}} \cdot \dot{\mathbf{w}}^{\text{micro}} \rangle \equiv \dot{\mathbf{w}} \cdot \mathbf{T}^C \cdot \dot{\mathbf{w}}; \quad \forall \dot{\mathbf{w}}, \dot{\mathbf{w}} \cdot (\mathbf{T}^C - \mathbf{1}) \cdot \dot{\mathbf{w}} \geq 0 \quad (18)$$

Note that the inequality requires that both $\mathbf{T}^C$ and $(\mathbf{T}^C - \mathbf{1})$ be positive definite. The tortuosity tensor $\mathbf{T}^C$ depends on both the current geometry of the porous network through which the flow occurs and on the fluid viscosity. Its value is $\mathbf{1}$ for a fluid ($\phi = 1$), due to a lack of solid surface contact area for the generation of relative velocities; the fluid velocity field is homogeneous in this case.

The expression in Eq. (12) for K^f is simplified by using Eq. (2) and Eq. (13) to eliminate $\dot{\mathbf{u}}$; thus,

$$\frac{K^f}{\Omega_p} = \frac{1}{2} \rho^f \dot{\mathbf{U}}^2 + \frac{1}{2} \rho^f \dot{\mathbf{w}} \cdot (\mathbf{T}^C - \mathbf{1}) \cdot \dot{\mathbf{w}} \quad (19)$$

We return now to the question of the relationship between the approach of Biot [2] represented by Eq. (3) and the approach in Eq. (6) of Coussy [1] for the averaging process for the microvelocity field in an RVE. Noting the equality, which follows from Eqs. (13) and (17):

$$\dot{\mathbf{w}} \cdot \mathbf{M} \cdot \dot{\mathbf{w}} = \dot{\mathbf{w}} \cdot \mathbf{T}^C \cdot \dot{\mathbf{w}} \quad (20)$$

it follows, with the use of Eq. (15), that

$$\mathbf{T}^C = \mathbf{M} = \mathbf{J}^T \cdot \mathbf{J} \quad (21)$$

This result shows that Coussy's tortuosity tensor is the same as our modified Biot's $\mathbf{M}$ tensor and that our modified Biot's $\mathbf{M}$ tensor is equal to the product $\mathbf{J}^T \cdot \mathbf{J}$ of our modified Biot's $\mathbf{J}$ tensor introduced by Eq. (3). The result in Eq. (21) expressed in terms of Biot's original definitions m_{ij} and a_{ij} is given by

$$T_{ij}^C = \frac{\phi}{\rho_f} m_{ij} = \phi^2 a_{ki} a_{kj} \quad (22)$$

7 Practical Considerations

It is interesting to consider how the various terms introduced by the hypotheses of Coussy [1] and Biot [2] for averaging the microvelocity in an RVE of a saturated porous matrix might be measured. Consider first the microvelocity $\dot{\mathbf{w}}^{\text{micro}}$ of the pore fluid in an RVE. Local measurements of the microvelocity field $\dot{\mathbf{w}}^{\text{micro}}$ can be obtained using a microparticles imaging velocimetry (micro-PIV) approach. In this technology, an epifluorescent microscope is used to visualize fluorescing particles (several hundred nm in diameter). A laser light is directed through an objective lens that focuses on the point of interest and illuminates a regional volume. The fluorescence emission from moving particles, along with reflected laser light, shines back through the objective and through an emission filter that blocks the laser light. Two-dimensional planes are acquired at different depths within the sample to produce a 3D reconstruction of the microvelocity field. Optic choice is critical, requiring high numerical aperture objectives to capture the maximum emission light possible and a long z travel distance. A trabecular bone sample would require being decalcified (i.e., using formic acid) and cleared (by immersion on glycerin). When the $\dot{\mathbf{w}}^{\text{micro}}$ field is known, the value of $\dot{\mathbf{w}}$ for an RVE may be determined using Eq. (6) and an appropriate formula for $\langle \cdot \rangle$, e.g., Eq. (8).

Consider next the measurement of $\mathbf{T}^C$. Since the $\dot{\mathbf{w}}^{\text{micro}}$ field is known and, hence by Eq. (4), the value of $\dot{\mathbf{w}}$ is known, the first of Eq. (17) may be used to determine $\mathbf{T}^C$.

