Published in final edited form as:

Exp Mech. 2014 February 1; 54(2): 187–198. doi:10.1007/s11340-013-9778-6.

Contact-coupled impact of slender rods: analysis and experimental validation

Ira B. Tibbitts², Deepika Kakarla¹, Stephanie Siskey³, Jorge A. Ochoa⁴, Kevin L. Ong³, and Rebecca M. Brannon²

¹Ortho Development Corp., 12187 Business Park Dr., Draper, UT 84020

²University of Utah, Dept. of Mechanical Engineering, 2134 MEB, 50 S Central Campus Dr., Salt Lake City, UT 84112

³Exponent Inc., 3401 Market St., Suite 300, Philadelphia, PA, 19104

⁴Exponent Inc., 15375 SE 30th Place, Suite 250, Bellevue, WA 98007

Abstract

To validate models of contact mechanics in low speed structural impact, slender rods were impacted in a drop tower, and measurements of the contact and vibration were compared to analytical and finite element (FE) models. The contact area was recorded using a novel thin-film transfer technique, and the contact duration was measured using electrical continuity. Strain gages recorded the vibratory strain in one rod, and a laser Doppler vibrometer measured speed. The experiment was modeled analytically on a one-dimensional spatial domain using a quasi-static Hertzian contact law and a system of delay differential equations. The three-dimensional FE model used hexahedral elements, a penalty contact algorithm, and explicit time integration. A small submodel taken from the initial global FE model economically refined the analysis in the small contact region. Measured contact areas were within 6% of both models' predictions, peak speeds within 2%, cyclic strains within 12 $\mu\epsilon$ (RMS value), and contact durations within 2 μs . The global FE model and the measurements revealed small disturbances, not predicted by the analytical model, believed to be caused by interactions of the non-planar stress wavefront with the rod's ends. The accuracy of the predictions for this simple test, as well as the versatility of the diagnostic tools, validates the theoretical and computational models, corroborates instrument calibration, and establishes confidence that the same methods may be used in experimental and computational study of contact mechanics during impact of more complicated structures. Recommendations are made for applying the methods to a particular biomechanical problem: the edge-loading of a loose prosthetic hip joint which can lead to premature wear and prosthesis failure.

Keywords

Contact; finite element analysis; submodeling; experimental validation; edge loading; hip prosthesis

4 Introduction

The axial impact of two slender rods involves an approximately 1D domain, which is simple compared to the 3D domain typical of impacting structures in applications. Even so, basic capability for analyzing and measuring the effects of impact between slender rods can be a

¹Corresponding author: Anthony P. Sanders, tsanders@odev.com, phone: 801-619-3436, fax: 801-619-8936.

technology platform for studying the impact of more complicated 3D structures, because both involve numerous details, such as material properties, contact mechanics, and wave mechanics, which may be dealt with analytically and experimentally using similar techniques in both domain types. A difficulty in common between both domain types is that key phenomena of interest may occur on greatly differing scales, such as contact mechanics that generally encompass the local scale proximate to the impact point in contrast with stress wave mechanics that involve the global scale of both entire bodies. Difficulty arises because analysis and measurement techniques well-suited to one scale may be ill-suited to the other. Accordingly, the overall aim of this study is to demonstrate, by way of a two-rod impact problem, techniques for studying structural impact for more complex 3D structures that are effective for both analysis and measurement, on both the local and global scales.

Classical impact analysis involving slender rods has elucidated the basic characteristics of stress wave propagation and reflection [1]. Laplace transformation has been applied to solve the differential equations of motion for an impacted rod having various boundary conditions [2]. Numerical integration has been applied to solve the equations of motion in two axially impacting rods, with the contact force described by a Hertzian contact law, yielding a prediction of the contact force as a function of time [3]. Results from a substructure analysis demonstrated close agreement with results from an analytical solution for axial impact of a rigid body on a flexible rod [4]. Modal analysis combined with a Hertzian contact law has predicted post-impact dynamic behavior of impacted rods and beams [5,6]. Finite element analysis (FEA) has been applied to the case of a steel sphere impacting an aluminum rod, results agreeing closely with those from continuous analytical models and laboratory experiments [7,8]. FEA has also produced close fidelity with analytical results (based on stress wave theory) in a problem of two impacting rods [9].

A chief example of an experiment involving impact between slender rods is the split Hopkinson pressure bar (SHPB) technique [10], which has been widely used to determine dynamic material properties. Yet, the apparatus and its underlying theory are not commonly applied in the study of local contact mechanics in situations where the contact area is small relative to the impacting surfaces, which is a frequent practical situation. Furthermore, the simplicity of the SHPB technique is a severe limitation because its theory and apparatus cannot generally be applied to impact of more complicated, practical engineering structures. Impact of a compact striker upon beams and plates has also been studied [11,12]. In these studies, a frequent aim has been to determine the contact force as a function of time, which is typically accomplished indirectly, using a combination of analytical and experimental results, since direct measurement would interfere with the contact itself. The contact duration has been directly measured experimentally, e.g., via electrical continuity between the striker and the structure [13]. Yet, for more thorough validation of accompanying models that predict the contact mechanics (e.g., force and stress histories) on the local scale, additional direct measurements are desirable. Further, studies involving impact by a compact striker often neglect the influence of stress waves in the striker [13,14], a simplification likely to be unwarranted in the impact of practical structures.

Approaches such as FEA are useful for analyzing impact between structures more complicated than simple rods, beams, and plates. However, when the focus of study is the contact mechanics local to the point of impact, sufficient element mesh refinement in the case of small contact dimensions, e.g. Ø1 mm, may entail very small elements, e.g. 10 µm. The prospect of such tiny elements on much larger bodies raises doubts about computational efficiency, but the submodeling technique in FEA provides an efficient approach when the important results depend on both global and local-scale accuracy. In submodeling, the results of an initial global model using coarse elements are applied as boundary conditions to a smaller model (a *submodel*) of an area of interest, the submodel having a greatly refined

mesh. Submodeling has been applied in FEA where a contact area of interest was a small portion of a larger model [15,16]. Accuracy with this technique depends upon proper transference of boundary conditions from the global model to the submodel, and it has been recommended that multiple submodels with successively refined meshes be employed to study convergence [15]. The benefits of using a submodeling approach can be improved accuracy within the region of interest and reduced computation time; a tradeoff can be increased effort required for model setup.

