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On a multiphase multicomponent model of biofilm growth

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Abstract

Biofilms are formed when free-floating bacteria attach to a surface and secrete polysaccharide to form an extracellular polymeric matrix (EPS). A general model of biofilm growth needs to include the bacteria, the EPS, and the solvent within the biofilm region $\Omega(t)$, and the solvent in the surrounding region $D(t)$. The interface between the two regions, $\Gamma(t)$, is a free boundary. In this paper, we consider a mathematical model, which consists of a Stokes equation for the EPS with bacteria attached to it, and a Stokes equation for the solvent in $\Omega(t)$ and a different one for the solvent in $D(t)$. The volume fraction of the EPS is another unknown satisfying a reaction-diffusion equation. The entire system is coupled nonlinearly within $\Omega(t)$, and across the free surface $\Gamma(t)$. We prove the existence and uniqueness of solution, with a smooth surface $\Gamma(t)$, for a small time interval.

1 Introduction

Biofilms are defined as communities of microorganisms, typically bacteria, that are attached to a surface. The bacteria within the biofilms are usually embedded within a matrix consisting of polysaccharides which is made by the bacteria and called extracellular polymeric substance (EPS). The EPS shields the bacteria from attack, making them, in particular, resistant to therapeutic drugs [34]. Biofilms account for over 80% of infections in the body. Examples include plaques on teeth and dental implants, infections of gastrointestinal tracks, urinary tracks, infections of eyes, nose, and ears, and infections of skin wounds [1]. The bacterium *Pseudomonas aeruginosa*, an organism that can cause life threatening infections in diseases such as cystic fibrosis, is especially adept in resisting antimicrobial therapy [34]. Improved understanding of the physical organization of a biofilm matrix may help in the development of new drugs aimed at disrupting the biofilm.

Mathematical models of biofilm go back to the 1980's. Two comprehensive reviews of the mathematical models, up to 2010, were published by Klapper and Dockery [25], and Wang and Zhang [44]. The most comprehensive models use two-phase flows to describe biofilm growth. The variables in these models include the volume fractions and velocities of the EPS and the solvent within the biofilm, and the velocity of the fluid surrounding the biofilm. These variables, as well as the concentration of bacteria and nutrients, satisfy a system of PDEs within the biofilm, and a system of PDEs outside the biofilm. These two systems are coupled through the common boundary $\Gamma(t)$, which is a free boundary. Klapper and Dockery

[25] explored the linear stability of the surface $\Gamma(t)$ in case $\Gamma(t)$ is initially flat and the linear perturbations are periodic. However, except for numerical results [6], there have been no rigorous mathematical results that address the questions of existence, uniqueness and properties of the full free boundary problem of the biofilm model.

In this paper, motivated by models of biofilms, we consider a general two-phase free boundary problem. In one phase, $D(t)$, there is an incompressible viscous fluid, and in the other phase, $\Omega(t)$, there is a mixture of two incompressible fluids, which represent the viscous fluid and the polymeric network (with bacteria attached to it) associated with a biofilm; see Figure 1. The interface $\Gamma = \Gamma(t)$ is a free boundary which evolves in time. In $D(t)$, the fluid satisfies a Stokes system for the velocity and pressure of the fluid, and in $\Omega(t)$, the mixture satisfies a system of Stokes equations for the velocities of the fluid and the network and their common pressure. We prove that this system has a unique solution with smooth free boundary, for a small time interval $0 \leq t \leq T$.

There has been a lot of work in the area of Stokes systems. Regularity for solutions of elliptic and evolution problems are well known. L^p estimates and Schauder estimates have been established for domains with smooth boundary as well as Lipschitz domains [29,30,23,39,40,42,41].

When it comes to free boundary problems in the Stokes systems, issues of regularity need to be resolved in order to apply estimates for this type of systems. Free boundary problems similar to the one studied in this paper involving a Stokes system was studied in [8,7]. That system of equations was derived from conservation of energy and, consequently, it was possible to formulate the free boundary condition in a weak form. This enabled the system to be solved in an appropriate Sobolev space with the free boundary conditions automatically built into the system. However, in general, free boundary problems for Stokes systems do not have a weak formulation. Instead, one may transform the free boundary to a fixed boundary either by a change of variables as in [21,33], or by introducing Lagrange variables, as in the work of Solonnikov [37,38]. This latter approach was also used in [13–15] to prove local existence for a free boundary problem for a system which couples Stokes equation with several diffusion equations modeling tumor growth. Global existence for solutions with initial domain near a sphere was proved in [20,19,36] for viscous drops. The proofs in [20,19] use an expansion of the solution in terms of vector spherical harmonics. Symmetry-breaking bifurcations for a coupled system of Stokes equation and diffusion equations modeling tumor growth, were established in [16,17]. A more complex system, modeling wound healing, which includes the Stokes equation coupled to diffusion and hyperbolic equations was considered in [18] where local existence was established.

The present system for biofilm, however, is significantly more complex. It involves several Stokes flows in two domains with a free interface. Furthermore, on the free boundary, the conditions of continuity of velocities and forces are not standard. In addition, a coefficient θ_n appears in the Stokes equations and in the boundary conditions at the free boundary, and θ_n satisfies a diffusion equation coupled with the Stokes equations.

The main result of this paper is a proof of existence and uniqueness of a smooth solution for a short time interval. The proof is based on the Schauder estimates and consists of three main steps presented in Sections 3–5. In Section 3, we consider the situation in which the fluids are in two fixed domains and θ_n is a given function, and use the following procedure to prove the existence and uniqueness of solution for this fixed boundary problem: flatten the boundary and reflect the system in one domain locally into the other domain, use local Schauder estimates, and then a partition of unity to derive global Schauder estimates; apply the *a priori* estimates to a special system to establish existence; and finally by the method of

continuity establish existence for the general system for the fixed boundary problem with given θ_n . In Section 4, we use the approach of [18] to extend the results in Section 3 to the situation with moving boundaries but still with a given function θ_n , using a fixed point theorem. In Section 5, we extend the analysis to include the variable θ_n using a fixed point theorem once more. In Section 6 we show how our general result can be used to establish existence and uniqueness, in small time, for a general biofilm model.

2 The general mathematical model

In this section, we state the general two-phase free boundary problem. We consider the geometry given in Figure 1, where a growing gel, modeled as a mixture of fluids, occupies a domain $\Omega(t)$ outside a solid substrate G , and is surrounded by an incompressible fluid in a domain $D(t)$. The interface $\Gamma = \Gamma(t)$ between $\Omega(t)$ and $D(t)$ is a free boundary which evolves in time.

The PDE system in $D(t)$ is given by,

$$\nabla \cdot (\nabla \mathbf{v}_1 + \nabla \mathbf{v}_1^T) - \nabla P_1 = \mathbf{f}_1 \quad \text{in } D(t), \quad (2.1)$$

$$\nabla \cdot \mathbf{v}_1 = h_1 \quad \text{in } D(t), \quad (2.2)$$

where $\mathbf{v}_1$ and P_1 are the velocity and pressure of the fluid in the domain $D(t)$, $\mathbf{f}_1$ is the body force acting on the fluid, and h_1 is included for generality.

In the $\Omega(t)$ phase, we have a more complicated system of PDEs,

$$\frac{\partial \theta_n}{\partial t} + \nabla \cdot (\theta_n \mathbf{v}_2) = \varepsilon \Delta \theta_n + G_n \quad \text{in } \Omega(t), \quad (2.3)$$

$$\nabla \cdot [\theta_n (\nabla \mathbf{v}_2 + \nabla \mathbf{v}_2^T)] - \theta_n \nabla P_2 = \mathbf{f}_2 \quad \text{in } \Omega(t), \quad (2.4)$$

$$\nabla \cdot [(1 - \theta_n) \eta (\nabla \mathbf{v}_3 + \nabla \mathbf{v}_3^T)] - (1 - \theta_n) \nabla P_2 = \mathbf{f}_3 \quad \text{in } \Omega(t), \quad (2.5)$$

$$\nabla \cdot (\theta_n \mathbf{v}_2 + (1 - \theta_n) \mathbf{v}_3) = h_2 \quad \text{in } \Omega(t), \quad (2.6)$$

where θ_n is the volume fraction of the polymer, $(1 - \theta_n)$ is the volume fraction of the solvent, G_n is the rate of mass conversion from solvent to polymer network, $\mathbf{v}_2$ and $\mathbf{v}_3$ are the velocities of the polymer and fluid, P_2 is the pressure of the mixture in the domain $\Omega(t)$, and $\mathbf{f}_2$ and $\mathbf{f}_3$ are the body force acting on the two components of the solution including the friction forces between the them, and h_2 is included for generality. We note that the total mass of the polymer network is not conserved if the conversion rate G_n is nonzero. However, if $h_1 = h_2 = 0$, as we consider in the actual biofilm model in Sec. 6, the total mass of the solvent in $D(t) \cup \Omega(t)$ and the polymer in $\Omega(t)$ is conserved, reflecting conservation of mass in the entire domain.

The boundary conditions are

$$(\nabla \mathbf{v}_2 \cdot \mathbf{n} + \nabla \mathbf{v}_2^T \cdot \mathbf{n} - P_2 \mathbf{n}) - (\nabla \mathbf{v}_1 \cdot \mathbf{n} + \nabla \mathbf{v}_1^T \cdot \mathbf{n} - P_1 \mathbf{n}) = \mathbf{g}_1 \quad \text{on } \Gamma(t), \quad (2.7)$$

$$(\eta \nabla \mathbf{v}_3 \cdot \mathbf{n} + \eta \nabla \mathbf{v}_3^T \cdot \mathbf{n} - P_2 \mathbf{n}) - (\nabla \mathbf{v}_1 \cdot \mathbf{n} + \nabla \mathbf{v}_1^T \cdot \mathbf{n} - P_1 \mathbf{n}) = \mathbf{g}_2 \quad \text{on } \Gamma(t), \quad (2.8)$$

$$\mathbf{v}_1 = \theta_n \mathbf{v}_2 + (1 - \theta_n) \mathbf{v}_3 \quad \text{on } \Gamma(t), \quad (2.9)$$

$$\mathbf{v}_1 = 0 \quad \text{on } S \quad \text{and} \quad \mathbf{v}_2 = \mathbf{v}_3 = 0 \quad \text{on } \partial G, \quad (2.10)$$

$$\frac{\partial \theta_n}{\partial \mathbf{n}} = 0 \quad \text{on } \partial \Omega(t), \quad (2.11)$$

where $\mathbf{n}$ is the outward normal, and $\mathbf{g}_1$ and $\mathbf{g}_2$ are additional forces acting on the moving interface $\Gamma(t)$, including, e.g., surface tension. The boundary conditions (2.7)–(2.9) represents balance of forces and velocities across the free boundary. The velocity of the boundary $\Gamma(t)$ in the normal direction is thus given by

$$V_{\Gamma(t)} = \mathbf{v}_n \cdot \mathbf{n} \quad \text{on } \Gamma(t). \quad (2.12)$$

The initial condition for θ_n is

$$\theta_n|_{t=0} = \theta_0 \quad \text{in } \Omega(0). \quad (2.13)$$

3 Existence and uniqueness of solution for the static reduced problem

Given θ_n , consider the static reduced problem for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, P$:

$$\nabla \cdot (\nabla \mathbf{v}_1 + \nabla \mathbf{v}_1^T) - \nabla P_1 = \mathbf{f}_1 \quad \text{in } D, \quad (3.1)$$

$$\nabla \cdot \mathbf{v}_1 = h_1 \quad \text{in } D, \quad (3.2)$$

$$\nabla \cdot [\theta_n (\nabla \mathbf{v}_2 + \nabla \mathbf{v}_2^T)] - \theta_n \nabla P_2 = \mathbf{f}_2 \quad \text{in } \Omega, \quad (3.3)$$

$$\nabla \cdot [(1 - \theta_n) \eta (\nabla \mathbf{v}_3 + \nabla \mathbf{v}_3^T)] - (1 - \theta_n) \nabla P_2 = \mathbf{f}_3 \quad \text{in } \Omega, \quad (3.4)$$

$$\nabla \cdot (\theta_n \mathbf{v}_2 + (1 - \theta_n) \mathbf{v}_3) = h_2 \quad \text{in } \Omega, \quad (3.5)$$

with the boundary conditions

$$(\nabla \mathbf{v}_2 \cdot \mathbf{n} + \nabla \mathbf{v}_2^T \cdot \mathbf{n} - P_2 \mathbf{n}) - (\nabla \mathbf{v}_1 \cdot \mathbf{n} + \nabla \mathbf{v}_1^T \cdot \mathbf{n} - P_1 \mathbf{n}) = \mathbf{g}_1 \quad \text{on } \Gamma, \quad (3.6)$$

$$(\eta \nabla \mathbf{v}_3 \cdot \mathbf{n} + \eta \nabla \mathbf{v}_3^T \cdot \mathbf{n} - P_2 \mathbf{n}) - (\nabla \mathbf{v}_1 \cdot \mathbf{n} + \nabla \mathbf{v}_1^T \cdot \mathbf{n} - P_1 \mathbf{n}) = \mathbf{g}_2 \quad \text{on } \Gamma, \quad (3.7)$$

$$\mathbf{v}_1 = \theta_n \mathbf{v}_2 + (1 - \theta_n) \mathbf{v}_3 \quad \text{on } \Gamma, \quad (3.8)$$

$$\mathbf{v}_1 = 0 \quad \text{on } S \quad \text{and} \quad \mathbf{v}_2 = \mathbf{v}_3 = 0 \quad \text{on } \partial G. \quad (3.9)$$

In this section we prove that this problem has a unique classical solution, given that

$$\mathbf{f}_1 \in C^\alpha(\overline{D}), \mathbf{f}_2 \in C^\alpha(\overline{\Omega}), \mathbf{f}_3 \in C^\alpha(\overline{\Omega}), h_1 \in C^{1+\alpha}(\overline{D}), h_2 \in C^{1+\alpha}(\overline{\Omega}), \quad (3.10)$$

$$\mathbf{g}_j \in C^{1+\alpha}(\Gamma), j=1, 2, \quad (3.11)$$

$$S, \partial G \text{ and } \Gamma \text{ are in } C^{2+\alpha}, \quad (3.12)$$

$$\theta_n \in C^{2+\alpha}(\overline{\Omega}), 0 < \theta_n < 1 \quad \text{in } \overline{\Omega}, \quad (3.13)$$

and η is a positive constant.