Alternatively, in the case when the velocity of the solid $\dot{\mathbf{u}}$ is zero, it follows from Eq. (16) that

$$\frac{K^f}{\Omega_p} = \frac{\rho^f}{2} \dot{\mathbf{w}} \cdot \mathbf{T}^C \cdot \dot{\mathbf{w}} \quad (23)$$

Since all the quantities in the integrand of this integral, except $\mathbf{T}^C$, are known, the measurement of the kinetic energy would allow $\mathbf{T}^C$ to be determined.

There does not appear to be a similar simple approach to the measurement of the terms introduced by the hypothesis of Biot [2] for averaging the microvelocity in an RVE of a saturated porous matrix. $\mathbf{M}$ can be determined using $\mathbf{T}^C = \mathbf{M}$, which is part of Eq. (21). It is also possible to solve Eq. (21) for part of $\mathbf{J}$ given $\mathbf{M}$; an orthogonal matrix component of $\mathbf{J}$ remains undetermined. From the matrix polar decomposition theorem, $\mathbf{J}$ may be written as a product of an orthogonal matrix $\mathbf{Q}$ and the positive definite matrix $\sqrt{\mathbf{M}}$; thus,

$$\mathbf{J} = \mathbf{Q} \cdot \sqrt{\mathbf{M}} = \mathbf{Q} \cdot \sqrt{\mathbf{J}^T \cdot \mathbf{J}} \quad (24)$$

$\mathbf{Q}$ represents a rotation matrix associated with the transformation between the principal axes of $\mathbf{M}$ and the reference coordinate system used for $\mathbf{J}$. The matrix $\mathbf{J}$ appears in Biot's formulation of poroelastic wave propagation [11] and may sometimes be considered to be a matrix of complex numbers. The result of Eq. (24) still applies and holds in the more general conditions when the matrix $\mathbf{Q}$ is unitary and the matrix $\sqrt{\mathbf{M}}$ is positive-semidefinite Hermitian matrix.

8 Fabric Dependence of Microvelocity Averaging

Biot [2] noted that the coefficients $\mathbf{J}$ in Eq. (3) depend upon the coordinates in the pores and the pore geometry. Recall that the RVE for poroelasticity is considered to have a length or size larger than the length scale of the pores (Fig. 2). It is assumed that the pores represent a lesser length scale that is sub-RVE. The length of the RVE is the length of the material structure over which the porous microstructure is averaged or "homogenized" in the process of forming a continuum model. In poroelasticity, the RVE is of sufficient size so that three sets of elastic constants (the drained and the undrained and those of the matrix material) may be represented as well as the porosity and the pore structure fabric tensor, $\mathbf{F}$. Pore structure fabric is a quantitative stereological measure of the degree of structural anisotropy in the pore architecture of a porous medium [14–34]. The governing equations for anisotropic poroelasticity were developed and extended to include the dependence of the constitutive relations upon pore structure fabric (Cowin [35,36]). Dynamic poroelasticity was extended by Cardoso and Cowin [37] and Cowin and Cardoso [38] to include the pore structure fabric tensor as a variable. The pore structure of the RVE is assumed to be characterized by porosity and a pore structure fabric tensor $\mathbf{F}$; thus, both $\mathbf{J}$ and $\mathbf{T}^C = \mathbf{M}$ are functions of $\mathbf{F}$ and ϕ . The representation of $\mathbf{T}^C$ as a function of $\mathbf{F}$ and ϕ is obtained from the Cayley–Hamilton theorem; thus,

$$\mathbf{T}^C = \mathbf{M} = g_1 \mathbf{1} + g_2 \mathbf{F} + g_3 \mathbf{F}^2 \quad (25)$$

where g_1 , g_2 , and g_3 are functions of μ , ϕ , II , and III , where II and III are the second and third invariants of $\mathbf{F}$; the first invariant is normalized to one. An analogous formula holds for $\sqrt{\mathbf{M}}$,

$$\sqrt{\mathbf{M}} = h_1 \mathbf{1} + h_2 \mathbf{F} + h_3 \mathbf{F}^2 \quad (26)$$

where h_1 , h_2 , and h_3 are again functions of μ , ϕ , II , and III . It follows from Eqs. (24) and (26) that

$$\mathbf{J} = \mathbf{Q} \cdot (h_1 \mathbf{1} + h_2 \mathbf{F} + h_3 \mathbf{F}^2) \quad (27)$$

In earlier work, the governing equations for quasistatic (Cowin [36]) and dynamic (Cowin and Cardoso [38]) linear theories of anisotropic poroelastic materials were developed and extended to include the dependence of the constitutive relations upon pore structure fabric (Cowin [35,36]). The dynamic tortuosity tensor $\mathbf{T}^D$ and the fabric tensor in the case of harmonic wave propagation are related by

$$\mathbf{T}^D = \frac{i\mu\phi}{\omega\rho_o} (j_1 \mathbf{1} + j_2 \mathbf{F} + j_3 \mathbf{F}^2) \quad (28)$$

9 Conclusion

With the objective of determining the best expression for the tortuosity of cancellous bone, the procedures for averaging the microvelocity over the RVE of poroelastic media by Coussy [1] and Biot [2] were investigated. A significant connection in Eq. (21) between these two procedures was established. Success was achieved in identifying the Coussy approach as the more attractive

expression for the tortuosity of cancellous bone and one for which the key variables might be measured.