This article describes an analysis and laboratory validation of axial impact between two slender rods, undertaken to develop the FEA simulation and experimental techniques needed to determine the contact mechanics in a complicated biomechanical scenario. Specifically, a prosthetic hip joint may experience impact loading between non-conforming surfaces when the joint connection is loose and allows the ball to slide partly out of the socket [17]. The impact loading can occur when heel-strike forces the dislocated ball into the edge of the socket (before the ball relocates into its intended, fully seated position). An understanding of the contact conditions arising during the brief moment of socket edge loading is important because the edge loading may cause premature wear of the prosthetic joint and eventual prosthesis failure [18,19]. When both the socket and ball are made of ceramic or metallic materials, the contact area during edge loading is likely to be very small relative to the implants and the bones in which they reside [20]. As a precursor to simulating this physiologic scenario using FEA and validating the model experimentally, we performed simulations and experiments of impacting rods with convex tips designed to generate small contact areas. A continuous analytical model (CAM) coupling the stress wave and contact mechanics of the rods was derived, and on its basis, a laboratory experiment was designed for the primary objective of validating a finite element model (FEM) for the same impact problem. The FEM used submodeling to efficiently resolve the contact mechanics. Both models were experimentally validated on the global scale using strain gages and a laser Doppler vibrometer; and on the local scale by measuring the contact timing via electrical continuity, and the contact area using a novel “fingerprinting” technique [20]. The results show close agreement among the two models and the laboratory measurements, considering both local-scale (contact area and timing) and global-scale (speed and strain) effects. Implications for subsequent application of the developed techniques to the hip joint impact problem are discussed.

5 Theory

5.1 Continuous analytical model (CAM)

In the CAM, Rod 1 (Fig. 1) with speed s_1 travels along a straight line and impacts Rod 2, which is initially stationary. The rods are parallel, and the impact is centric, meaning that the point of impact is on each rod's axis. Rod 1 has diameter d_1 , a spherical tip with radius r_1 at the impacted end, a flat distal end, and length L_1 (measured from the apex of r_1). Further, Rod 1 has density ρ_1 , elastic modulus E_1 , and Poisson's ratio ν_1 , and it is assumed to be homogeneous, linear elastic, and isotropic. Rod 1's material coordinate, x_1 , is measured along its axis inward from the apex of r_1 . Rod 2 is defined likewise.

The elementary approach to formulating the equation of motion in each rod is to assume that stress components other than the axial one (σ_x) are negligible and that σ_x is uniform across the rod's cross section [1]. The condition of dynamic equilibrium applied to an infinitesimally thin cross sectional element of Rod 2 then yields a 1D wave equation:

$$\frac{\partial^2 u_2}{\partial t^2} = c_2^2 \frac{\partial^2 u_2}{\partial x_2^2} \quad (1)$$

Here, u_2 denotes the displacement of the thin section, and $c_2 = \sqrt{E_2/\rho_2}$ is the speed of the uniform, planar displacement wave as it travels along the rod. Similar equations apply to Rod 1. The general solution to Eqn. (1) is due to d'Alembert [21], and we use the form given by Drumheller [22]:

$$u_2(x_2, t) = f_2\left(t - \frac{x_2}{c_2}\right) + g_2\left(t + \frac{x_2}{c_2}\right) \quad (2)$$

This solution represents two waves, one traveling rightward given by unknown function $f_2(\cdot)$ and one traveling leftward given by unknown function $g_2(\cdot)$ [1]. A similar equation applies to Rod 1, although with a reversal of the left/right description since x_1 is defined as positive leftward.

The stress at the free end of Rod 2 is always zero, and likewise the strain. The strain boundary condition is written as:

$$\left. \frac{\partial u_2}{\partial x_2} \right|_{x_2=L_2} = -f_2'\left(t - \frac{L_2}{c_2}\right) + g_2'\left(t + \frac{L_2}{c_2}\right) = 0 \quad (3)$$

$$g_2'\left(t + \frac{L_2}{c_2}\right) = f_2'\left(t - \frac{L_2}{c_2}\right)$$

Here, the prime symbol (') denotes differentiation with respect to the entire function argument. Since this expression holds throughout time, it holds at an offset instant, t_0 , chosen as $t_0 = t - L_2/c_2$ to simplify Eqn. (3):

$$g_2'\left[\left(t - \frac{L_2}{c_2}\right) + \frac{L_2}{c_2}\right] = f_2'\left[\left(t - \frac{L_2}{c_2}\right) - \frac{L_2}{c_2}\right] \quad (4)$$

$$g_2'(t) = f_2'\left(t - \frac{2L_2}{c_2}\right)$$

At the impacted end ($x_2=0$), the strain is related to the stress via Hooke's law:

$$\left. \frac{\partial u_2}{\partial x_2} \right|_{x_2=0} = f_2'(t) \left(-\frac{1}{c_2}\right) + g_2'(t) \left(\frac{1}{c_2}\right) = \frac{1}{E_2} \frac{F(t)}{A_2} \quad (5)$$

$$f_2'(t) = g_2'(t) - \frac{c_2 F(t)}{E_2 A_2}$$

Here, $F(t)$ is a function representing the contact force during impact, and A_2 is the cross-sectional area of Rod 2. This relation is written with the assumption that the contact force is uniformly distributed across the impacted end, which is a simplification in the case of rounded ends. Rationale for this assumption comes from St. Venant's principle, from which it is assumed that in a long, slender bar, only a small portion of the bar close to the impacted end experiences a non-uniform stress distribution. This simplification notwithstanding, the contact force is related to the displacement at the impacted end by a Hertzian contact law [23]:

$$F(t) = -K[\delta(t)]^{\frac{1}{2}} \quad (6)$$

Here, K is the contact stiffness (addressed below), and $\delta(t)$ is the positive-valued compression due to impact; the negative sign is employed so that positive (inward) compression will yield negative (compressive) force and stress. The compression $\delta(t)$ is the difference in rod displacements at the impacted ends:

$$\begin{aligned}\delta(t) &= -u_1(0, t) - u_2(0, t) \\ &= -f_1(t) - g_1(t) - f_2(t) - g_2(t)\end{aligned}\quad (7)$$