Remark: Note that θ_n occurs in Equations (3.3)–(3.5). After rewriting (3.5) in the form

$$\theta_n \nabla \mathbf{v}_2 + (1 - \theta_n) \nabla \mathbf{v}_3 = h_2 + \tilde{h},$$

where $\tilde{h}$ includes terms with $\nabla \cdot \theta_n$, we see that $\theta_n \in C^{2+\alpha}(\overline{\Omega})$ is needed to ensure that the function $\tilde{h}$ belongs to $C^{1+\alpha}(\overline{\Omega})$.

Structure of the proof: To prove this result, we consider the following parametrized problem:

$$\nabla \cdot (\nabla \mathbf{v}_1 + \lambda \nabla \mathbf{v}_1^T) - \nabla P_1 = \mathbf{f}_1 \quad \text{in } D, \quad (3.14)$$

$$\nabla \cdot \mathbf{v}_1 = h_1 \quad \text{in } D, \quad (3.15)$$

$$\nabla \cdot [\theta_n (\nabla \mathbf{v}_2 + \lambda \nabla \mathbf{v}_2^T)] - \theta_n \nabla P_2 = \mathbf{f}_2 \quad \text{in } \Omega, \quad (3.16)$$

$$\nabla \cdot [(1 - \theta_n) \eta (\nabla \mathbf{v}_3 + \lambda \nabla \mathbf{v}_3^T)] - (1 - \theta_n) \nabla P_2 = \mathbf{f}_3 \quad \text{in } \Omega, \quad (3.17)$$

$$\nabla \cdot (\theta_n \mathbf{v}_2 + (1 - \theta_n) \mathbf{v}_3) = h_2 \quad \text{in } \Omega, \quad (3.18)$$

with the boundary conditions

$$(\nabla \mathbf{v}_2 \cdot \mathbf{n} + \lambda \nabla \mathbf{v}_2^T \cdot \mathbf{n} - P_2 \mathbf{n}) - (\nabla \mathbf{v}_1 \cdot \mathbf{n} + \lambda \nabla \mathbf{v}_1^T \cdot \mathbf{n} - P_1 \mathbf{n}) = \mathbf{g}_1 \quad \text{on } \Gamma, \quad (3.19)$$

$$(\eta \nabla \mathbf{v}_3 \cdot \mathbf{n} + \eta \lambda \nabla \mathbf{v}_3^T \cdot \mathbf{n} - P_2 \mathbf{n}) - (\nabla \mathbf{v}_1 \cdot \mathbf{n} + \lambda \nabla \mathbf{v}_1^T \cdot \mathbf{n} - P_1 \mathbf{n}) = \mathbf{g}_2 \quad \text{on } \Gamma, \quad (3.20)$$

$$\mathbf{v}_1 = \theta_n \mathbf{v}_2 + (1 - \theta_n) \mathbf{v}_3 \quad \text{on } \Gamma, \quad (3.21)$$

$$\mathbf{v}_1 = 0 \quad \text{on } S \quad \text{and} \quad \mathbf{v}_2 = \mathbf{v}_3 = 0 \quad \text{on } \partial G, \quad (3.22)$$

with $0 \leq \lambda \leq 1$. For easy reference we denote this system of equations and the boundary conditions, respectively, by

$$\begin{aligned} L_\lambda \mathbf{v} &= \mathbf{F}, \\ B_\lambda \mathbf{v} &= \mathbf{G}. \end{aligned}$$

We first prove uniqueness of solutions for L_λ, B_λ for all $0 \leq \lambda \leq 1$ by deriving an energy equality. We then establish Schauder estimates as follows: (i) we consider the case of a planar interface, reflect one system across the interface and derive the Schauder estimates for the combined system in the reflected domain; (ii) using partition of unity $\{\chi_k\}$ of a neighborhood of the interface and flattening the free boundary in a small region containing the support of χ_k for each k , we use (i) to extend the Schauder estimates to general fixed domains. We next prove the existence of solutions for L_0, B_0 . This is done by applying Lax-Milgram lemma to establish existence of a unique solution in H^1 , and taking finite differences we extend the regularity to H^2, H^3 , etc, and eventually to $C^{2+\alpha}$. Finally, by a continuity method, we derive existence and uniqueness of solutions for the system (3.1)–(3.9) which corresponds to the parameterized problem with $\lambda = 1$.

3.1 Uniqueness of solutions

We shall prove that the only solution to the corresponding homogeneous problem of (3.14)–(3.22) is zero. We multiply $v_{1,j}$ to the j -th equation of (3.14) and apply integration by part. We obtain,

$$\begin{aligned} \int_D \mathbf{v}_1 \cdot \mathbf{f}_1 d\mathbf{x} &= \int_D \sum_{j=1}^3 v_{1,j} \left[\sum_{i=1}^3 \frac{\partial}{\partial x_i} \left(\frac{\partial v_{1,j}}{\partial x_i} + \lambda \frac{\partial v_{1,i}}{\partial x_j} \right) - \frac{\partial P_1}{\partial x_j} \right] d\mathbf{x} \\ &= \int_{\partial D} \sum_{j=1}^3 v_{1,j} \left[\sum_{i=1}^3 \left(\frac{\partial v_{1,j}}{\partial x_i} + \lambda \frac{\partial v_{1,i}}{\partial x_j} \right) n_i - P_1 n_j \right] dS - \int_D \sum_{i,j=1}^3 \frac{\partial v_{1,j}}{\partial x_i} \left(\frac{\partial v_{1,j}}{\partial x_i} + \lambda \frac{\partial v_{1,i}}{\partial x_j} \right) d\mathbf{x} + \int_D P_1 \nabla \cdot \mathbf{v}_1 d\mathbf{x}. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_\Omega \mathbf{v}_2 \cdot \mathbf{f}_2 d\mathbf{x} &= \int_\Omega \sum_{j=1}^3 v_{2,j} \left[\sum_{i=1}^3 \frac{\partial}{\partial x_i} \left(\theta_n \left(\frac{\partial v_{2,j}}{\partial x_i} + \lambda \frac{\partial v_{2,i}}{\partial x_j} \right) \right) - \theta_n \frac{\partial P_2}{\partial x_j} \right] d\mathbf{x} \\ &= \int_{\partial \Omega} \sum_{j=1}^3 v_{2,j} \theta_n \left[\sum_{i=1}^3 \left(\frac{\partial v_{2,j}}{\partial x_i} + \lambda \frac{\partial v_{2,i}}{\partial x_j} \right) n_i - P_2 n_j \right] dS - \int_\Omega \sum_{i,j=1}^3 \theta_n \left(\frac{\partial v_{2,j}}{\partial x_i} \right) \left(\frac{\partial v_{2,j}}{\partial x_i} + \lambda \frac{\partial v_{2,i}}{\partial x_j} \right) d\mathbf{x} + \int_\Omega P_2 \nabla \cdot (\theta_n \mathbf{v}_2) d\mathbf{x}. \end{aligned}$$

$$\begin{aligned} \int_{\Omega} \mathbf{v}_3 \cdot \mathbf{f}_3 d\mathbf{x} &= \int_{\Omega} \sum_{j=1}^3 v_{3,j} \left[\sum_{i=1}^3 \frac{\partial}{\partial x_i} \left((1-\theta_n) \eta \left(\frac{\partial v_{3,j}}{\partial x_i} + \lambda \frac{\partial v_{3,i}}{\partial x_j} \right) \right) - (1-\theta_n) \frac{\partial P_2}{\partial x_j} \right] d\mathbf{x} \\ &= \int_{\partial\Omega} \sum_{j=1}^3 v_{3,j} (1-\theta_n) \left[\sum_{i=1}^3 \eta \left(\frac{\partial v_{3,j}}{\partial x_i} + \lambda \frac{\partial v_{3,i}}{\partial x_j} \right) n_i - P_2 n_j \right] dS - \int_{\Omega} \sum_{i,j=1}^3 (1-\theta_n) \eta \left(\frac{\partial v_{3,j}}{\partial x_i} \right) \left(\frac{\partial v_{3,i}}{\partial x_j} + \lambda \frac{\partial v_{3,i}}{\partial x_j} \right) d\mathbf{x} + \int_{\Omega} P_2 \nabla \cdot ((1-\theta_n) \mathbf{v}_3) d\mathbf{x}. \end{aligned}$$

Taking the sum of the above three equations, we obtain

$$I_d = I_b + I_p - I_f, \quad (3.23)$$

where I_b is the sum of the boundary integrals

$$\begin{aligned} I_b &= \int_{\partial D} \sum_{j=1}^3 v_{1,j} \left[\sum_{i=1}^3 \left(\frac{\partial v_{1,j}}{\partial x_i} + \lambda \frac{\partial v_{1,i}}{\partial x_j} \right) n_i - P_1 n_j \right] dS \\ &\quad + \int_{\partial\Omega} \sum_{j=1}^3 v_{2,j} \theta_n \left[\sum_{i=1}^3 \left(\frac{\partial v_{2,j}}{\partial x_i} + \lambda \frac{\partial v_{2,i}}{\partial x_j} \right) n_i - P_2 n_j \right] dS \\ &\quad + \int_{\partial\Omega} \sum_{j=1}^3 v_{3,j} (1-\theta_n) \eta \left[\sum_{i=1}^3 \left(\frac{\partial v_{3,j}}{\partial x_i} + \lambda \frac{\partial v_{3,i}}{\partial x_j} \right) n_i - P_2 n_j \right] dS, \end{aligned}$$

and

$$\begin{aligned} I_d &= \int_D \sum_{i,j=1}^3 \frac{\partial v_{1,j}}{\partial x_i} \left(\frac{\partial v_{1,j}}{\partial x_i} + \lambda \frac{\partial v_{1,i}}{\partial x_j} \right) d\mathbf{x} + \int_{\Omega} \sum_{i,j=1}^3 \theta_n \frac{\partial v_{2,j}}{\partial x_i} \left(\frac{\partial v_{2,j}}{\partial x_i} + \lambda \frac{\partial v_{2,i}}{\partial x_j} \right) d\mathbf{x} + \int_{\Omega} \sum_{i,j=1}^3 (1-\theta_n) \eta \frac{\partial v_{3,j}}{\partial x_i} \left(\frac{\partial v_{3,j}}{\partial x_i} + \lambda \frac{\partial v_{3,i}}{\partial x_j} \right) d\mathbf{x}, \\ I_p &= \int_D P_1 \nabla \cdot \mathbf{v}_1 d\mathbf{x} + \int_{\Omega} P_2 \nabla \cdot (\theta_n \mathbf{v}_2 + (1-\theta_n) \mathbf{v}_3) d\mathbf{x}, \\ I_f &= \int_D \mathbf{v}_1 \cdot \mathbf{f}_1 d\mathbf{x} + \int_{\Omega} \mathbf{v}_2 \cdot \mathbf{f}_2 + \mathbf{v}_3 \cdot \mathbf{f}_3 d\mathbf{x}. \end{aligned}$$

From (3.19)–(3.21) we see that

$$\begin{aligned} I_b &= - \int_{\Gamma} \sum_{j=1}^3 (\theta_n v_{2,j} + (1-\theta_n) v_{3,j}) \left[\sum_{i=1}^3 \left(\frac{\partial v_{1,j}}{\partial x_i} + \lambda \frac{\partial v_{1,i}}{\partial x_j} \right) n_i - P_1 n_j \right] dS \\ &\quad + \int_{\Gamma} \sum_{j=1}^3 v_{2,j} \theta_n \left[\sum_{i=1}^3 \left(\frac{\partial v_{2,j}}{\partial x_i} + \lambda \frac{\partial v_{2,i}}{\partial x_j} \right) n_i - P_2 n_j \right] dS \\ &\quad + \int_{\Gamma} \sum_{j=1}^3 v_{3,j} (1-\theta_n) \left[\sum_{i=1}^3 \eta \left(\frac{\partial v_{3,j}}{\partial x_i} + \lambda \frac{\partial v_{3,i}}{\partial x_j} \right) n_i - P_2 n_j \right] dS \\ &= \int_{\Gamma} \sum_{j=1}^3 \theta_n v_{2,j} \left[\sum_{i=1}^3 \left(\frac{\partial v_{2,j}}{\partial x_i} + \lambda \frac{\partial v_{2,i}}{\partial x_j} \right) n_i - P_2 n_j - \sum_{i=1}^3 \left(\frac{\partial v_{1,j}}{\partial x_i} + \lambda \frac{\partial v_{1,i}}{\partial x_j} \right) n_i + P_1 n_j \right] \\ &\quad + \int_{\Gamma} \sum_{j=1}^3 (1-\theta_n) v_{3,j} \left[\sum_{i=1}^3 \eta \left(\frac{\partial v_{3,j}}{\partial x_i} + \lambda \frac{\partial v_{3,i}}{\partial x_j} \right) n_i - P_2 n_j - \sum_{i=1}^3 \eta \left(\frac{\partial v_{1,j}}{\partial x_i} + \lambda \frac{\partial v_{1,i}}{\partial x_j} \right) n_i + P_1 n_j \right] \\ &= \int_{\Gamma} \theta_n \mathbf{v}_2 \cdot \mathbf{g}_1 + (1-\theta_n) \mathbf{v}_3 \cdot \mathbf{g}_2 dS, \end{aligned}$$

and from (3.2), (3.5), we obtain

$$I_p = \int_{\Omega} P_1 \mathbf{h}_1 + P_2 \mathbf{h}_2 d\mathbf{x}.$$

Notice next that

$$\sum_{i,j=1}^3 \left[\left(\frac{\partial v_{1,j}}{\partial x_i} \right)^2 + \lambda \left(\frac{\partial v_{1,j}}{\partial x_i} \right) \left(\frac{\partial v_{1,i}}{\partial x_j} \right) \right] = \sum_{i=1}^3 (1+\lambda) \left(\frac{\partial v_{1,i}}{\partial x_i} \right)^2 + \sum_{i,j=1, i \neq j}^3 (1-\lambda) \left(\frac{\partial v_{1,i}}{\partial x_j} \right)^2 + \sum_{i,j=1, i \neq j}^3 \frac{\lambda}{2} \left(\frac{\partial v_{1,j}}{\partial x_i} + \frac{\partial v_{1,i}}{\partial x_j} \right)^2,$$