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References

- [1] Coussy, O., 1995, *Mechanics of Porous Continua*, Wiley, New York, Chap. 2.
- [2] Biot, M. A., 1962, "Mechanics of Deformation and Acoustic Propagation in Porous Media," *J. Appl. Phys.*, **33**, pp. 1482–1498.
- [3] Parfitt, A. M., Mathews, C. H. E., Villanueva, A. R., Kleerekoper, M., Frame, B., and Rao, D. S., 1983, "Relationships Between Surface, Volume, and Thickness of Iliac Trabecular Bone in Aging and in Osteoporosis. Implications for the Microanatomic and Cellular Mechanisms of Bone Loss," *J. Clin. Invest.*, **72**, pp. 1396–1409.
- [4] Rehman, M. T., Hoyland, J. A., Denton, J., and Freemont, A. J., 1994, "Age Related Histomorphometric Changes in Bone in Normal British Men and Women," *J. Clin. Pathol.*, **47**(6), pp. 529–534.
- [5] Hildebrand, T., Laib, A., Müller, R., Dequeker, J., and Rüegsegger, P., 1999, "Direct Three-Dimensional Morphometric Analysis of Human Cancellous Bone: Microstructural Data From Spine, Femur, Iliac Crest, and Calcaneus," *J. Bone Miner. Res.*, **14**, pp. 1167–1174.
- [6] Glorieux, F. H., Travers, R., Taylor, A., Bowen, J. R., Rauch, F., Norman, M., and Parfitt, A. M., 2000, "Normative Data for Iliac Bone Histomorphometry in Growing Children," *Bone*, **26**(2), pp. 103–109.
- [7] Bear, J., 1972, *Dynamics of Fluids in Porous Media*, Dover, New York, Chap. 4.
- [8] Biot, M. A., 1941, "General Theory of Three-Dimensional Consolidation," *J. Appl. Phys.*, **12**, pp. 155–164.
- [9] Biot, M. A., 1956, "Theory of Propagation of Elastic Waves in a Fluid-Saturated Porous Solid. I. Low Frequency Range," *J. Acoust. Soc. Am.*, **28**, pp. 168–178.
- [10] Biot, M. A., 1956, "Theory of Propagation of Elastic Waves in a Fluid-Saturated Porous Solid. II. Higher Frequency Range," *J. Acoust. Soc. Am.*, **28**, pp. 179–191.
- [11] Biot, M. A., 1962, "Generalized Theory of Acoustic Propagation in Porous Dissipative Media," *J. Acoust. Soc. Am.*, **34**, pp. 1254–1264.
- [12] Carman, P. C., 1937, "Fluid Flow Through Granular Beds," *Trans. Inst. Chem. Eng.*, **15**, pp. 150–156.
- [13] Norris, A. N., 1986, "On the Viscoelastic Operator in Biot's Equations of Poroelasticity," *J. Wave-Mater. Interact.*, **1**, pp. 365–380, <http://rci.rutgers.edu/~norris/papers.html>
- [14] Costa, U. M. S., Andrade, J. S., Makse, H. A., and Stanley, H. E., 1999, "Inertial Effects on Fluid Flow Through Disordered Porous Media," *Phys. Rev. Lett.*, **82**, pp. 5249–5252.
- [15] Johnson, D. L., Koplik, J., and Dashen, R., 1987, "Theory of Dynamic Permeability and Tortuosity in Fluid-Saturated Porous Media," *J. Fluid Mech.*, **176**, pp. 379–402.
- [16] Perrot, C., Chevillotte, F., Panneton, R., Allard, J. F., and Lafarge, D., 2008, "On the Dynamic Viscous Permeability Tensor Symmetry," *J. Acoust. Soc. Am. Express Lett.*, **124**, pp. 210–217.
- [17] Darcy, H., 1856, *Les Fontaines Publiques de la Ville de Dijon*, Dalmont, Paris.
- [18] Hilliard, J. E., 1967, "Determination of Structural Anisotropy," *Stereology—Proc. 2nd Int. Congress for Stereology, Chicago, 1967*, Springer, Berlin, p. 219.
- [19] Whitehouse, W. J., 1974, "The Quantitative Morphology of Anisotropic Trabecular Bone," *J. Microsc.*, **101**, pp. 153–168.