This relation assumes that across each rod's impacted end, the displacement is uniform, notwithstanding the use of a Hertzian contact law.

In Rod 1, the rate of strain is given by differentiating with respect to position and time:

$$\dot{\epsilon}_1(x_1, t) = \frac{\partial u_1}{\partial x_1 \partial t} = -\frac{1}{c_1} f_1'' \left(t - \frac{x_1}{c_1} \right) + \frac{1}{c_1} g_1'' \left(t + \frac{x_1}{c_1} \right) \quad (8)$$

Since the strain is always zero at the distal end (where $x_1 = L_1$), so is the strain rate, which gives:

$$\dot{\epsilon}_1(L_1, t) = 0 = -\frac{1}{c_1} f_1'' \left(t - \frac{L_1}{c_1} \right) + \frac{1}{c_1} g_1'' \left(t + \frac{L_1}{c_1} \right) \quad (9)$$

Applying a time-offset technique (like that applied to simplify Eqn. (4)) simplifies Eqn. (9):

$$g_1''(t) = f_1'' \left(t - \frac{2L_1}{c_1} \right) \quad (10)$$

Evaluating Eqn. (8) at the impacted end (where $x_1 = 0$) yields:

$$\begin{aligned}\dot{\epsilon}_1(0, t) &= -\frac{1}{c_1} f_1''(t) + \frac{1}{c_1} g_1''(t) \frac{F'(t)}{E_1 A_1} \\ f_1''(t) &= g_1''(t) + \frac{c_1 K}{E_1 A_1} \frac{3}{2} [\delta(t)]^{\frac{1}{2}} \dot{\delta}(t)\end{aligned}\quad (11)$$

The rate of compression in Eqn. (11) is derived from Eqn. (7) and is given by:

$$\dot{\delta}(t) = -f_1'(t) - g_1'(t) - f_2'(t) - g_2'(t) \quad (12)$$

The governing differential equations of the system are thus given by Eqns. (11), (10), (5) (with Eqn. (6)), (4), and (12), arranged as follows:

$$\begin{aligned}f_1''(t) &= g_1''(t) + \frac{3c_1 K}{2E_1 A_1} \sqrt{\delta(t)} \dot{\delta}(t) \\ g_1''(t) &= f_1'' \left(t - \frac{2L_1}{c_1} \right) \\ f_2'(t) &= g_2'(t) + \frac{c_2 K}{E_2 A_2} [\delta(t)]^{\frac{3}{2}} \\ g_2'(t) &= f_2' \left(t - \frac{2L_2}{c_2} \right) \\ \dot{\delta}(t) &= -f_1'(t) - g_1'(t) - f_2'(t) - g_2'(t)\end{aligned}\quad (13)$$

The initial conditions for this system are obtained from the rods' states at the instant of impact, $t=0$. Since at $t=0$, neither rod has yet any impact-induced displacements, the initial conditions are:

$$f_1(0)=0, \quad g_1(0)=0, \quad f_2(0)=0, \quad g_2(0)=0 \quad (14)$$

The velocity of Rod 1 at $t=0$ is constant, $-s_1$, along its entire length. This may be written as the rate of displacement in this rod:

$$\frac{\partial u_1(x_1, 0)}{\partial t} = f_1' \left(0 - \frac{x_1}{c_1} \right) + g_1' \left(0 + \frac{x_1}{c_1} \right) = -s_1 \quad (15)$$

Equation (15) gives the velocity of Rod 1 as a sum of contributions from $f_1'(-x_1)$ and $g_1'(x_1)$. Since Rod 1 has no negative-valued material coordinates, it is deduced that only $g_1'(x_1)$ contributes to Rod 1's velocity at the instant of impact. From this come the remaining initial conditions:

$$f_1'(0)=0, \quad g_1'(0)=-s_1 \quad (16)$$

The Hertzian stiffness in Eqn. (6) is computed from material properties and the contacting radii [24]:

$$K = \frac{4}{3} E^* \sqrt{R^*} \quad (17)$$

Where the effective modulus, E^* , and the effective radius, R^* , are:

$$E^* = \left(\frac{1-v_1^2}{E_1} + \frac{1-v_2^2}{E_2} \right)^{-1} \quad \text{and} \quad R^* = \frac{r_1 r_2}{r_1 + r_2} \quad (18)$$

Various kinematic quantities may be determined using derivatives of the wave equations in each rod. For instance, the velocity at a point on Rod 2 is found by differentiating Eqn. (2) with respect to time:

$$\dot{u}_2(x_2, t) = \frac{\partial u_2}{\partial t} = f_2' \left(t - \frac{x_2}{c_2} \right) + g_2' \left(t + \frac{x_2}{c_2} \right) \quad (19)$$

Evaluating this expression at the midpoint gives the velocity as a function of time at $x_2=L_2/2$:

$$\dot{u}_2 \left(\frac{L_2}{2}, t \right) = f_2' \left(t - \frac{L_2}{2c_2} \right) + g_2' \left(t + \frac{L_2}{2c_2} \right) \quad (20)$$