and that similar formulas hold for $\mathbf{v}_2$ and $\mathbf{v}_3$. Hence we can rewrite I_d in the form

$$\begin{aligned} I_d = & \int_D \left[\sum_{i=1}^3 (1+\lambda) \left(\frac{\partial v_{1,i}}{\partial x_i} \right)^2 + \sum_{i,j=1, i \neq j}^3 (1-\lambda) \left(\frac{\partial v_{1,i}}{\partial x_j} \right)^2 + \sum_{i,j=1, i \neq j}^3 \frac{\lambda}{2} \left(\frac{\partial v_{1,j}}{\partial x_i} + \frac{\partial v_{1,i}}{\partial x_j} \right)^2 \right] dx \\ & + \int_{\Omega} \theta_n \left[\sum_{i=1}^3 (1+\lambda) \left(\frac{\partial v_{2,i}}{\partial x_i} \right)^2 + \sum_{i,j=1, i \neq j}^3 (1-\lambda) \left(\frac{\partial v_{2,i}}{\partial x_j} \right)^2 + \sum_{i,j=1, i \neq j}^3 \frac{\lambda}{2} \left(\frac{\partial v_{2,j}}{\partial x_i} + \frac{\partial v_{2,i}}{\partial x_j} \right)^2 \right] dx \\ & + \int_{\Omega} (1-\theta_n) \eta \left[\sum_{i=1}^3 (1+\lambda) \left(\frac{\partial v_{3,i}}{\partial x_i} \right)^2 + \sum_{i,j=1, i \neq j}^3 (1-\lambda) \left(\frac{\partial v_{3,i}}{\partial x_j} \right)^2 + \sum_{i,j=1, i \neq j}^3 \frac{\lambda}{2} \left(\frac{\partial v_{3,j}}{\partial x_i} + \frac{\partial v_{3,i}}{\partial x_j} \right)^2 \right] dx. \end{aligned}$$

Given $\mathbf{g}_i = \mathbf{h}_i = \mathbf{f}_i = 0$, we then have $I_b = I_p = I_f = 0$, so that $I_d = 0$. If $0 < \lambda < 1$, then

$$\frac{\partial v_{k,i}}{\partial x_j} \equiv 0, \quad a.e., \quad i, j, k=1, 2, 3.$$

Since $\theta_n \in (0, 1)$ in $\bar{\Omega}$. Using the zero boundary condition we find that $\mathbf{v}_k \equiv 0$. If $\lambda = 1$, then from the vanishing of I_d we only obtain

$$\frac{\partial v_{k,i}}{\partial x_i} \equiv 0, \quad \text{and} \quad \frac{\partial v_{k,j}}{\partial x_i} + \frac{\partial v_{k,i}}{\partial x_j} \equiv 0 \quad a.e., \quad i, j, k=1, 2, 3. \quad (3.24)$$

By (3.22), on the fixed boundary S , we have $v_{1,i} = 0$, and in the domain D , $\frac{\partial v_{1,i}}{\partial x_i} = 0$ a.e.. Therefore $v_{1,i} \equiv 0$ a.e. on the sets of points that can be connected by a line segment in x_i direction to the boundary S . Denote this set by D_{1i}^0 . Since $v_{1,j} = 0$ a.e. in

D_{1j}^0 , $\frac{\partial v_{1,i}}{\partial x_j} = -\frac{\partial v_{1,j}}{\partial x_i} = 0$. Thus $v_{1,i} = 0$ a.e. in the portion of D_{1j}^0 that is connected by a line segment in x_j direction to D_{1i}^0 . Denote this set by D_{1ij}^0 and let $D_{1i}^1 = D_{1i}^0 \cup \left(\bigcup_{j \neq i} D_{1ij}^0 \right)$. If Γ is convex, then D_{1i}^1 coincides with D . Otherwise, we continue this process and at each step increase the set where $v_1 = 0$. Since Γ is continuously differentiable, after a finite number of steps we arrive at the set D , so that $\mathbf{v}_1 = 0$ a.e. in D .

Similarly, we can show that $\mathbf{v}_2 = \mathbf{v}_3 = 0$ a.e. in Ω . Thus the homogeneous problem only has the zero solution for $\mathbf{v}$. From the equation (3.1), we see that $P = 0$ up to a constant.

3.2 $C^{2+\alpha}$ a priori estimates

For any vector valued functions $\mathbf{v} = (v^1, v^2, v^3)$, we shall use the notation $\mathbf{v} \in H$ if all the components $v^j \in H$, and we introduce the notation

$$\|\mathbf{v}\|_H = \sqrt{\|v^1\|_H^2 + \|v^2\|_H^2 + \|v^3\|_H^2}.$$

For simplicity, we consider the case with $\lambda = 1$, but the estimates derived below hold true for $0 < \lambda < 1$, although the details are a bit more complex. The local Schauder estimates in any domain bounded away from Γ is standard. In this section we shall derive estimates that are also valid in a neighborhood of Γ . It will be convenient to use the variables (x, y, z) instead of (x_1, x_2, x_3) .

3.2.1 Case I: $\Gamma = \{z = 0\}$ and $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3 = \mathbf{0}$ for $\sqrt{|x|^2 + |y|^2 + |z|^2} \geq d$ and small d
 —We assume that Γ is given by $\{z = 0\}$, and $\theta_n(\mathbf{x})$, $(1 - \theta_n(\mathbf{x}))$ are uniformly positive. To derive *a priori* estimates, we use the following reflection:

$$\begin{aligned} v_{1,i}^r(x, y, z) &= v_{1,i}(x, y, -z), \quad i=1, 2, \\ v_{1,3}^r(x, y, z) &= -v_{1,3}(x, y, -z), \\ P_1^r(x, y, z) &= P_1(x, y, -z), \\ f_{1,i}^r(x, y, z) &= f_{1,i}(x, y, -z), \quad i=1, 2, \\ f_{1,3}^r(x, y, z) &= -f_{1,3}(x, y, -z), \\ h_1^r(x, y, z) &= h_1(x, y, -z). \end{aligned}$$

Then $\mathbf{v}_1^r$ and P_1^r satisfy the following equations in Ω ,

$$\begin{aligned} \nabla \cdot (\nabla \mathbf{v}_1^r + \nabla (\mathbf{v}_1^r)^T) - \nabla P_1^r &= \mathbf{f}_1^r, \\ \nabla \cdot \mathbf{v}_1^r &= h_1^r(x, y, z). \end{aligned}$$

Using the reflected variables, the original system becomes

$$\nabla \cdot (\nabla \mathbf{v}_1^r + \nabla (\mathbf{v}_1^r)^T) - \nabla P_1^r = \mathbf{f}_1^r \quad \text{in } \Omega, \quad (3.25)$$

$$\nabla \cdot \mathbf{v}_1^r = h_1^r(x, y, z) \quad \text{in } \Omega, \quad (3.26)$$

$$\nabla \cdot [\theta_n(\nabla \mathbf{v}_2 + \nabla \mathbf{v}_2^T)] - \theta_n \nabla P_2 = \mathbf{f}_2 \quad \text{in } \Omega, \quad (3.27)$$

$$\nabla \cdot [(1 - \theta_n)\eta(\nabla \mathbf{v}_3 + \nabla \mathbf{v}_3^T)] - (1 - \theta_n) \nabla P_2 = \mathbf{f}_3 \quad \text{in } \Omega, \quad (3.28)$$

$$\nabla \cdot (\theta_n \mathbf{v}_2 + (1 - \theta_n) \mathbf{v}_3) = h_2 \quad \text{in } \Omega, \quad (3.29)$$

with the boundary conditions

$$\frac{\partial v_{1,1}^r(x, y, 0)}{\partial z} + \frac{\partial v_{1,3}^r(x, y, 0)}{\partial x} + \frac{\partial v_{2,1}(x, y, 0)}{\partial z} + \frac{\partial v_{2,3}(x, y, 0)}{\partial x} = g_{11} \quad \text{on } \Gamma, \quad (3.30)$$

$$\frac{\partial v_{1,2}^r(x,y,0)}{\partial z} + \frac{\partial v_{1,3}^r(x,y,0)}{\partial y} + \frac{\partial v_{2,2}(x,y,0)}{\partial z} + \frac{\partial v_{2,3}(x,y,0)}{\partial y} = g_{12} \quad \text{on } \Gamma, \quad (3.31)$$

$$-2 \frac{\partial v_{1,3}^r(x,y,0)}{\partial z} + 2 \frac{\partial v_{2,3}(x,y,0)}{\partial z} + P_1 - P_2 = g_{13} \quad \text{on } \Gamma, \quad (3.32)$$

$$\frac{\partial v_{1,1}^r(x,y,0)}{\partial z} + \frac{\partial v_{1,3}^r(x,y,0)}{\partial x} + \eta \frac{\partial v_{3,1}(x,y,0)}{\partial z} + \eta \frac{\partial v_{3,3}(x,y,0)}{\partial x} = g_{21} \quad \text{on } \Gamma, \quad (3.33)$$

$$\frac{\partial v_{1,2}^r(x,y,0)}{\partial z} + \frac{\partial v_{1,3}^r(x,y,0)}{\partial y} + \eta \frac{\partial v_{3,2}(x,y,0)}{\partial z} + \eta \frac{\partial v_{3,3}(x,y,0)}{\partial y} = g_{22} \quad \text{on } \Gamma, \quad (3.34)$$

$$-2 \frac{\partial v_{1,3}^r(x,y,0)}{\partial z} + 2 \eta \frac{\partial v_{3,3}(x,y,0)}{\partial z} + P_1 - P_2 = g_{23} \quad \text{on } \Gamma, \quad (3.35)$$

$$v_{1,1}^r(x,y,0) - \theta_n v_{2,1}(x,y,0) - (1 - \theta_n) v_{3,1}(x,y,0) = 0 \quad \text{on } \Gamma, \quad (3.36)$$

$$v_{1,2}^r(x,y,0) - \theta_n v_{2,2}(x,y,0) - (1 - \theta_n) v_{3,2}(x,y,0) = 0 \quad \text{on } \Gamma, \quad (3.37)$$

$$v_{1,3}^r(x,y,0) + \theta_n v_{2,3}(x,y,0) + (1 - \theta_n) v_{3,3}(x,y,0) = 0 \quad \text{on } \Gamma, \quad (3.38)$$

$$\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{v}_3 \equiv 0 \quad \text{for } |x_1|^2 + |x_2|^2 + |z|^2 \geq d^2. \quad (3.39)$$

In the sequel, we drop the superscript “r” when there is no confusion.

Ellipticity: We consider the ellipticity, in the sense of [2, p. 39], of the system (3.25)–(3.29) for the variables $\mathbf{v}_1, P_1, \mathbf{v}_2, \mathbf{v}_3, P_2$. The matrix l' corresponding to the principal part of the operators in these equations is

$$l'(\Xi) = \begin{pmatrix} L_1 & \Xi^T & 0 & 0 & 0 \\ \Xi & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_n L_1 & 0 & \theta_n \Xi^T \\ 0 & 0 & 0 & (1 - \theta_n) L_1 & (1 - \theta_n) \Xi^T \\ 0 & 0 & \theta_n \Xi & (1 - \theta_n) \Xi & 0 \end{pmatrix},$$

where

$$\Xi = (\xi_1, \xi_2, \xi_3), \quad L_1 = |\Xi|^2 + \Xi^T \Xi,$$

and the determinant of l' is

$$L(\Xi) = \det(l'(\Xi)) = 2\eta^3 \theta_n^3 (1 - \theta_n)^3 |\Xi|^{18} (\theta_n + (1 - \theta_n)/\eta) > 0, \iff |\Xi| > 0.$$

Therefore the system is elliptic.

Supplementary condition: Since we consider a three dimensional problem, the supplementary condition is automatically satisfied [2, p. 39].

Complementing boundary condition: In the following we show that the boundary conditions on Γ satisfy the Complementing Boundary Condition [2, p.42].

We begin with some computations. Let $\mathbf{m}$ be any unit tangential vector on the boundary.

Using the fact that $\mathbf{m} \cdot \mathbf{n} = 0$, we find that the roots τ_k^+ of $L(\mathbf{m} + \tau \mathbf{n}) = 0$ are

$$\tau_j^+ = i, \quad j = 1, 2, \dots, 9.$$

As in [2, p.42], we set

$$M^+(\tau) = \prod_{j=1}^6 (\tau - \tau_j^+) = (\tau - i)^9. \quad (3.40)$$

The matrix $B'(\Xi)$ corresponding to the principal part of the boundary operators is

$$B'(\Xi) = \begin{pmatrix} \xi_3 & 0 & \xi_1 & 0 & \xi_3 & 0 & \xi_1 & 0 & 0 & 0 & 0 \\ 0 & \xi_3 & \xi_2 & 0 & 0 & \xi_3 & \xi_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\xi_3 & 1 & 0 & 0 & 2\xi_3 & 0 & 0 & 0 & -1 \\ \xi_3 & 0 & \xi_1 & 0 & 0 & 0 & 0 & \eta\xi_3 & 0 & \eta\xi_1 & 0 \\ 0 & \xi_3 & \xi_2 & 0 & 0 & 0 & 0 & 0 & \eta\xi_3 & \eta\xi_2 & 0 \\ 0 & 0 & -2\xi_3 & 1 & 0 & 0 & 0 & 0 & 0 & 2\eta\xi_3 & -1 \\ 1 & 0 & 0 & 0 & -\theta_n & 0 & 0 & -(1-\theta_n) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\theta_n & 0 & 0 & -(1-\theta_n) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \theta_n & 0 & 0 & (1-\theta_n) & 0 \end{pmatrix} \quad (3.41)$$

For $\mathbf{n} = (0, 0, 1)$, $\mathbf{m} = (m_1, m_2, 0)$, we have $\mathbf{m} \cdot \mathbf{n} = 0$, $\Xi = \mathbf{m} + \tau \mathbf{n} = (m_1, m_2, \tau)$. Using MATLAB symbolic computations we found that

$$B'(\Xi) \text{adj } l'(\Xi) = \sum_{j=0}^8 \tau^j \tilde{Q}_j, \quad \text{mod } M^+(\tau),$$

with $\tilde{Q}_0$ a matrix with 9 rows and 11 columns. The determinant of the 9×9 submatrix $\tilde{Q}_{01}$ which consists the 1st, and 3rd–10th columns of $\tilde{Q}_0$, and the 9×9 matrix $\tilde{Q}_{02}$ which consists the 2nd–10th columns of $\tilde{Q}_0$, can be computed to be

$$\begin{aligned}\det \tilde{Q}_{01} &= -3 \cdot 2^{78} m_2 \eta^{27} \theta_n^{24} (1-\theta_n)^{24} (1+\theta_n+(1-\theta_n)/\eta)^3 (\theta_n+(1-\theta_n)/\eta)^9, \\ \det \tilde{Q}_{02} &= -3 \cdot 2^{78} m_1 \eta^{27} \theta_n^{24} (1-\theta_n)^{24} (1+\theta_n+(1-\theta_n)/\eta)^3 (\theta_n+(1-\theta_n)/\eta)^9.\end{aligned}$$

Since $m_1^2 + m_2^2 = 1$, $\det \tilde{Q}_{01}$ and $\det \tilde{Q}_{02}$ cannot vanish simultaneously. Hence the rows of $\tilde{Q}_0$ are linearly independent.