- [20] Whitehouse, W. J. and Dyson, E. D., 1974, "Scanning Electron Microscope Studies of Trabecular Bone in The Proximal End of the Human Femur," *J. Anat.*, **118**, pp. 417–444, <http://www.ncbi.nlm.nih.gov/pmc/articles/PMC1231543/>
- [21] Oda, M., 1976, "Fabrics and Their Effects on the Deformation Behaviors of Sand," Special Report, Dept. of Foundation Engr., Saitama University, Saitama City, Japan, p. 59.
- [22] Oda, M., Konishi, J., and Nemat-Nasser S., 1980, "Some Experimentally Based Fundamental Results on the Mechanical Behavior of Granular Materials," *Geotechnique*, **30**, pp. 479–495.
- [23] Oda, M., Nemat-Nasser, S., and Konishi, J., 1985, "Stress Induced Anisotropy in Granular Masses," *Soils Found.*, **25**, pp. 85–97.
- [24] Cowin, S. C., and Satake, M., eds., 1978, *Continuum Mechanical and Statistical Approaches in the Mechanics of Granular Materials*, Gakujutsu Bunken Fukyu-Kai, Tokyo.
- [25] Satake, M., 1982, "Fabric Tensor in Granular Materials," *Deformation and Failure of Granular Materials*, P. A. Vermeer and H. J. Luger, eds., Balkema, Rotterdam, pp. 63–68.
- [26] Kanatani, K., 1983, "Characterization of Structural Anisotropy by Fabric Tensors and Their Statistical Test," *J. Jpn. Soil Mech. Found. Eng.*, **23**, pp. 171–177.
- [27] Kanatani, K., 1984, "Distribution of Directional Data and Fabric Tensors," *Int. J. Eng. Sci.*, **22**, pp. 149–164.
- [28] Kanatani, K., 1984, "Stereological Determination of Structural Anisotropy," *Int. J. Eng. Sci.*, **22**, pp. 531–546.
- [29] Kanatani, K., 1985, "Procedures for Stereological Estimation of Structural Anisotropy," *Int. J. Eng. Sci.*, **23**(5), pp. 587–598.
- [30] Harrigan, T. P., Jasty, M., Mann, R. W., and Harris, W. H., 1988, "Limitations of the Continuum Assumption in Cancellous Bone," *J. Biomech.*, **21**, pp. 269–275.
- [31] Odgaard, A., 1997, "Three-Dimensional Methods for Quantification of Cancellous Bone Architecture," *Bone*, **20**, pp. 315–328.
- [32] Odgaard, A., Kabel, J., van Rietbergen, B., Dalstra, M., and Huiskes, R., 1997, "Fabric and Elastic Principal Directions of Cancellous Bone are Closely Related," *J. Biomech.*, **30**, pp. 487–495.
- [33] Odgaard, A., 2001, "Quantification of Cancellous Bone Architecture," *Bone Mechanics Handbook*, S. C. Cowin, ed., CRC, Boca Raton, FL.
- [34] Matsuura, M., Eckstein, F., Lochmüller, E. M., and Zysset, P. K., 2008, "The Role of Fabric in the Quasi-Static Compressive Mechanical Properties of Human Trabecular Bone From Various Anatomical Locations," *Biomech. Model. Mechanobiol.*, **7**, pp. 27–42.
- [35] Cowin, S. C., 1985, "The Relationship Between the Elasticity Tensor and the Fabric Tensor," *Mech. Mater.*, **4**, pp. 137–147.
- [36] Cowin, S. C., 2004, "Anisotropic Poroelasticity: Fabric Tensor Formulation," *Mech. Mater.*, **36**, pp. 666–677.
- [37] Cardoso, L., and Cowin, S. C., 2011, "Fabric Dependence of Quasi-Waves in Anisotropic Porous Media," *J. Acoust. Soc. Am.*, **129**, pp. 3302–3316.
- [38] Cowin, S. C., and Cardoso, L., 2011, "Fabric Dependence of Poroelastic Wave Propagation in Anisotropic Porous Media," *Biomech. Model. Mechanobiol.*, **10**, pp. 39–65.