Applying Eqn. (4) and a suitable time offset, this may also be written entirely in terms of $f_2'(\cdot)$:

$$\begin{aligned} g_2' \left(t + \frac{L_2}{c_2} \right) &= f_2' \left[\left(t - \frac{2L_2}{c_2} \right) + \frac{L_2}{c_2} \right], \text{ so} \\ \dot{u}_2 \left(\frac{L_2}{2}, t \right) &= f_2' \left(t - \frac{L_2}{2c_2} \right) + f_2' \left(t - \frac{3L_2}{2c_2} \right) \end{aligned} \quad (21)$$

This equation is interpreted to mean that the velocity of Rod 2 at point $x_2=L_2/2$, at any instant t , is the sum of two time-delayed values of $f_2'(t)$, one delayed by $L_2/2c_2$ and the other by $3L_2/2c_2$. Therefore, once the differential equations in Eqn. (13) have been solved, the velocity at this material point may be computed by summing the appropriate time-delayed values of $f_2'(t)$ from a table of results across the entire solution period. When the time delay points to a negative time instant, the function value is taken as zero. For a second example, the strain at the midpoint of Rod 2 is given by:

$$\begin{aligned}\varepsilon_2\left(\frac{L_2}{2}, t\right) &= -\frac{1}{c_2} f_2'\left(t - \frac{L_2}{2c_2}\right) + \frac{1}{c_2} g_2'\left(t + \frac{L_2}{2c_2}\right) \\ &= \frac{1}{c_2} \left[f_2'\left(t - \frac{3L_2}{2c_2}\right) - f_2'\left(t - \frac{L_2}{2c_2}\right) \right]\end{aligned}\quad (22)$$

5.2 Solution method

Solutions to Eqns. (13) with initial conditions (14) and (16) were computed using numerical integration using the commercial software Simulink (Mathworks, Natick, MA). In this software, the solver was constructed in a block-diagram format, wherein the variables such as $f(\cdot)$ and $f'(\cdot)$ were represented as signals, and operations such as integration and time-shifting were performed by modular function blocks (Fig. 2). Integration was performed using the MATLAB function *ode45*, which implements an explicit Runge-Kutta method using the Dormand-Prince (4,5) pair [25]. The time delay was implemented using the software's *Transport Delay* function block; this block creates a memory buffer that stores prior values of its input, and for its output, it provides values interpolated from the buffer that are offset from the current time step by a specified time delay. Besides integrating the governing equations, the program computed kinematic quantities such as those in Eqns. (20) and (22).

5.3 Contact analysis

The local-scale quantities of interest were the contact dimensions and the contact stresses. For the circular contacts studied in this work, the contact radius, a , and the peak contact pressure within the contact patch, P_0 , relate to the contact force as follows [24]:

$$a = \sqrt[3]{\frac{3FR^*}{4E^*}} \quad P_0 = \frac{3F}{2\pi a^2} \quad (23)$$

To compute the components of contact stress, analytical formulas [26] were implemented in a custom MATLAB program. Further, the experiments in this work were designed to maintain contact stresses within the material's linear elastic region, which is an assumption of Hertzian contact theory. Thus, we used the criterion $P_0 < 1.1S_y$, where S_y is the uniaxial yield stress [26]. Based on the Tresca theory of plastic yield, this criterion avoids plastic stresses in the subsurface location where shear stresses are maximal.

5.4 Finite element model

The global FEM was a 3D model completely encompassing both rods (rod details below, in Sec. 6.1). The meshes were generated using HyperMesh (Altair, Troy, MI) and consisted of hexahedral elements with an axial edge length of 1.0 mm and an average cross-section edge length of 0.25 mm (Fig. 3a). The submodel included only the first 14 mm (measured from the impact tips) of both rods. This length corresponded with the x coordinates in both rods where the stresses remained uniformly at near-zero magnitude at the instant of peak contact force in the global model results. Three mesh refinements – *Coarse*, *Mid*, and *Fine* (Fig. 3)

were used to examine submodel convergence. In each, the element aspect ratio was approximately 1:1:1, and the average element edge length was successively halved in the refinement steps. The Fine submodel was simplified by using a half-symmetry model. Basic details describing all models are in Table 1. The material model was linear elastic to represent the steel from which the rods were made (Sec. 6.1). Contact constraints were implemented using a penalty algorithm. The finite element solver Abaqus/Explicit (Abaqus v. 6.8, Simulia, Providence, RI) was used for computation. The 8-node linear hexahedral element type with uniform strain and hourglass control (C3D8R; reduced integration element) was used.

6 Experimental techniques

6.1 Specimen configuration

The physical configuration of the study is summarized in Table 2. The particular design resulted from iterative evaluation of Eqns. (13) (using the solver), using varied values of the rod dimensions and impact speed, to ensure that the contact stress criterion $P_0 < 1.1S_y$ would be met and that measured responses (strain and vibration speed) would be within the ranges of the sensors (which are described below). Both rods were made from a single lot of precision-ground A2 tool steel drill rod. The impact tips were lapped to form and polished to a mirror-like finish. There were three specimens of Rod 1, and one of Rod 2. The rods were hardened and tempered to R_c 60. Their density was determined from the volume and the mass of a Rod 1 specimen whose length was measured using a height gauge, diameter using a micrometer, and mass using an analytic balance. The elastic properties were measured using the impulse excitation method, ASTM E 1876, using a Grindosonic MK5 instrument (Lemmens, Lueven, BLG). The impact speed in Table 2, 2.197 m/s, was a value recorded during an individual experimental trial.