The system is said to satisfy the complementing boundary condition if the rows of the matrix $B'(\Xi) \text{adj } l'(\Xi)$ are linearly independent mod $M^+(\bar{v})$; that is, if

$$\sum_{k=1}^6 C_k \sum_{j=0}^5 \tau^j \tilde{Q}_j^k \equiv 0 \pmod{M^+(\tau)},$$

where $\tilde{Q}_j^k$ is the k -th row vector of the matrix $\tilde{Q}_j$, then all the constants C_k are zero. Suppose not all the C_k are zero. Then the row vectors of $\sum_{j=0}^5 \tau^j \tilde{Q}_j$ are linearly dependent, for all τ , and, in particular, for $\tau = 0$; that is, the row vectors of $\tilde{Q}_0$ are linearly dependent, which is a contradiction. Thus the complementing boundary condition on Γ has been verified.

Schauder estimates: Since the reflected system satisfies the supplementary condition and complementing boundary condition, we conclude by Theorem 9.3 on page 74 of [2], that the following Schauder estimates hold:

$$\begin{aligned}& \|\mathbf{v}_1\|_{C^{2+\alpha}(\bar{D})} + \sum_{j=2}^3 \|\mathbf{v}_j\|_{C^{2+\alpha}(\bar{D})} + \|P_1\|_{C^{1+\alpha}(\bar{D})} + \|P_2\|_{C^{1+\alpha}(\bar{D})} \\ & \leq C \left(\|\mathbf{f}_1\|_{C^\alpha(\bar{D})} + \sum_{i=2}^3 \|\mathbf{f}_i\|_{C^\alpha(\bar{D})} + \|h_1\|_{C^{1+\alpha}(\bar{D})} + \|h_2\|_{C^{1+\alpha}(\bar{D})} + \sum_{i=1}^2 \|\mathbf{g}_i\|_{C^{1+\alpha}(\Gamma)} \right) \\ & + C \left(\|\mathbf{v}_1\|_{L^\infty(\bar{D})} + \sum_{j=2}^3 \|\mathbf{v}_j\|_{L^\infty(\bar{D})} + \|P_1\|_{L^\infty(D)} + \|P_2\|_{L^\infty(\Omega)} \right).\end{aligned}\quad (3.42)$$

3.2.2 Case II: $\Gamma = \{z = \varphi(x, y)\}$, $\nabla \varphi$ small in the C^α norm—In this section, we extend the results of Section 3.2.1 to the case where Γ is given by $z = \varphi(x, y)$ with $\nabla \varphi$ small in the C^α norm, and the $\mathbf{v}_i$ have bounded support.

We flatten the boundary by taking

$$x' = x, \quad y' = y, \quad \text{and} \quad z' = z - \phi(x, y).$$

Under this change of variables, we have, for any function $U'(x, y, z') = U(x, y, z)$,

$$\begin{aligned}
U_x &= U'_x + U'_{z'}(-\phi_x), & U_y &= U'_y + U'_{z'}(-\phi_y), & U_z &= U'_{z'} \\
U_{xx} &= U'_{xx} + 2U'_{xz'}(-\phi_x) + U'_{z'z'}(-\phi_x)^2 + U'_{z'}(-\phi_{xx}), \\
U_{yy} &= U'_{yy} + 2U'_{yz'}(-\phi_y) + U'_{z'z'}(-\phi_y)^2 + U'_{z'}(-\phi_{yy}), \\
U_{xy} &= U'_{xy} + U'_{xz'}(-\phi_y) + U'_{yz'}(-\phi_x) + U'_{z'z'}(-\phi_x)(-\phi_y) + U'_{z'}(-\phi_{xy}), \\
U_{xz} &= U'_{xz'} + U'_{z'z'}(-\phi_x), & U_{yz} &= U'_{yz'} + U'_{z'z'}(-\phi_y), \\
U_{zz} &= U'_{zz}.
\end{aligned}$$

We apply the formulas to the equations for the components of $\mathbf{v}_1$, and for simplicity drop the primes. The equations for $\mathbf{v}_1$ in the new variables are:

$$\begin{aligned}
&2(v_{11})_{xx} + (v_{11})_{yy} + (1 + 2\phi_x^2 + \phi_y^2)(v_{11})_{zz} - 4\phi_x(v_{11})_{xz} - 2\phi_y(v_{11})_{yz} \\
&- (2\phi_{xx} + \phi_{yy})(v_{11})_z + (v_{12})_{xy} - \phi_y(v_{12})_{xz} - \phi_x(v_{12})_{yz} + \phi_x\phi_y(v_{12})_{zz} \\
&- \phi_{xy}(v_{12})_z + (v_{13})_{xz} - \phi_x(v_{13})_{zz} - (P_1)_x + \phi_x(P_1)_z = f_{11},
\end{aligned}$$

$$\begin{aligned}
&(v_{11})_{xy} - \phi_y(v_{11})_{xz} - \phi_x(v_{11})_{yz} + \phi_x\phi_y(v_{11})_{zz} - \phi_{xy}(v_{11})_z + (v_{12})_{xx} + 2(v_{12})_{yy} \\
&+ (1 + \phi_x^2 + 2\phi_y^2)(v_{12})_{zz} - 2\phi_x(v_{12})_{xz} - 4\phi_y(v_{12})_{yz} - (\phi_{xx} + 2\phi_{yy})(v_{12})_z + (v_{13})_{yz} \\
&- \phi_y(v_{13})_{zz} - (P_1)_y + \phi_y(P_1)_z = f_{12},
\end{aligned}$$

$$\begin{aligned}
&(v_{11})_{xz} - \phi_x(v_{11})_{zz} + (v_{12})_{yz} - \phi_y(v_{12})_{zz} + (v_{13})_{xx} + (v_{13})_{yy} + (2 + \phi_x^2 + \phi_y^2)(v_{13})_{zz} \\
&- 2\phi_x(v_{13})_{xz} - 2\phi_y(v_{13})_{yz} - (\phi_{xx} + \phi_{yy})(v_{13})_z - (P_1)_z = f_{13},
\end{aligned}$$

$$(v_{11})_x - \phi_x(v_{11})_z + (v_{12})_y - \phi_y(v_{12})_z + (v_{13})_z = h_1,$$

or

$$\begin{aligned}
\nabla \cdot (\nabla \mathbf{v}_1 + \nabla \mathbf{v}_1^T) - \nabla P_1 &= \mathbf{f}_1 + F_1^1(D^2 \mathbf{v}_1, \nabla P_1, \nabla \phi) + F_1^2(\nabla \mathbf{v}_1, D^2 \phi) \quad \text{in } D, \\
\nabla \cdot \mathbf{v}_1 &= h_1 + H_1(\nabla \phi, \nabla \mathbf{v}_1) \quad \text{in } D,
\end{aligned}$$

where

$$\begin{aligned}
\|F_1^1\|_{C^\alpha(\bar{\Omega})} &\leq C \left(\|D^2 \mathbf{v}_1\|_{C^\alpha(\bar{\Omega})} + \|\nabla P_1\|_{C^\alpha(\bar{\Omega})} \right) \|\nabla \phi\|_{C^\alpha(\bar{\Omega})}, \\
\|F_1^2\|_{C^\alpha(\bar{\Omega})} &\leq C \|\nabla \mathbf{v}_1\|_{C^\alpha(\bar{\Omega})} \|D^2 \phi\|_{C^\alpha(\bar{\Omega})}, \\
\|H_1\|_{C^{1+\alpha}(\bar{\Omega})} &\leq C \left(\|\nabla \mathbf{v}_1\|_{C^{1+\alpha}(\bar{\Omega})} \|\nabla \phi\|_{C^\alpha(\bar{\Omega})} + \|\nabla v_1\|_{C^\alpha(\bar{\Omega})} \|D^2 \phi\|_{C^\alpha(\bar{\Omega})} \right).
\end{aligned}$$

Similarly, under the new variables,

$$\begin{aligned} \nabla \cdot [\theta_n(\nabla \mathbf{v}_2 + \nabla \mathbf{v}_2^T)] - \theta_n \nabla P_2 &= \mathbf{f}_2 + F_2^1(D^2 \mathbf{v}_2, \nabla P_2, \nabla \phi) + F_2^2(\nabla \mathbf{v}_2, D^2 \phi), \\ \nabla \cdot [(1-\theta_n)\eta(\nabla \mathbf{v}_3 + \nabla \mathbf{v}_3^T)] - (1-\theta_n) \nabla P_2 &= \mathbf{f}_3 + F_3^1(D^2 \mathbf{v}_3, \nabla P_2, \nabla \phi) + F_3^2(\nabla \mathbf{v}_3, D^2 \phi), \\ \nabla \cdot (\theta_n \mathbf{v}_2 + (1-\theta_n) \mathbf{v}_3) &= h_2 + H_2(\nabla \mathbf{v}_2, \nabla \mathbf{v}_3, \nabla \phi), \end{aligned}$$

The boundary conditions are transformed in a similar way.

We now use the reflections as in Section 3.2.1, and thus derive that the following estimates for any solution of the system:

$$\begin{aligned} & \sum_{j=1}^3 \|\mathbf{v}_j\|_{C^{2+\alpha}(\bar{\Omega})} + \sum_{j=1}^2 \|P_j\|_{C^{1+\alpha}(\bar{\Omega})} \\ & \leq C \left\{ \sum_{i=1}^3 \|\mathbf{f}_i\|_{C^\alpha(\bar{\Omega})} + \|\nabla \phi\|_{C^\alpha(\bar{\Omega})} \|D^2 \mathbf{v}_j\|_{C^\alpha(\bar{\Omega})} + \|D^2 \phi\|_{C^\alpha(\bar{\Omega})} \|\nabla \mathbf{v}_j\|_{C^\alpha(\bar{\Omega})} \right. \\ & \quad + \sum_{i=1}^2 \|h_i\|_{C^{1+\alpha}(\bar{\Omega})} + \|\mathbf{g}_i\|_{C^{1+\alpha}(\bar{\Omega})} + \|\nabla \phi\|_{C^\alpha(\bar{\Omega})} \|\nabla \mathbf{v}_j\|_{C^{1+\alpha}(\bar{\Omega})} \\ & \quad \left. + \|\mathbf{v}_1\|_{L^\infty(\bar{\Omega})} + \sum_{j=2}^3 \|\mathbf{v}_j\|_{L^\infty(\bar{\Omega})} + \|P_1\|_{L^\infty(D)} + \|P_2\|_{L^\infty(\Omega)} \right\}. \end{aligned}$$

Using the assumption that $\|\nabla \phi\|_{C^{\alpha}(\bar{\Omega})}$ is small, and the interpolation inequality

$$\|\mathbf{v}_j\|_{C^{1+\alpha}(\bar{\Omega})} \leq \varepsilon \|\mathbf{v}_j\|_{C^{2+\alpha}(\bar{\Omega})} + C_\varepsilon \|\mathbf{v}_j\|_{L^\infty},$$

and going back to the original domain, we obtain the Schauder estimates (3.42). We note that the constant C depends on θ_n .

3.2.3 Case III: arbitrary $\Gamma = \{z = \varphi(x, y)\}$ —Let χ_k ($k = 1, \dots, N$) be a partition of unity of Ω , such that $\chi_1, \dots, \chi_{N_1}$ form a covering of Γ , $\chi_{N_1+1}, \dots, \chi_{N_2}$ form a covering of G , and the support of the remaining χ_k lie in the interior of Ω . The support of each χ_k is taken as a ball B_k with a small radius $< d$. By translation and rotation, we may assume that $B_k \cap \Gamma$, for any $1 \leq k \leq N_1$, has the form $z = \psi(x, y)$ where $|\nabla \psi|_{C^\alpha}$ is small, as in Section 3.2.2. Then the Schauder estimates can be applied in this ball. Similarly one can derive Schauder estimates in the balls which cover G . The Schauder estimates hold also in each ball B_k for $k > N_2$, and they also hold for any compact subset D_1 of D which does not intersect Γ .