6.2 Equipment

The experiments were performed in a drop-tower impact test machine (Dynatup 8250, Instron, Massachusetts, USA) with the rods oriented vertically (Fig. 4). The machine provided a motorized latch block that suspended a sled. Upon computer command, the sled could be released from the latch block into a free fall guided by twin columns. To the sled was mounted a tubular fixture that suspended Rod 1 (named per Fig. 1); the rod was spaced off the fixture's interior walls by 2 oiled, lightly compressed o-rings. The rod's weight was suspended by a thin ring of tape whose diameter was slightly greater than the tube's inner diameter. Otherwise, Rod 1 was distally unconstrained. Rod 2 was suspended in a tubular fixture attached to the base of the test machine; this fixture also spaced its rod from the interior walls via oiled o-rings. Each fixture provided ~6 cm of clearance past its rod's distal end, space into which the rod could slide freely after impact. Rod 2 was partly suspended by friction from its oiled o-rings, and partly supported on its distal end by a polyethylene plug lightly press fit into the tube; the plug could fall freely into the fixture's clearance space when Rod 2 was impacted. The position of Rod 2's fixture was adjustable, which permitted manual alignment of the rods to achieve parallel, centric impact. The velocity of the sled was measured using an infrared sensor (fixed to the drop tower) that sensed passage of a flag mounted to the sled. The sensor was positioned to detect velocity at the impact position, and it was assumed that Rod 1's velocity was equal to the sled's velocity. Both rods were wired into an electrical circuit by taping to each a 24 gauge multi-filament wire. The circuit charged Rod 1 to ~3 V relative to Rod 2 using an electrical power supply. Continuity between the rods during impact created a voltage across a 10 k Ω resistor, and the voltage was used to trigger data acquisition and to measure the duration of impact.

Measurements were taken using the following equipment. The velocity of Rod 2 was measured at its midpoint using a 3D laser Doppler vibrometer (CLV-3D, Polytec, Germany). The vibrometer provided 3 separate, orthogonal velocity components, but it was oriented perpendicular to the rod's surface, and only the component parallel to the rod's surface was recorded. Strain was measured at Rod 2's midpoint using two 1000 Ω foil strain gages (WC-06-125AC-W/C, MicroMeasurements, Raleigh, NC, USA), oriented to measure axial strain and wired into opposing arms of a Wheatstone bridge. This circuit design doubled the bridge sensitivity compared to a circuit with only one active gage. The bridge was powered and its output signal was amplified using a high bandwidth signal conditioner (2310B, MicroMeasurements). Use of 1000 Ω gages permitted maximal excitation of the bridge (15 V); so, the amplifier gain was set relatively low (~ 110), which enhanced the amplifier's frequency response quality (-3 dB bandwidth of 230 kHz). The bridge and amplifier were calibrated using a shunt calibration procedure [27]. The three measurement signals were recorded at 443 kHz using a 16 bit analog-to-digital (A/D) converter (USB1604HS, Measurement Computing, Norton, MA, USA) controlled by a laptop computer. To enhance A/D accuracy, the input range of each A/D channel was programmed to limits just greater than the maximal signal value; thus, the ranges for velocity and strain were ± 10 V, and ± 0.5 V, respectively.

The contact area was recorded using a "fingerprinting" technique [28]. The tip of Rod 2 was wiped with a cotton swab to give it a thin coat of bearing grease. The grease layer was wiped repeatedly (16 times), each time using a clean section of a paper towel, to evenly distribute the layer and to reduce its thickness. After this wiping, the remaining grease was barely discernible to the naked eye at the border between greased and ungreased portions of the surface. The tip of Rod 1 was cleaned with warm, soapy water and thoroughly rinsed. After impact, the entire tip of Rod 1 was sprinkled liberally with black photocopier toner powder. The powder was blown off with low pressure compressed air from an aerosol can. Afterwards, some powder remained (the "fingerprint") where it adhered to a thin layer of grease that had been transferred from the greased tip of Rod 2 during the impact test. The adherent powder was then microscopically measured and photographed using an optical coordinate measuring machine ('CMM', Nexiv VMR 3020, Nikon, JPN). The CMM detected edge points by analyzing contrast levels in the digital image of the contact patch; 64 points were found at uniform spacing around the patch's perimeter, and these were used to compute the radius and circularity of a best-fit circle.

7 Results

The fingerprinting technique provided a clear record of the contact patch, which agreed closely in size with the analyses' predictions (Fig. 5, Table 3, Fig. 6). In 2 trials of each Rod 1 specimen, the radius predicted by the CAM averaged 3.2% less than measured. The measured patches' circularity (as defined by ASME Y14.5M) was 3–4% of the diameter with Specimens 1 and 3 (of Rod 1), but 5–6% with Specimen 2, perhaps indicating that Specimen 2 had slightly greater form error to its spherical tip. (By definition, circularity of the contact patch was zero in the CAM.) The finite element analyses were performed for a single instance of the impact speed (2.197 m/s), and the predicted maximum contact area approached the measured value (3.86 mm²) with refinement of the submodel (Table 4, Fig. 7). Likewise, the maximum FEM contact force approached the value predicted by the CAM; however, the Global FEM showed the least error (0.1%) with respect to the CAM-predicted contact force.

The measured speed, strain, and contact duration all exhibited close agreement with the CAM and the Global FEM; a typical comparative result from an early portion of the data is shown in Fig. 8. In Fig. 8a and 8b, the vertical gray lines indicate the instant of impact; also,

in Fig. 8b, the experiment's strain signal exhibits small spikes (indicated by arrows) at the start and end of the impact event. These spikes were perhaps due to momentary imbalance in current leaking from Rod 2 into each strain gage when the trigger circuit was closed and opened according to rod contact. In future studies, such spikes could be useful as indicators of contact duration. The peak speed at the midpoint of Rod 2, consisting of values 1.0 mm/s , was averaged over the first five plateaus in the record; the maximum difference from the CAM was 2.2% (Table 5, Fig. 9). The root-mean-square (RMS) of the difference in strain was computed over the first five periods; the maximum RMS difference from the CAM was $11.6 \mu\epsilon$ (Table 5). The trigger voltage demarcated the duration of impact (Fig. 8c), and the result was within $2 \mu\text{s}$ of that from both analyses. The data acquisition rate of 443 kHz was ample for high fidelity measurements (Fig. 8d). In later portions of the data, the Global FEM and experimental results showed a phenomenon not present in the CAM results, namely small ripples, which grew in magnitude with time, on the plateaus of the speed and strain records (Fig. 10).