Define $\mathbf{u}^k = \chi_k \mathbf{u}$, $\mathbf{v}^k = \chi_k \mathbf{v}$, $\mathbf{w}^k = \chi_k \mathbf{w}$, $P_j^k = \chi_k P_j$, so that

$$\mathbf{v}_j = \sum_{k=1}^N \mathbf{v}_j^k, \quad j=1, 2, 3, \quad \text{and} \quad P_j^k = \sum_{k=1}^N P_j^k, \quad j=1, 2.$$

For each k , we have the following equations,

$$\begin{aligned}
\nabla \cdot (\nabla \mathbf{v}_1^k + (\nabla \mathbf{v}_1^k)^T) - \nabla P_1^k &= \chi_k \mathbf{f}_1 + D^2 \chi_k \cdot \mathbf{v}_1 + \Delta \chi_k \mathbf{v}_1 + \nabla \chi_k (h_1 + P_1) + \nabla \chi_k \cdot (2 \nabla \mathbf{v}_1 + \nabla \mathbf{v}_1^T) \quad \text{in } D, \\
\nabla \cdot \mathbf{v}_1^k &= \chi_k h_1 + \nabla \chi_k \cdot \mathbf{v}_1 \quad \text{in } D, \\
\nabla \cdot [\theta_n (\nabla \mathbf{v}_2^k + (\nabla \mathbf{v}_2^k)^T)] - \theta_n \nabla P_2^k &= \chi_k \mathbf{f}_2 + \theta_n D^2 \chi_k \cdot \mathbf{v}_2 + \theta_n \Delta \chi_k \mathbf{v}_2 \\
&\quad + \theta_n \nabla \chi_k \cdot (\nabla \mathbf{v}_2 + \nabla \mathbf{v}_2^T) + \nabla \chi_k (\nabla \cdot (\theta_n \mathbf{v}_2)) + \nabla \chi_k \cdot \nabla (\theta_n \mathbf{v}_2) + \theta_n P_2 \nabla \chi_k \quad \text{in } \Omega, \\
\nabla \cdot [(1-\theta_n) \eta (\nabla \mathbf{v}_3 + \nabla \mathbf{v}_3^T)] - (1-\theta_n) \nabla P_2 &= \chi_k \mathbf{f}_3 + \eta (1-\theta_n) D^2 \chi_k \cdot \mathbf{v}_3 + \eta (1-\theta_n) \Delta \chi_k \mathbf{v}_3 \\
&\quad + \eta (1-\theta_n) \nabla \chi_k \cdot (\nabla \mathbf{v}_3 + \nabla \mathbf{v}_3^T) + \eta \nabla \chi_k (\nabla \cdot ((1-\theta_n) \mathbf{v}_3)) \\
&\quad + \eta \nabla \chi_k \cdot \nabla ((1-\theta_n) \mathbf{v}_3) + \eta (1-\theta_n) P_2 \nabla \chi_k \quad \text{in } \Omega, \\
\nabla \cdot (\theta_n \mathbf{v}_2 + (1-\theta_n) \mathbf{v}_3) &= \chi_k h_2 + \nabla \chi_k \cdot (\theta_n \mathbf{v}_2 + (1-\theta_n) \mathbf{v}_3) \quad \text{in } \Omega,
\end{aligned}$$

with corresponding boundary conditions whose right hand side may involve $\nabla \chi_k$.

By applying (3.42) we obtain, for each k ,

$$\begin{aligned}
&\sum_{j=1}^3 \|\mathbf{v}_j^k\|_{C^{2+\alpha}(\overline{\Omega})} + \sum_{j=1}^2 \|P_j^k\|_{C^{1+\alpha}(\overline{\Omega})} \\
&\leq C \left(\sum_{i=1}^3 \|\mathbf{f}_i\|_{C^\alpha(\overline{\Omega})} + \sum_{i=1}^2 \|h_i\|_{C^{1+\alpha}(\overline{\Omega})} + \|\mathbf{g}_i\|_{C^{1+\alpha}(\overline{\Omega})} \right) \\
&+ C \left(\sum_i \|\nabla \chi_k\|_{L^\infty} (\|\mathbf{v}_j\|_{C^{1+\alpha}(\overline{\Omega})} + \|P_j\|_{C^\alpha(\overline{\Omega})}) + \|D^2 \chi_k\|_{L^\infty} \|\mathbf{v}_j\|_{C^\alpha(\overline{\Omega})} \right). \quad (3.43)
\end{aligned}$$

Summing over k , and combining the Schauder estimates for $\mathbf{v}_i, P_i$, we obtain

$$\begin{aligned}
&\|\mathbf{v}_1\|_{C^{2+\alpha}(\overline{\Omega})} + \sum_{j=2}^3 \|\mathbf{v}_j\|_{C^{2+\alpha}(\overline{\Omega})} + \|P_1\|_{C^{1+\alpha}(\overline{\Omega})} + \|P_2\|_{C^{1+\alpha}(\overline{\Omega})} \\
&\leq C \left(\|\mathbf{f}_1\|_{C^\alpha(\overline{\Omega})} + \sum_{i=2}^3 \|\mathbf{f}_i\|_{C^\alpha(\overline{\Omega})} + \|h_1\|_{C^{1+\alpha}(\overline{\Omega})} + \|h_2\|_{C^{1+\alpha}(\overline{\Omega})} + \sum_{i=1}^2 \|\mathbf{g}_i\|_{C^{1+\alpha}(\Gamma)} \right) \\
&+ C \left(\|\mathbf{v}_1\|_{C^{1+\alpha}(\overline{\Omega})} + \sum_{j=2}^3 \|\mathbf{v}_j\|_{C^{1+\alpha}(\overline{\Omega})} + \|P_1\|_{C^\alpha(\overline{\Omega})} + \|P_2\|_{C^\alpha(\overline{\Omega})} \right). \quad (3.44)
\end{aligned}$$

Using again interpolation, we again obtain the estimate (3.42).

Noting that the solution is unique, up to constants for P_j , we proceed to show that the terms involving the L^∞ norms on the right-hand side of (3.42) can be dropped when the $1 + \alpha$ norm of P_j is replaced by α norm of ∇P_j . Clearly, the estimate (3.42) is valid when P_j is replaced by $P_j - \mu$, for any constant μ . We take a point $M \in \Gamma$ and set $\mu = P_1(M)$. Then, by Poincaré's inequality,

$$\|P_1 - \mu\|_{L^\infty} \leq C \|\nabla P_1\|_{L^\infty},$$

and, by (3.19),

$$\begin{aligned}
\|P_2 - \mu\|_{L^\infty} &\leq \|P_2 - P_2(M)\|_{L^\infty} + \|P_2(M) - P_1(M)\|_{L^\infty} \\
&\leq C \|\nabla P_2\|_{L^\infty} + C \left(\|g_1\|_{L^\infty} + \|\nabla \mathbf{v}_1\|_{L^\infty} + \|\nabla \mathbf{v}_2\|_{L^\infty} \right). \quad (3.45)
\end{aligned}$$

From (3.42) it then follows that

$$\begin{aligned}
& \|\mathbf{v}_1\|_{C^{2+\alpha}(\overline{D})} + \sum_{j=2}^3 \|\mathbf{v}_j\|_{C^{2+\alpha}(\overline{D})} + \|\nabla P_1\|_{C^\alpha(\overline{D})} + \|\nabla P_2\|_{C^\alpha(\overline{D})} \\
& \leq C \left(\|\mathbf{f}_1\|_{C^\alpha(\overline{D})} + \sum_{i=2}^3 \|\mathbf{f}_i\|_{C^\alpha(\overline{D})} + \|h_1\|_{C^{1+\alpha}(\overline{D})} + \|h_2\|_{C^{1+\alpha}(\overline{D})} + \sum_{i=1}^2 \|\mathbf{g}_i\|_{C^{1+\alpha}(\Gamma)} \right) \\
& \quad + C \left(\|\mathbf{v}_1\|_{L^\infty} + \sum_{j=2}^3 \|\mathbf{v}_j\|_{L^\infty} + \|\nabla P_1\|_{L^\infty} + \|\nabla P_2\|_{L^\infty} \right). \quad (3.46)
\end{aligned}$$

Next, we shall prove, by contradiction, that the L^∞ norms of the $\mathbf{v}_i$ can be eliminated by the right-hand side of (3.46), so that

$$\begin{aligned}
& \|\mathbf{v}_1\|_{C^{2+\alpha}(\overline{D})} + \sum_{j=2}^3 \|\mathbf{v}_j\|_{C^{2+\alpha}(\overline{D})} + \|\nabla P_1\|_{C^\alpha(\overline{D})} + \|\nabla P_2\|_{C^\alpha(\overline{D})} \\
& \leq C \left(\|\mathbf{f}_1\|_{C^\alpha(\overline{D})} + \sum_{i=2}^3 \|\mathbf{f}_i\|_{C^\alpha(\overline{D})} + \|h_1\|_{C^{1+\alpha}(\overline{D})} + \|h_2\|_{C^{1+\alpha}(\overline{D})} + \sum_{i=1}^2 \|\mathbf{g}_i\|_{C^{1+\alpha}(\Gamma)} \right). \quad (3.47)
\end{aligned}$$

Assume that this is not true. Then there exist sequences of data $\mathbf{f}_i^n, h_i^n, \mathbf{g}_i^n$ bounded in their respective norms, and solutions $\mathbf{v}_j^n$ such that $Q_n \equiv \{\text{left-hand side of (3.47)}\}$ corresponding to

$\mathbf{v}_j^n \rightarrow \infty$. Hence $Q_n^0 \equiv \left(\|\mathbf{v}_1^n\|_{L^\infty} + \sum_{j=2}^3 \|\mathbf{v}_j^n\|_{L^\infty} + \|\nabla P_1^n\|_{L^\infty} + \|\nabla P_2^n\|_{L^\infty} \right) \rightarrow \infty$. We normalize the solution and data by Q_n^0 . Since the system of equations is linear, we get $\mathbf{f}_i^n, h_i^n, \mathbf{g}_i^n \rightarrow 0$, so that, by (3.47), a subsequence of the solutions $(\mathbf{v}_1^n, \mathbf{v}_2^n, \mathbf{v}_3^n, \nabla P_1^n, \nabla P_2^n)$ converges uniformly in their respective domains to a solution $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \nabla P_1, \nabla P_2)$ which is, by uniqueness, identically zero. Since, however

$$\|\mathbf{v}_1\|_{L^\infty} + \sum_{j=2}^3 \|\mathbf{v}_j\|_{L^\infty} + \|\nabla P_1\|_{L^\infty} + \|\nabla P_2\|_{L^\infty} = 1, \text{ this is a contradiction.}$$

We note that the above proof is valid for any $0 < \lambda < 1$ and the constant C in (3.47) can be chosen to be independent of λ .

3.3 $C^{3+\alpha}$ a priori estimates

The arguments used to derive the $C^{2+\alpha}$ Schauder estimate can also be applied to show that if $\Gamma \in C^{3+\alpha}$, $\theta_n \in C^{3+\alpha}$, $\mathbf{f}_j \in C^{1+\alpha}$, $\mathbf{g}_i, h_i \in C^{2+\alpha}$ in their respective sets, then the following $C^{3+\alpha}$ Schauder estimate holds:

$$\begin{aligned}
& \|\mathbf{v}_1\|_{C^{3+\alpha}(\overline{D})} + \sum_{j=2}^3 \|\mathbf{v}_j\|_{C^{3+\alpha}(\overline{D})} + \|\nabla P_1\|_{C^{1+\alpha}(\overline{D})} + \|\nabla P_2\|_{C^{1+\alpha}(\overline{D})} \\
& \leq C \left(\|\mathbf{f}_1\|_{C^{1+\alpha}(\overline{D})} + \sum_{i=2}^3 \|\mathbf{f}_i\|_{C^{1+\alpha}(\overline{D})} + \|h_1\|_{C^{2+\alpha}(\overline{D})} + \|h_2\|_{C^{2+\alpha}(\overline{D})} + \sum_{i=1}^2 \|\mathbf{g}_i\|_{C^{2+\alpha}(\Gamma)} \right). \quad (3.48)
\end{aligned}$$

3.4 Existence of solution for $\lambda = 0$

In this section, we take $\lambda = 0$, and prove that if Γ is in C^∞ , $\mathbf{f}_1, h_1 \in C^\infty(D)$, $\mathbf{f}_2, \mathbf{f}_3, h_2 \in C^\infty(\mathcal{Q})$, and $\mathbf{g}_1, \mathbf{g}_2 \in C^\infty(\Gamma)$, then there exists a weak solution, $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, P_1, P_2) \in H^1(D) \times H^1(\mathcal{Q}) \times H^1(\mathcal{Q}) \times L^2(D) \times L^2(\mathcal{Q})$ to the system (3.14)–(3.17) with boundary conditions (3.19)–(3.22). Next we shall improve the regularity and show that the solution is in $C^{2+\alpha}$. Finally, we shall prove the existence of solution in $C^{2+\alpha}$ under the assumptions (3.10)–(3.13).

3.4.1 Existence of weak solution—We first consider the case that $h_1 = h_2 = 0$ in $\mathcal{Q}$, and $\mathbf{g}_1 = \mathbf{g}_2 = 0$ on Γ . The general case will be considered later.

We introduce the linear space $\mathbf{H}$ of vectors $(\phi_1, \phi_2, \phi_3) \in \mathbf{H}$ where each ϕ_i is a 3D vector, as follows,

$$\mathbf{H} = \{(\phi_1 \in H^1(D), \phi_2 \in H^1(\Omega), \phi_3 \in H^1(\Omega)) : (\phi_1, \phi_2, \phi_3) \text{ satisfies (3.15), (3.18), (3.21), (3.22)}\}.$$

We define a norm $\|\cdot\|_H$ on $\mathbf{H}$ by

$$\|(\phi_1, \phi_2, \phi_3)\|_H = (\|\phi_1\|_{H^1(D)}^2 + \|\phi_2\|_{H^1(\Omega)}^2 + \|\phi_3\|_{H^1(\Omega)}^2)^{1/2}$$

and an inner product by

$$\langle (\phi_1, \phi_2, \phi_3), (\psi_1, \psi_2, \psi_3) \rangle = \langle \phi_1, \psi_1 \rangle_{H^1(D)} + \langle \phi_2, \psi_2 \rangle_{H^1(\Omega)} + \langle \phi_3, \psi_3 \rangle_{H^1(\Omega)}.$$

One can easily show that $\mathbf{H}$ is a Hilbert space.