The FEM contact stresses were evaluated to examine local results of interest and to further assess results convergence (Fig. 11). The maximum principal stress was examined in elements located on the impacted tip surface of Rod 2, along a radial line from the axis to the outer surface. The stress was taken at the instant of maximum contact force and from the element integration points located closest to the surface (beneath the surface, not directly on it). These stresses were compared to the same computed using analytical formulas [26] at identical points and using the maximum contact force (5,543 N) from the CAM, determined for the same impact speed (2.197 m/s) used in the FEMs. The integration points approached the surface with decreasing element size; hence, the CAM results differ among the graphs in Fig. 11 because the computation points were closer to the surface with successively smaller elements. The errors (of FEM vs. CAM) appeared to trend smaller with mesh refinement. However, the RMS value of error over the evaluated line increased from the Global FEM to the Coarse submodel, and numerical anomalies that arose in the Fine submodel (Fig. 11) led to a 1 MPa increase in RMS error from the Mid to the Fine submodel.

8 Discussion

The experimental results showed that both global models (CAM and Global FEM) gave high fidelity predictions of the structural response. The CAM predicted the peak Rod 2 midpoint speed within 0.7–2.2% error (Table 5). Likewise, the CAM's strain prediction had an RMS error (over 5 periods of the vibration) of 3.4–11.6 $\mu\epsilon$ (Table 5). The Global FEM yielded similar fidelity, with the plots of its results nearly overlying those from the CAM (Fig. 8). The error of both models' contact duration predictions was 1–2 μs , which was on the order of the data sampling period. Similarly accurate predictions of global structural response, from both analytical and FE models, have been reported for the case of a ball striking a long rod [14,23,8], though without direct measurement of the contact duration and contact area. The small ripples observed in the experiment and FEM strain and speed records (Fig. 10) may have been effects from the interaction of stress waves with the ends of Rod 2. The CAM modeled the stress as propagating in a 1D domain, meaning that it was assumed to be uniform on a cross-section of the rod (which is justified on the basis of St. Venant's Principle and mostly validated by the results), and no such ripples developed with time. However, judging by the small size of the contact patch relative to the rod diameter (Fig. 5a) and by stress contours from the FEM (Fig. 12), the stress entered the rod non-uniformly, thus it would not have been entirely uniform across a section, even one distant from the impacted tip. It is hypothesized that 3D effects occurred when the non-uniform stress waves reflected from the rod ends in the experiment and the FEM and that these effects magnified with successive reflections, causing the small strain and speed ripples that grew with time.

The chief aim of submodeling was to provide a refined FEA focused on the contact mechanics. The peak contact radius during impact was ~ 1 mm, so the $0.25 \times 0.25 \times 1.0$ mm elements in the Global FEM (Fig. 3a) were expected to yield relatively coarse resolution of the contact stress and area. The use of three submodel mesh refinements with successively halved element lengths has been previously recommended [15]. In the Coarse submodel, the contact force magnitude decreased by 330 N (-6%) from the Global model; the change occurred because mesh refinement reduced the stiffness of the contact surfaces while nodal displacements from the Global model were applied to the submodel boundary. This effect diminished with subsequent mesh refinements, the Fine submodel yielding a contact force <1% different from the CAM and contact area 6.2% different from the experimental measurement. A limitation of the software prevented applying stresses from the Global FEM as the submodel boundary conditions. However, it is expected that if stress-based boundary conditions were applied, the contact force would have been more consistent between the Global and Coarse models, so that error relative to the CAM would have decreased in that step, rather than increase as seen in Table 4 and Fig. 11. The generous length of the submodels, which included 14 mm from the tips of both rods, probably prevented the discrepancies between the Global and Coarse models from being greater. However, in similar future analyses where it would be advantageous to set the submodel boundary closer to the point of impact (for instance, to reduce the number of elements in each submodeling step), then it could be important for accuracy to employ stress boundary conditions in a step during which the contact stiffness is simultaneously changed by refining the element mesh near the impact point.

The value of submodeling in an FEA is illustrated by the large differences seen in the maximum principal stress, comparing the Global FEM to the Mid and Fine submodels (Fig. 11). The submodels predicted stresses >130% higher in the elements closest to the surface; these differences occurred because the Global model's elements were too large to provide good resolution of the contact mechanics in this problem, where the contact area was tiny relative to the overall size of the impacting rods. Had the Global model been meshed as densely as the Mid submodel, it would have contained ~ 51 million elements, or ~ 210 million if meshed as densely as the Fine submodel. Thus, the submodeling approach can provide improved resolution while avoiding outsized FE models, and so it may be particularly useful in impact analyses whose results of interest span greatly differing size scales.

The “fingerprinting” technique proved to be an effective means for recording the maximal contact area, rendering an easily observed and measured darkened area on one rod. However, the technique will be inapplicable in the planned experiments for hip prosthesis impact loading because there, a lubricant simulating natural joint fluid should be used to represent physiologic conditions. Furthermore, the lubricant will be electrically conductive, so the present method of recording contact timing via electrical continuity will also be ineffective. These impediments heighten the importance of improving the modeling techniques, particularly the strategy for imposing submodel boundary conditions, to avoid the above-mentioned inconsistencies in the contact force. In addition, the sensors used for global-scale measurements, the strain gages and vibrometer, should be placed close to the contact area, and one or more (ideally, all) of the submodels should encompass their measurement locations, to validate that the submodels yield outputs consistent with the global model and the experiment.