We multiply the j -th equation of (3.14), (3.16) and (3.17) by vector functions ϕ_{1j} , ϕ_{2j} , and ϕ_{3j} , then take the sum over j , and integrate over the domain D for (3.14), and over the domain Ω for (3.16) and (3.17). After integration by parts, we obtain,

$$\begin{aligned} \int_D \phi_1 \cdot \mathbf{f}_1 d\mathbf{x} &= \int_D \sum_{j=1}^3 \phi_{1,j} \left[\sum_{i=1}^3 \frac{\partial}{\partial x_i} \left(\frac{\partial v_{1,j}}{\partial x_i} \right) - \frac{\partial P_1}{\partial x_j} \right] d\mathbf{x} \\ &= \int_{\partial D} \sum_{j=1}^3 \phi_{1,j} \left[\sum_{i=1}^3 \left(\frac{\partial v_{1,j}}{\partial x_i} \right) n_i - P_1 n_j \right] dS - \int_D \sum_{i,j=1}^3 \frac{\partial \phi_{1,j}}{\partial x_i} \left(\frac{\partial v_{1,j}}{\partial x_i} \right) d\mathbf{x} + \int_D P_1 \nabla \cdot \phi_1 d\mathbf{x}. \end{aligned}$$

$$\begin{aligned} \int_\Omega \phi_2 \cdot \mathbf{f}_2 d\mathbf{x} &= \int_\Omega \sum_{j=1}^3 \phi_{2,j} \left[\sum_{i=1}^3 \frac{\partial}{\partial x_i} \left(\theta_n \left(\frac{\partial v_{2,j}}{\partial x_i} \right) \right) - \theta_n \frac{\partial P_2}{\partial x_j} \right] d\mathbf{x} \\ &= \int_{\partial \Omega} \sum_{j=1}^3 \phi_{2,j} \theta_n \left[\sum_{i=1}^3 \left(\frac{\partial v_{2,j}}{\partial x_i} \right) n_i - P_2 n_j \right] dS - \int_\Omega \sum_{i,j=1}^3 \theta_n \left(\frac{\partial \phi_{2,j}}{\partial x_i} \right) \left(\frac{\partial v_{2,j}}{\partial x_i} \right) d\mathbf{x} + \int_\Omega P_2 \nabla \cdot (\theta_n \phi_2) d\mathbf{x}. \end{aligned}$$

$$\begin{aligned} \int_\Omega \phi_3 \cdot \mathbf{f}_3 d\mathbf{x} &= \int_\Omega \sum_{j=1}^3 \phi_{3,j} \left[\sum_{i=1}^3 \frac{\partial}{\partial x_i} \left((1-\theta_n) \eta \left(\frac{\partial v_{3,j}}{\partial x_i} \right) \right) - (1-\theta_n) \frac{\partial P_2}{\partial x_j} \right] d\mathbf{x} \\ &= \int_{\partial \Omega} \sum_{j=1}^3 \phi_{3,j} (1-\theta_n) \left[\sum_{i=1}^3 \eta \left(\frac{\partial v_{3,j}}{\partial x_i} \right) n_i - P_2 n_j \right] dS - \int_\Omega \sum_{i,j=1}^3 (1-\theta_n) \eta \left(\frac{\partial \phi_{3,j}}{\partial x_i} \right) \left(\frac{\partial v_{3,j}}{\partial x_i} \right) d\mathbf{x} + \int_\Omega P_2 \nabla \cdot ((1-\theta_n) \phi_3) d\mathbf{x}. \end{aligned}$$

We take the sum of the above three equations, and obtain the relation

$$I_d = I_b + I_p - I_f, \quad (3.49)$$

where I_b is the sum of the boundary integrals

$$\begin{aligned}
I_b = & \int_{\partial D} \sum_{j=1}^3 \phi_{1,j} \left[\sum_{i=1}^3 \left(\frac{\partial v_{1,j}}{\partial x_i} \right) n_i - P_1 n_j \right] dS \\
& + \int_{\partial \Omega} \sum_{j=1}^3 \phi_{2,j} \theta_n \left[\sum_{i=1}^3 \left(\frac{\partial v_{2,j}}{\partial x_i} \right) n_i - P_2 n_j \right] dS \\
& + \int_{\partial \Omega} \sum_{j=1}^3 \phi_{3,j} (1 - \theta_n) \left[\sum_{i=1}^3 \eta \left(\frac{\partial v_{3,j}}{\partial x_i} \right) n_i - P_2 n_j \right] dS,
\end{aligned}$$

and

$$\begin{aligned}
I_d = & \int_D \sum_{i,j=1}^3 \frac{\partial \phi_{1,j}}{\partial x_i} \left(\frac{\partial v_{1,j}}{\partial x_i} \right) d\mathbf{x} + \int_{\Omega} \sum_{i,j=1}^3 \theta_n \frac{\partial \phi_{2,j}}{\partial x_i} \left(\frac{\partial v_{2,j}}{\partial x_i} \right) d\mathbf{x} + \int_{\Omega} \sum_{i,j=1}^3 (1 - \theta_n) \eta \frac{\partial \phi_{3,j}}{\partial x_i} \left(\frac{\partial v_{3,j}}{\partial x_i} \right) d\mathbf{x}, \\
I_p = & \int_D P_1 \nabla \cdot \phi_1 d\mathbf{x} + \int_{\Omega} P_2 \nabla \cdot (\theta_n \phi_2 + \theta_3 \phi_3) d\mathbf{x}, \\
I_f = & \int_D \phi_1 \cdot \mathbf{f}_1 d\mathbf{x} + \int_{\Omega} (\phi_2 \cdot \mathbf{f}_2 + \phi_3 \cdot \mathbf{f}_3) d\mathbf{x}.
\end{aligned}$$

If $(\phi_1, \phi_2, \phi_3) \in \mathbf{H}$, then

$$I_p = 0,$$

and

$$\begin{aligned}
I_b = & - \int_{\Gamma} \sum_{j=1}^3 (\theta_n \phi_{2,j} + (1 - \theta_n) \phi_{3,j}) \left[\left(\frac{\partial v_{1,j}}{\partial x_i} \right) n_i - P_1 n_j \right] dS \\
& + \int_{\Gamma} \sum_{j=1}^3 \phi_{2,j} \theta_n \left[\left(\frac{\partial v_{2,j}}{\partial x_i} \right) n_i - P_2 n_j \right] dS + \int_{\Gamma} \sum_{j=1}^3 \phi_{3,j} (1 - \theta_n) \left[\eta \left(\frac{\partial v_{3,j}}{\partial x_i} \right) n_i - P_2 n_j \right] dS \\
= & \int_{\Gamma} \sum_{j=1}^3 \theta_n \phi_{2,j} \left[\left(\frac{\partial v_{2,j}}{\partial x_i} \right) n_i - P_2 n_j - \left(\frac{\partial v_{1,j}}{\partial x_i} \right) n_i + P_1 n_j \right] dS \\
& + \int_{\Gamma} \sum_{j=1}^3 (1 - \theta_n) \phi_{3,j} \left[\eta \left(\frac{\partial v_{3,j}}{\partial x_i} \right) n_i - P_2 n_j - \left(\frac{\partial v_{1,j}}{\partial x_i} \right) n_i + P_1 n_j \right] dS \\
= & \int_{\Gamma} (\theta_n \phi_2 \cdot \mathbf{g}_1 + (1 - \theta_n) \phi_3 \cdot \mathbf{g}_2) dS = 0.
\end{aligned}$$

Define the bilinear form

$$\begin{aligned}
B[\phi, \mathbf{v}] = I_d = & \int_D \sum_{i,j=1}^3 \frac{\partial \phi_{1,j}}{\partial x_i} \left(\frac{\partial v_{1,j}}{\partial x_i} \right) d\mathbf{x} + \int_{\Omega} \sum_{i,j=1}^3 \theta_n \frac{\partial \phi_{2,j}}{\partial x_i} \left(\frac{\partial v_{2,j}}{\partial x_i} \right) d\mathbf{x} \\
& + \int_{\Omega} \sum_{i,j=1}^3 (1 - \theta_n) \eta \frac{\partial \phi_{3,j}}{\partial x_i} \left(\frac{\partial v_{3,j}}{\partial x_i} \right) d\mathbf{x}.
\end{aligned} \tag{3.50}$$

By the Cauchy-Schwartz inequality,

$$|B[\phi, \mathbf{v}]| \leq C \|\phi\|_H \cdot \|\mathbf{v}\|_H. \tag{3.51}$$

Also,

$$\begin{aligned}
B[\mathbf{v}, \mathbf{v}] &= \int_D \sum_{i,j=1}^3 \left(\frac{\partial v_{1,i}}{\partial x_j} \right)^2 d\mathbf{x} + \int_\Omega \theta_n \sum_{i,j=1}^3 \left(\frac{\partial v_{2,i}}{\partial x_j} \right)^2 d\mathbf{x} + (1-\theta_n) \eta \sum_{i,j=1}^3 \left(\frac{\partial v_{3,i}}{\partial x_j} \right)^2 d\mathbf{x} \\
&\geq C \left(\|\nabla \mathbf{v}_1\|_{L^2(D)}^2 + \|\nabla \mathbf{v}_2\|_{L^2(\Omega)}^2 + \|\nabla \mathbf{v}_3\|_{L^2(\Omega)}^2 \right), \quad (C > 0).
\end{aligned}$$

We can then use the Poincaré's inequality (since $\mathbf{v} = 0$ on the fixed boundary, by (3.22)), to deduce that

$$B[\mathbf{v}, \mathbf{v}] \geq C \|\mathbf{v}\|_H^2, \quad (3.52)$$

with another positive constant C . In view of (3.51) and (3.52), we can apply the Lax-Milgram lemma, to conclude that there exists a unique element $\mathbf{v} \in \mathbf{H}$ such that

$$B[\phi, \mathbf{v}] = -I_f(\phi), \quad \forall \phi \in \mathbf{H}. \quad (3.53)$$

Next we show that there exist scalar functions $P_1 \in L^2(D)$, $P_2 \in L^2(\Omega)$ such that

$$B[\phi, \mathbf{v}] = I_p(\phi) - I_f(\phi), \quad \forall \phi \in \hat{\mathbf{H}}, \quad (3.54)$$

where $\mathbf{v}$ is the weak solution of (3.53), and

$$\hat{\mathbf{H}} = \{\phi_1 \in H^1(D), \phi_2 \in H^1(\Omega), \phi_3 \in H^1(\Omega) : (\phi_1, \phi_2, \phi_3) \text{ satisfies (3.21), (3.22)}\}.$$

This will prove that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, P_1, P_2)$ is a weak solution in case the h_i and $\mathbf{g}_i$ are identically zero.

We will need Propositions 1.1 and 1.2 (ii) of [43], which we restate as a lemma:

Lemma 1

- i. Let Ω be an open set of $\mathbb{R}^n$ and $\mathbf{f} = (f_1, \dots, f_n)$, $f_i \in \mathcal{D}'(\Omega)$, $i = 1, \dots, n$. A necessary and sufficient condition that $\mathbf{f} = \nabla p$, for some p in $\mathcal{D}'(\Omega)$, is that $\langle \mathbf{f}, \nu \rangle = 0$, for all $\nu \in C_0^\infty(\Omega)$ with $\nabla \cdot \nu = 0$.

Let Ω be a bounded Lipschitz open set in $\mathbb{R}^n$. If a distribution p has all its first derivatives $D_i p$, $1 \leq i \leq n$, in $H^{-1}(\Omega)$, then $p \in L^2(\Omega)$ and $\|p\|_{L^2(\Omega)/\mathbb{R}} \leq c(\Omega) \|\nabla p\|_{H^{-1}(\Omega)}$.

In the statement (i), the notation $\langle \mathbf{f}, \nu \rangle$ denotes the application of the distribution $\mathbf{f}$ to a test function ν .

To any $\Phi \in H_0^1(D \cup \Gamma \cup \Omega)$, we correspond a vector ϕ by

$$\phi_1 = \Phi|_D, \quad \phi_2 = \phi_3 = \Phi|_\Omega, \quad \phi = (\phi_1, \phi_2, \phi_3).$$

It is clear that $\nabla \cdot \phi = 0$ if and only if $\nabla \cdot \phi_1 = 0$ and $\nabla \cdot (\theta_n \phi_2 + (1-\theta_n) \phi_3) = \nabla \cdot ((\theta_n + (1-\theta_n)) \Phi) = \nabla \cdot \Phi = 0$. We define the distribution $\mathbf{F}$ by

$$\mathbf{F}(\Phi) = B[\phi, \mathbf{v}] + I_f(\phi), \text{ where } \phi = (\phi_1, \phi_2, \phi_3).$$

Then $\mathbf{F}(\Phi) = I_p(\Phi) = 0$ if $\Phi \in C_0^\infty(D \cup \Gamma \cup \Omega)$ with $\nabla \cdot \Phi = 0$. Thus, by Lemma 1 (i), there exists a distribution P such that

$$\mathbf{F} = \nabla P.$$

Further, since

$$|\mathbf{F}(\Phi)| \leq |B[\phi, \mathbf{v}]| + |I_f(\phi)| \leq C \|\Phi\|_{H^1(D \cup \Gamma \cup \Omega)},$$

where C is a positive constant depending on $\mathbf{f}_i, \mathbf{v}_i, \theta_n$, we deduce that $\mathbf{F} = \nabla P \in H^{-1}(D \cup \Gamma \cup \Omega)$, and by Lemma 1 (ii), we conclude that $P \in L^2(D \cup \Gamma \cup \Omega)$.

We next define $P_1 = P|_D, P_2 = P|_\Omega$ so that $P_1 \in L^2(D), P_2 \in L^2(\Omega)$, and proceed to show that (P_1, P_2) is a solution to (3.54), i.e., for any $\varphi \in \hat{\mathbf{H}}, \varphi_i \in C^\infty$, the equation (3.54) holds. We introduce a function $\Phi \in H_0^1(D \cup \Gamma \cup \Omega)$ by

$$\Phi|_D = \phi_1, \quad \Phi|_\Omega = \theta_n \phi_2 + (1 - \theta_n) \phi_3.$$

The conditions (3.21), (3.22) imply that $\Phi \in H_0^1(D \cup \Gamma \cup \Omega)$. We define $\tilde{\varphi} = (\tilde{\varphi}_D, \tilde{\varphi}_\Omega)$; then $\tilde{\varphi} = \varphi - \varphi \in \mathbf{H}$ (again using $\theta_n \Phi + (1 - \theta_n) \Phi = \Phi$).