The impact of slender rods provides a means for examining fundamental characteristics of the transient dynamics of impacting bodies. These include material property effects, speeds of wave propagation, and contact mechanics. A basic understanding of these dynamic phenomena, as they occur in the approximately 1D domain of slender rods, is an important

pre-requisite to advanced impact analysis and testing involving more complicated structures. In our laboratory, this study has served as a means to verify and validate analysis and laboratory techniques for studying the transient dynamics of prosthetic hip joints, which may experience impact loading after small-scale dislocation [17]. The contact force cannot be directly measured in this scenario, but it is important to understand as a key input for implant wear tests, so it shall be determined using experimentally validated FEA models. The outcomes of the present study of two-rod impact have suggested strategies to follow in the biomechanical models to bolster their accuracy, and in the accompany experiments to optimize their efficacy for model validation. The engineering approach to impact problems in other fields may benefit similarly by preparatory testing and analysis of the low speed impact of slender rods.

9 Conclusions

The analytical model for Hertzian contact-coupled axial impact of two slender rods and the Global FEM were validated by experimental measurements of vibrational speed and strain and of contact area and duration. Small ripples in the stress waves, not predicted by the analytical model, were detected by the speed and strain measurements and predicted by the Global FEM.

Submodel FEMs provided greatly improved resolution of the contact stresses, which converged towards the analytically predicted stresses with mesh refinement. However, the technique of applying Global FEM nodal displacements as submodel boundary conditions led to a 6% change in peak contact force in the first submodeling step (Coarse). In future similar studies, use of stress-based boundary conditions should be considered to avoid this problem, which arose because the contact stiffness changed with mesh refinement.

A relatively simple structural impact study such as the present study of impacting slender rods can serve as a platform for developing the technology and strategies needed to successfully undertake more complicated impact analysis and testing.

Acknowledgments

This research was supported by the National Institutes of Health/NIAMS, Grant No. 1R21AR056374. The authors thank Jeff Kessler for laboratory assistance, particularly with the drop tower test machine.

References

1. Timoshenko, SP.; Goodier, JN. *Theory of Elasticity*. 3. McGraw Hill; New York: 1970.
2. Thomson, WT. *Laplace Transformation*. Prentice Hall; Englewood Cliffs: 1960.
3. Goldsmith, W. *Impact: the theory and physical behaviour of colliding solids*. Edward Arnold, Ltd; London: 1960.
4. Guo A, Batzer S. Substructure Analysis of a Flexible System Contact-Impact Event. *J Vib Acoust*. 2004; 126 (1):126–131.
5. Marghitu, DB.; Boghiu, D. Spatial impact of a flexible link using a nonlinear contact force. American Society of Mechanical Engineers, Design Engineering Division (Publication) DE; ASME International Mechanical Engineering Congress; Atlanta. New York, NY, USA: ASME; 1996. p. 103-11.
6. Schiehlen W, Seifried R. Three approaches for elastodynamic contact in multibody systems. *Multibody Sys Dyn*. 2004; 12 (1):1–16.
7. Trowbridge DA, Grady JE, Aiello RA. Low velocity impact analysis with NASTRAN. *Comput Struct*. 1991; 40 (4):977–984.
8. Seifried R, Hu B. Numerical and experimental investigation of radial impacts on a half-circular plate. *Multibody Sys Dyn*. 2003; 9 (3):265–281.

9. Wei H, Yida Z. Finite element analysis on collision between two moving elastic bodies at low velocities. *Comput Struct.* 1995; 57 (3):379–382.
10. Frew DJ, Forrestal MJ, Chen W. Pulse shaping techniques for testing brittle materials with a split Hopkinson pressure bar. *Exp Mech.* 2002; 42 (1):93–106.10.1177/0018512002042001794
11. Doyle JF. Wavelet deconvolution method for impact force identification. *Exp Mech.* 1997; 37 (4): 403–408.
12. Yang IY, Im KH. Impulsive stress analysis of plates at the concentrated impact loading point by the three-dimensional dynamic theory of elasticity. *Exp Mech.* 1996; 36 (4):421–429.
13. Doyle JF. Determining the contact force during the transverse impact of plates. *Exp Mech.* 1987; 27 (1):68–72.
14. Hu B, Eberhard P. Simulation of longitudinal impact waves using time delayed systems. *J Dyn Syst Meas Contr.* 2004; 126 (3):644–649.
15. Cormier NG, Smallwood BS, Sinclair GB, et al. Aggressive submodelling of stress concentrations. *Int J Numer Methods Eng.* 1999; 46:889–909.
16. Rajasekaran R, Nowell D. On the finite element analysis of contacting bodies using submodelling. *J Strain Anal Eng Des.* 2005; 40:95–106.
17. Nevelos J, Ingham E, Doyle C, et al. Microseparation of the centers of alumina-alumina artificial hip joints during simulator testing produces clinically relevant wear rates and patterns. *J Arthroplasty.* 2000; 15 (6):793–795. [PubMed: 11021457]
18. Walter WL, Insley GM, Walter WK, et al. Edge loading in third generation alumina ceramic-on-ceramic bearings: stripe wear. *J Arthroplasty.* 2004; 19 (4):402–413. [PubMed: 15188097]
19. Kwon YM, Glyn-Jones S, Simpson DJ, et al. Analysis of wear of retrieved metal-on-metal hip resurfacing implants revised due to pseudotumours. *J Bone Joint Surg Br.* 2010; 92 (3):356–361. [PubMed: 20190305]
20. Sanders AP, Brannon RM. Assessment of the applicability of the Hertzian contact theory to edge-loaded prosthetic hip bearings. *J Biomech.* 2011; 44 (16):2802–2808. [PubMed: 21962465]
21. Kreyszig, E. *Advanced Engineering Mathematics.* 6. Wiley and Sons; New York: 1988.
22. Drumheller, DS. *Introduction to wave propagation in nonlinear fluids and solids.* Cambridge University Press; Cambridge: 1998.
23. Hu B, Schiehlen W, Eberhard P. Comparison of analytical and experimental results for longitudinal impacts on elastic rods. *J Vib Control.* 2003; 9 (1–2):157–174.
24. Johnson, KL. *Contact Mechanics.* Cambridge University Press; Cambridge: 1985.
25. Shampine LF, Reichelt MW. *The MATLAB ODE Suite.* SIAM J Sci Comput. 1997; 18 (1):1.
26. Fischer-Cripps, AC. *Mechanical Engineering Series.* Springer-Verlag; New York: 2000. Introduction to contact mechanics.
27. Tech Note TN-514: Shunt calibration of strain gage instrumentation. Vishay Micro-Measurements; Rayleigh, North Carolina: 2007.
28. Sanders AP, Brannon RM. Determining a surrogate contact pair in a Hertzian contact problem. *J Tribol.* 2011; 133 (2):024502–024506.10.1115/1.4003492

Fig. 1.
Schematic for the axial, centric impact of two slender rods

Fig. 2.

Small portion of Simulink block diagram solver for Eqns. (13). At (a), signal $f'_2()$ enters; at (b), a time delay is implemented, transforming the signal into $g'_2()$ at (c) per the third of Eqns. (13); at (d), $g'_2()$ is integrated; and at (e) the signal $g_2()$ is passed out for further computations as needed

Fig. 3. Finite element model illustrations. a) Global model, truncated to the first 2 mm of both rods. Cross sections of sequentially refined submodel meshes: b) Coarse, c) Mid, and d) Fine ($\frac{1}{2}$ symmetry model)

Fig. 4. Schematic of drop tower impact test machine, with both rods and their fixtures. Data acquisition and strain gage circuit not illustrated

Fig. 5. Contact patch. a) and b) show a typical contact patch, $\sim\varnothing 2.2$ mm, revealed by fingerprinting technique using black photocopier powder: a) from handheld camera; b) from optical CMM, original magnification 37X. c) Plot of contact pressure from Global FEM

Fig. 6.
Contact radii, showing data from Table 3

Fig. 7.
Contact force and contact area, showing data from Table 4

Fig. 8.

Results for a trial with impact speed 2.197 m/s; FEM results are from Global model. a) Speed at Rod 2 midpoint. b) Strain at Rod 2 midpoint; arrows indicate voltage spikes at start and end of impact. c) Contact force from analyses along with experiment's contact trigger voltage. d) Samples of non-scaled strain and speed voltage data

Fig. 9.
Average peak speed at Rod 2 midpoint, showing data from Table 5

Fig. 10.

Small portions of data from *Mid* and *Late* periods of speed and strain at Rod 2 midpoint. FEM results exhibit small, growing ripples not present in the CAM results but detected by the sensors

Fig. 11.

Maximum principal stress vs. radial coordinate, near the impacted surface of Rod 2 (z =mean distance beneath surface), showing convergence of FEM results towards CAM result. RMS=root-mean-square. Numerical anomalies arose in Fine submodel close to the rod's axis (black arrow)

Fig. 12.
Portion of Fine submodel, showing contours of minimum principal stress at the instant of peak contact force

Table 1

Finite element model details

Model	Avg. element edge length (mm)		Nodes $\times 10^6$	Elements $\times 10^6$	Symmetry applied
	axial	radial and circumferential			
Global	1.0	0.25	2.4	1.4	No
Coarse submodel	0.25	0.25	0.2	0.2	No
Mid submodel	0.125	0.125	1.5	1.5	No
Fine submodel	0.062	0.062	6.3	6.2	$1/2$ symmetry

Table 2

Physical configuration details

Rod 1 length	Rod 1 tip radius	Rod 2 length	Rod 2 tip	Diameter, both rods	Density	Elastic modulus	Poisson's ratio	Impact speed
250.09 mm	35 mm	700.99 mm	flat	12.70 mm	7.803 g/cc	204.3 GPa	0.30	2.197 m/s

Table 3
Experiment contact radius and circularity ('O') along with radius from CAM. 'Δ' compares analysis radii to experimental radii

Rod 1 specimen	Trial	Impact speed (m/s)	Experiment		CAM	
			Radius (mm)	O (mm)	Radius (mm)	Δ (%)
1	1	2.197	1.11	0.09	1.09	-1.8
	2	2.208	1.13	0.10	1.09	-3.5
2	1	2.178	1.14	0.14	1.09	-4.5
	2	1.848	1.06	0.12	1.03	-2.8
3	1	2.166	1.12	0.09	1.08	-3.6
	2	2.128	1.12	0.08	1.08	-3.6
		Average	1.11	0.10	1.08	-3.3

Table 4

Maximum contact force and area from FEMs

Source	Contact force (N)		Contact area (mm ²)	
	Magnitude	Error ^a	Magnitude	Error ^b
Experiment	n.a.	–	3.86	–
CAM	5,543	–	3.73	–3.4%
Global FEM	5,549	0.1%	4.61	19.4%
Coarse submodel	5,219	–5.8%	4.80	24.4%
Mid submodel	5,368	–3.2%	4.44	15.0%
Fine submodel	5,492	–0.9%	4.10	6.2%

^a Error with respect to CAM result,^b Error with respect to Experiment result

Table 5

Comparisons of speed and strain at Rod 2 midpoint to CAM results, over first five vibration periods. "RMS Δ " is the root-mean-square difference

Rod 1 specimen	Trial	Avg. peak speed (mm/s)		RMS Δ strain	
		Exp't	CAM	Δ (%)	($\mu\epsilon$)
1	1*	1.11	1.10	-0.9	6.0
2	1	1.10	1.10	0	11.6
	2	0.92	0.93	+1.1	3.4
3	1	1.11	1.09	-1.8	7.4
	2	1.08	1.07	-0.9	5.0

* Only 1 trial recorded