Since $\Phi \in H_0^1(D \cup \Gamma \cup \Omega)$, we obtain

$$B[\tilde{\varphi}, \mathbf{v}] + I_f(\tilde{\varphi}) = \mathbf{F}(\Phi) = \langle \nabla P, \Phi \rangle = \int_{D \cup \Omega} P \nabla \cdot \Phi \, d\mathbf{x} = I_p(\tilde{\varphi}),$$

and since $\varphi \in \hat{\mathbf{H}}$, we also have, by (3.53),

$$B[\hat{\varphi}, \mathbf{v}] + I_f(\hat{\varphi}) = 0.$$

Therefore, for $\varphi = \hat{\varphi} + \tilde{\varphi}$, we have

$$B[\varphi, \mathbf{v}] + I_f(\varphi) = (B[\hat{\varphi}, \mathbf{v}] + I_f(\hat{\varphi})) + (B[\tilde{\varphi}, \mathbf{v}] + I_f(\tilde{\varphi})) = 0 + I_p(\tilde{\varphi}) = I_p(\varphi),$$

and so the assertion (3.54) holds.

To summarize, we proved that there exist a weak solution $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, P_1, P_2) \in H^1(D) \times H^1(\Omega) \times H^1(\Omega) \times L^2(D) \times L^2(\Omega)$ to the static reduced problem, given $h_1 = h_2 = 0$ in Ω , $\mathbf{g}_1 = \mathbf{g}_2 = 0$ on Γ .

We next consider the case $\mathbf{g}_1 = 0, \mathbf{g}_2 = 0$ on Γ , but h_1, h_2 are arbitrary. We can construct functions $\mathbf{w}_1 \in H^1(D), \mathbf{w}_2 \in H^1(\Omega)$ such that

$$\begin{aligned} \nabla \cdot \mathbf{w}_1 &= h_1 & \text{in } D, \\ \nabla \cdot \mathbf{w}_2 &= h_2 & \text{in } \Omega, \\ \mathbf{w}_1 &= \mathbf{w}_2 & \text{on } \Gamma, \\ \mathbf{w}_1 &= 0, \mathbf{w}_2 = 0 & \text{on the fixed boundaries.} \end{aligned} \quad (3.55)$$

Indeed, if we let

$$\mathbf{w} = \mathbf{w}_1 \quad \text{in } D, \quad \mathbf{w} = \mathbf{w}_2 \quad \text{in } \Omega, \quad h = h_1 \quad \text{in } D, \quad h = h_2 \quad \text{in } \Omega,$$

then the problem of constructing the $\mathbf{w}_i$ is equivalent to solving

$$\nabla \cdot \mathbf{w} = h \quad \text{in } D \cup \Gamma \cup \Omega, \quad \mathbf{w} = 0 \quad \text{on } S \cup \partial G.$$

Given $h \in L^2(D \cup \Gamma \cup \Omega)$, one can solve the Stokes equation to obtain $\mathbf{w} \in H_0^1(D \cup \Gamma \cup \Omega)$; see [31]. By subtracting $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_2)$ from $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, we can eliminate h_1, h_2 , with new functions $\mathbf{f}_i$ and $\mathbf{g}_i$.

Finally, we consider the case where also $\mathbf{g}_1, \mathbf{g}_2$ are arbitrary. We want to introduce functions $\mathbf{u}_1, \mathbf{u}_3, Q_1, Q_2$, and subtract them from $\mathbf{v}_1, \mathbf{v}_3, P_1, P_2$, in a way that $\mathbf{g}_1, \mathbf{g}_2$ can be eliminated without changing any equations except for the functions $\mathbf{f}_i$.

We take $\mathbf{u}_1, \mathbf{u}_3, Q_1, Q_2$ such that

$$\begin{aligned} \nabla \cdot \mathbf{u}_1 &= 0 & \text{in } D, \\ \nabla \cdot ((1 - \theta_n)\mathbf{u}_3) &= 0 & \text{in } \Omega, \\ -\nabla \mathbf{u}_1 \cdot \mathbf{n} + Q_1 \mathbf{n} &= \mathbf{g}_1 & \text{on } \Gamma, \\ \eta \nabla \mathbf{u}_3 \cdot \mathbf{n} - Q_2 \mathbf{n} &= \mathbf{g}_2 - \mathbf{g}_1 & \text{on } \Gamma, \\ \mathbf{u}_1 &= (1 - \theta_n)\mathbf{u}_3 & \text{on } \Gamma, \\ \mathbf{u}_1 &= 0 & \text{on } S, \\ \mathbf{u}_3 &= 0 & \text{on } \partial G. \end{aligned} \quad (3.56)$$

The above system is under-determined, and we construct a solution in two steps as follows.

We begin by constructing a curvilinear orthogonal coordinate system (q^1, q^2, q^3) with scalar factors (h_1, h_2, h_3) in an ε -neighborhood of Γ , where ε is taken small such that the coordinate system is well-defined, with $\Gamma = \{q^3 = 0\}$, and we construct the solution in the region $|q_3| \leq \varepsilon$. Writing out the above equations (except the last two) in this new coordinate system near Γ , we have

$$\nabla \cdot \mathbf{u}_1 = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial(u_{11} h_2 h_3)}{\partial q^1} + \frac{\partial(u_{12} h_1 h_3)}{\partial q^2} + \frac{\partial(u_{13} h_1 h_2)}{\partial q^3} \right] = 0, \quad (3.57)$$

$$\nabla \cdot ((1-\theta_n) \mathbf{u}_3) = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial((1-\theta_n) u_{31} h_2 h_3)}{\partial q^1} + \frac{\partial((1-\theta_n) u_{32} h_1 h_3)}{\partial q^2} + \frac{\partial((1-\theta_n) u_{33} h_1 h_2)}{\partial q^3} \right] = 0, \quad (3.58)$$

$$\frac{1}{h_3} \frac{\partial u_{11}}{\partial q^3} - \frac{u_{13}}{h_1 h_3} \frac{\partial h_3}{\partial q^1} = g_{11} \quad \text{on } \Gamma, \quad (3.59)$$

$$\frac{1}{h_3} \frac{\partial u_{12}}{\partial q^3} - \frac{u_{13}}{h_2 h_3} \frac{\partial h_3}{\partial q^2} = g_{12} \quad \text{on } \Gamma, \quad (3.60)$$

$$\frac{1}{h_3} \frac{\partial u_{13}}{\partial q^3} + \frac{u_{11}}{h_1 h_3} \frac{\partial h_3}{\partial q^1} + \frac{u_{12}}{h_2 h_3} \frac{\partial h_3}{\partial q^2} - Q_1 = g_{13} \quad \text{on } \Gamma, \quad (3.61)$$

$$\eta \frac{1}{h_3} \frac{\partial u_{31}}{\partial q^3} - \eta \frac{u_{33}}{h_1 h_3} \frac{\partial h_3}{\partial q^1} = g_{21} - g_{11} \quad \text{on } \Gamma, \quad (3.62)$$

$$\eta \frac{1}{h_3} \frac{\partial u_{32}}{\partial q^3} - \eta \frac{u_{33}}{h_2 h_3} \frac{\partial h_3}{\partial q^2} = g_{22} - g_{12} \quad \text{on } \Gamma, \quad (3.63)$$

$$\eta \frac{1}{h_3} \frac{\partial u_{33}}{\partial q^3} + \eta \frac{u_{31}}{h_1 h_3} \frac{\partial h_3}{\partial q^1} + \eta \frac{u_{32}}{h_2 h_3} \frac{\partial h_3}{\partial q^2} - Q_2 = g_{23} - g_{13} \quad \text{on } \Gamma, \quad (3.64)$$

$$u_{11} = (1-\theta_n) u_{31}, \quad u_{12} = (1-\theta_n) u_{32}, \quad u_{13} = (1-\theta_n) u_{33} \quad \text{on } \Gamma. \quad (3.65)$$

If we take $\mathbf{u}_1 = \mathbf{u}_3 = 0$ on Γ , then (3.65) is automatically satisfied, and the boundary conditions (3.59) – (3.64) can be simplified to

$$\frac{1}{h_3} \frac{\partial u_{11}}{\partial q^3} = g_{11} \quad \text{on } \Gamma, \quad (3.66)$$

$$\frac{1}{h_3} \frac{\partial u_{12}}{\partial q^3} = g_{12} \quad \text{on } \Gamma, \quad (3.67)$$

$$\frac{1}{h_3} \frac{\partial u_{13}}{\partial q^3} - Q_1 = g_{13} \quad \text{on } \Gamma, \quad (3.68)$$

$$\eta \frac{1}{h_3} \frac{\partial u_{31}}{\partial q^3} = -g_{11} \quad \text{on } \Gamma, \quad (3.69)$$

$$\eta \frac{1}{h_3} \frac{\partial u_{32}}{\partial q^3} = -g_{12} \quad \text{on } \Gamma, \quad (3.70)$$

$$\eta \frac{1}{h_3} \frac{\partial u_{33}}{\partial q^3} - Q_2 = -g_{13} \quad \text{on } \Gamma, \quad (3.71)$$

In the region $\{|q^3| \leq \varepsilon\}$, we define

$$\begin{aligned} u_{11} &= g_{11}(q^1, q^2) \zeta(q^3) \int_0^{q^3} h_3(q^1, q^2, \tau) d\tau, \\ u_{12} &= g_{12}(q^1, q^2) \zeta(q^3) \int_0^{q^3} h_3(q^1, q^2, \tau) d\tau, \end{aligned}$$

where $\zeta(q^3)$ is a cutoff function that satisfies $\zeta(q^3) = 1$ if $|q^3| \leq \varepsilon/4$, and $\zeta(q^3) = 0$ if $|q^3| \geq \varepsilon/2$. We then solve (3.57) for u_{13} ,

$$u_{13} = -\frac{1}{h_1 h_2} \int_0^{q^3} \left(\frac{\partial(u_{11} h_2 h_3)}{\partial q^1} + \frac{\partial(u_{12} h_1 h_3)}{\partial q^2} \right) d\tau,$$

and, finally, we use (3.68) to define Q_1 . We also define $\mathbf{u}_3, Q_2$ in $\Omega \cap \{|q^3| \leq \varepsilon\}$ in a similar way.

We next extend the solution $\mathbf{u}_1$, and using a similar method, we can extend $\mathbf{u}_3$. We take a domain D_* with smooth boundary, and $D \setminus \{q^3 < \varepsilon\} \subset D_* \subset D \setminus \{q^3 < 3\varepsilon/4\}$; then $\mathbf{u}_1$ is already defined on $D_* \setminus \Gamma$. We redefine and extend $\mathbf{u}_1$ into $D \setminus D_*$ by solving the following problem,

$$\begin{aligned} \nabla \cdot \mathbf{u}_1 &= 0 \quad \text{in } D \setminus D_*, \\ \mathbf{u}_1 &= \mathbf{u}_1|_{\partial D_* \setminus \Gamma} \quad \text{on } \partial D_* \setminus \Gamma, \\ \mathbf{u}_1 &= 0 \quad \text{on } S. \end{aligned} \quad (3.72)$$

The problem (3.72) can be solved in a similar way as before.

The functions Q_1 and Q_2 can be extended in any way, provided that we preserve their regularity. The functions $\mathbf{u}_1, \mathbf{u}_3, Q_1, Q_3$ defined above thus form a solution to the problem (3.56). Hence the case of arbitrary $\mathbf{g}_1, \mathbf{g}_2$ can be reduced to the case of $\mathbf{g}_1 = \mathbf{g}_2 = 0$.

In conclusion, we have proved that there exists a weak solution $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, P_1, P_2)$ to the static reduced problem with $\lambda = 0$, and $\mathbf{v}_1 \in H^1(D)$, $\mathbf{v}_2, \mathbf{v}_3 \in H^1(\Omega)$, $P_1 \in L^2(D)$, $P_2 \in L^2(\Omega)$.

3.4.2 Regularity of solution—In this section we prove that the solution $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, P_1, P_2)$ is in C^∞ if the boundaries D, S, G and the inhomogeneous data are in C^∞ .

Interior regularity: Consider the bilinear form

$$B[\phi, \mathbf{v}] = -I_f(\phi), \quad \phi \in \mathbf{H}.$$

Given an open set $V \subset D$ with $V \subset \bar{D}$, we take open sets $W, \hat{W}$ such that $V \subset W, W \subset \hat{W}$, $\overline{\hat{W}} \subset D$, and select a smooth cutoff function ζ such that

$$\zeta \equiv 1 \quad \text{in } V, \quad \zeta \equiv 0 \quad \text{in } D \setminus W, \quad 0 \leq \zeta \leq 1.$$

We take $\phi_1 = -(\zeta^2(\mathbf{v}_1)^h)^{-h} - \chi_h^{-h}$, $\phi_2 = 0$, $\phi_3 = 0$, where

$$0 < h < \min\{d(\partial D, \partial \overline{\hat{W}}), d(\partial W, \partial \hat{W})\},$$

$$u^h = \frac{u(x + h\mathbf{e}_j) - u(x)}{h},$$

and a function χ_h which satisfies

$$\begin{aligned} \nabla \cdot \chi_h &= -\nabla \cdot (\zeta^2(\mathbf{v}_1)^h) \quad \text{in } D, \\ \chi_h &= 0 \quad \text{in } D \setminus \hat{W}. \end{aligned}$$

The function χ_h can be chosen such that $\|\chi_h\|_{H^1(D)} \leq C \|\nabla \mathbf{v}_1^h\|_{L^2(D)}$, where C does not depend on h .

If we substitute ϕ into the bilinear form (3.50), we obtain

$$\begin{aligned} B[\phi, \mathbf{v}] &= \sum_{i,j=1}^3 \int_D D_i [-(\zeta^2(v_{1,j})^h)^{-h} - (\chi_h)_j^{-h}] D_i(v_{1,j}) d\mathbf{x} \\ &= \sum_{i,j=1}^3 \int_D D_i [\zeta^2(v_{1,j})^h + (\chi_h)_j] D_i(v_{1,j})^h d\mathbf{x} \\ &= \sum_{i,j=1}^3 \int_D 2\zeta D_i \zeta (v_{1,j})^h D_i(v_{1,j})^h + \zeta^2 (D_i(v_{1,j})^h)^2 + D_i(\chi_h)_j D_i(v_{1,j})^h d\mathbf{x}. \end{aligned}$$

Also,

$$\begin{aligned} I_f(\phi) &= \sum_{j=1}^3 \int_D [-(\zeta^2(v_{1,j})^h)^{-h} - (\chi_h)_j^{-h}] f_{1,j} d\mathbf{x} \\ &= \sum_{j=1}^3 \int_D [\zeta^2(v_{1,j})^h + (\chi_h)_j] (f_{1,j})^h d\mathbf{x}. \end{aligned}$$

Since

$$\|(\mathbf{v}_1)^h\|_{L^2(D)} \leq C \|\mathbf{v}_1\|_{H^1(D)}, \quad \|D\chi_h\|_{L^2(D)} \leq C \|\mathbf{v}_1\|_{H^1(D)}, \quad \|\mathbf{f}_1^h\|_{L^2(\hat{W})} \leq C \|\mathbf{f}_1\|_{H^1(D)},$$

we have

$$B[\phi, \mathbf{v}] \geq \frac{1}{2} \|D(\mathbf{v}_1)^h\|_{L^2(V)}^2 - C_2 \|\mathbf{v}_1\|_{H^1(D)}^2$$

$$|I_f| \leq C(\|\mathbf{v}_1\|_{H^1(D)}^2 + \|\mathbf{f}_1\|_{H^1(D)}^2).$$

Thus

$$\|D\mathbf{v}_1^h\|_{L^2(V)} \leq C(\|\mathbf{v}_1\|_{H^1(D)} + \|\mathbf{f}_1\|_{H^1(D)}),$$

and, by Lemma 15.3 of [12], we conclude that $D\mathbf{v}_1 \in H^1(V)$, i.e., $\mathbf{v}_1 \in H_{loc}^2(D)$. Similarly, we can show that $\mathbf{v}_2, \mathbf{v}_3 \in H_{loc}^2(\Omega)$.

For any ϕ with $\phi_1 \in H_0^1(V)$, $\phi_2 = \phi_3 = 0$,

$$\left| \int_D P_1^h \nabla \cdot \phi_1 dx \right| = \left| B[\phi, \mathbf{v}^h] + \int_D \mathbf{f}_1^h \cdot \phi_1 dx \right|$$

$$\leq C \left(\|\mathbf{v}_1\|_{H^2(D)} + \|\mathbf{f}_1\|_{H^1(D)} \right) \|\phi_1\|_{H^1(D)}.$$

Thus by [43], we have $\|P_1^h\|_{L^2(V)} \leq C$, and by Lemma 15.3 of [12], $P_1 \in H_{loc}^1(D)$. Similarly, we can show that $P_2 \in H_{loc}^1(\Omega)$.

Boundary regularity: We next show that the velocities $\mathbf{v}_i$ are in H^2 along Γ . A simpler argument can establish the regularity near the fixed boundaries, and it is omitted here.

We use partition of unity to cover the boundary Γ by a finite number of balls B_ε with radius ε . We introduce new variables $x' = x$, $y' = y$, $z' = z - \zeta(x, y)$ to flatten the boundary about each of the local coordinate systems. The radius ε is taken small enough so that the variational formulation of the transformed system is coercive. Then we can use the same method as for the interior estimates to show that second order derivatives of $D_{x'}D\mathbf{v}_i$, $D_{y'}D\mathbf{v}_i$, and the first order derivatives $D_{x'}P_i$, $D_{y'}P_i$ are in L^2 in their respective domains. Derivatives in the other directions are obtained from the transformed equations. In particular, from the equation for $\mathbf{v}_1, P_1$, we obtain, in $\mathcal{D}'(B_\varepsilon \cap D)$,

$$(1+2\zeta_x^2+\zeta_y^2)(v_{11})_{zz}+\zeta_x(P_1)_z=:Y_1 \in L^2(B_\varepsilon \cap D),$$

$$(1+\zeta_x^2+2\zeta_y^2)(v_{12})_{zz}+\zeta_y(P_1)_z=:Y_2 \in L^2(B_\varepsilon \cap D),$$

$$(2+\zeta_x^2+\zeta_y^2)(v_{13})_{zz}-(P_1)_z=:Y_3 \in L^2(B_\varepsilon \cap D),$$

$$-\zeta_x(v_{11})_{zz}-\zeta_y(v_{12})_{zz}+(v_{13})_{zz}=:Y_4 \in L^2(B_\varepsilon \cap D).$$

Notice that this is a set of four linear equations for $(v_{11})_{zz}$, $(v_{12})_{zz}$, $(v_{13})_{zz}$, $(P_1)_z$, and the coefficients matrix is non-singular. Therefore we can solve the distributions $(v_{11})_{zz}$, $(v_{12})_{zz}$, $(v_{13})_{zz}$, $(P_1)_z$ as linear combinations of Y_i , $i = 1, 2, 3, 4$, and thus conclude that $(v_{11})_{zz}$, $(v_{12})_{zz}$, $(v_{13})_{zz}$, $(P_1)_z$ are in $L^2(B_\varepsilon \cap D)$. The argument for $D_{z'}^2 \mathbf{v}_2, D_{z'}^2 \mathbf{v}_3, D_{z'} P_2$ is similar.

By differentiating the equation for $\mathbf{v}_i$ and applying the above argument we can show that $\mathbf{v}_i$ are in H_{loc}^3 . The H^3 regularity near the boundary is established, as above, by differentiating first in the tangential directions and then in the normal directions. Similarly, ∇P_1 is in H^2 . Repeating this process step-by-step, we conclude that all the derivatives of the $\mathbf{v}_i$ are in H^2 up to the boundary, provided the $\mathbf{f}_i$, h_i , $\mathbf{g}_i$, θ_n , and the boundary are all in C^∞ . Hence, $\mathbf{v}_i$ and P_i are in C^∞ if the data are in C^∞ .

3.4.3 Classical solution—So far we have proved existence and uniqueness in the case $\lambda = 0$. Using the Schauder estimates (for smooth data) we can establish, by the standard method of continuity in the parameter λ , the existence and uniqueness of a solution for each λ , $0 < \lambda < 1$. Finally, by approximating Γ and the inhomogeneous data in the appropriate Hölder norms by data for which the right-hand side of (3.47) is uniformly bounded, we derive existence and uniqueness of a solution for $\lambda = 1$ with Γ in $C^{2+\alpha}$ and data for which the right-hand side of (3.47) is finite; the solution satisfies the estimate (3.47).

4 Existence and uniqueness of solution for the reduced free boundary problem

In this section we prove existence and uniqueness of solutions for the reduced free boundary problem (3.1)–(3.9) with Γ as a moving boundary at the normal velocity $\mathbf{V}_\Gamma$, but with θ_n as a given function. The free boundary $\Gamma = \Gamma(t)$ is moving by the velocity $\mathbf{v}_n$ of the network, so that

$$\mathbf{V}_\Gamma = \mathbf{v}_n \cdot \mathbf{n}. \quad (4.1)$$

If we represent $\Gamma(t)$ as $\Psi(t, x, y, z) = 0$, then the equation for $\Gamma(t)$ is,

$$\Psi_t + \mathbf{v}_n \cdot \nabla \Psi = 0 \quad \text{on } \Gamma(t), \quad \mathbf{n} = \frac{\nabla \Psi}{|\nabla \Psi|} \quad \text{on } \Gamma(t).$$

Structure of the proof: We use a method similar to our paper [18]. We introduce a parameterization of a free boundary $\mathbf{x} = \mathbf{X}(\lambda, t)$ and denote by $h(\lambda, t)$ the distance from $\mathbf{X}(\lambda, 0)$ to $\mathbf{X}(\lambda, t)$. For any family $\mathbf{X}(\lambda, t)$ we solve the fixed boundary problem and use the free boundary condition to define a corresponding function $h(\lambda, t)$. We shall prove that the mapping $h(\lambda, t) \rightarrow \tilde{h}(\lambda, t)$ is a contraction mapping. To do this we use the hyperbolic equation that $h(\lambda, t)$ satisfies to derive *a priori* estimates.

In the following we choose orthogonal curvilinear coordinates $\lambda = (\lambda_1, \lambda_2)$ on the initial free boundary $\mathbf{X}^0$, where λ varies in a region A , and denote by $(\mathbf{e}_n(\lambda), \mathbf{e}_1(\lambda), \mathbf{e}_2(\lambda))$ the orthonormal system along the initial boundary $\mathbf{X}^0(\lambda)$, with $\mathbf{e}_n(\lambda)$ the outward normal, and with $\mathbf{e}_1(\lambda), \mathbf{e}_2(\lambda)$ in the tangential plane. We assume that the free boundary can be represented in the form $\mathbf{x} = \mathbf{X}(t, \lambda)$, $\lambda \in A$. Set $\Omega_0 = \Omega(0)$, $\Gamma_0 = \Gamma(0)$, so that Γ_0 is given by $\mathbf{x} = \mathbf{X}(0, \lambda) = \mathbf{X}^0(\lambda)$. We assume that

$$\mathbf{X}^0(\lambda) \text{ is in } C^{3+\alpha}(A), \quad (4.2)$$

for some $\alpha \in (0, 1)$; if we use spherical coordinates $\lambda = (\theta, \varphi)$, then we need to require that $|\mathbf{X}^0(\lambda)| \leq C > 0$, $\forall \lambda \in A$. We also assume that for $0 \leq t \leq T$,

$$\begin{aligned} |\theta_n(\cdot, t)|_{C^{2+\alpha}(A)}, |\mathbf{f}_j(\cdot, t)|_{C^\alpha(A)}, |\mathbf{g}_j(\cdot, t)|_{C^{1+\alpha}(A)} < C, \\ 0 < \inf \theta_n \leq \sup \theta_n < 1. \end{aligned} \quad (4.3)$$

We represent the free boundary $\Gamma(t)$ in the form

$$\mathbf{X}(\lambda, t) = \mathbf{X}^0(\lambda) + h(t, \lambda) \mathbf{e}_n(\lambda), \quad (4.4)$$

and denote by $\Omega(t)$ the domain bounded by $\Gamma(t)$ and G and by $D(t)$ the domain bounded by $\Gamma(t)$ and S (see Figure 1).

Then (4.1) can be written as

$$\mathbf{X}_t(\lambda, t) = \mathbf{v}_n(\mathbf{X}(\lambda, t), t) - \sum_{i=1}^2 \mathbf{X}_{\lambda_i}(\lambda, t) \frac{\mathbf{v}_n(\mathbf{X}(\lambda, t), t) \cdot \mathbf{e}_i}{H_i}, \quad (4.5)$$

where $H_i = H_i(h, \lambda)$ are the Lamé coefficients associated with the coordinate system [35]. By (4.4), we have

$$\left(\frac{\partial \mathbf{X}(\lambda)}{\partial \lambda_i} \cdot \mathbf{e}_n \right) = \left(\frac{\partial \mathbf{X}^0(\lambda)}{\partial \lambda_i} + \frac{\partial h(\lambda, t)}{\partial \lambda_i} \mathbf{e}_n(\lambda) \right) \cdot \mathbf{e}_n + h(\lambda, t) \frac{\partial \mathbf{e}_n(\lambda)}{\partial \lambda_i} \cdot \mathbf{e}_n,$$

and the last term vanishes since $\mathbf{e}_n^2(\lambda) = 1$. By (4.4), we also have

$$\mathbf{X}_t(\lambda, t) \cdot \mathbf{e}_n(\lambda) = h_t(\lambda, t).$$

Hence, (4.5) is equivalent to

$$\frac{\partial h(\lambda, t)}{\partial t} + \sum_{i=1}^2 \frac{\mathbf{v}_n(\mathbf{X}(\lambda, t), t) \cdot \mathbf{e}_i}{H_i} \frac{\partial h(\lambda, t)}{\partial \lambda_i} = \mathbf{v}_n(\mathbf{X}(\lambda, t), t) \cdot \mathbf{e}_n - \sum_{i=1}^2 \left(\frac{\partial \mathbf{X}^0(\lambda)}{\partial \lambda_i} \cdot \mathbf{e}_n \right) \frac{\mathbf{v}_n(\mathbf{X}(\lambda, t), t) \cdot \mathbf{e}_i}{H_i} \quad (4.6)$$

Using (4.2) one can show that $\mathbf{e}_n(\lambda)$, $\mathbf{e}_i(\lambda)$ and H_i are in C^3 .

We introduce a class of functions $h(\lambda, t)$ by

$$W_{T,M} = \{h(\lambda, t); h(\lambda, 0) = 0, \|h\|_T \leq M\}, \quad M > 0,$$

where

$$\|h\|_T = \sup_{0 \leq t \leq T} \|h(\cdot, t)\|_{C^{2+\alpha}(A)} + \sup_{\lambda \in A} \sum_{j=0}^2 \|D_x^j h(\lambda, \cdot)\|_{C^\alpha[0, T]}.$$

We need the following lemma

Lemma 2: Consider the hyperbolic equation for $w = w(\lambda, t)$,

$$w_t + \mathbf{b}(\lambda, t) \cdot \nabla_\lambda w = G(\lambda, t), \quad \lambda \in A, \quad 0 \leq t \leq T, \quad T > 0 \quad (4.7)$$

with initial condition

$$w|_{t=0} = 0, \quad (4.8)$$

and assume that, for an integer $m \geq 1$,

$$\|\mathbf{b}(\cdot, t)\|_{C^{m+\alpha}(A)} + \|G(\cdot, t)\|_{C^{m+\alpha}(A)} \leq K, \quad \forall t \in [0, T].$$

Then there exists a unique solution of (4.7), (4.8) satisfying

$$\|w(\cdot, t)\|_{C^{m+\alpha}(A)} \leq C_1(K)T, \quad (4.9)$$

$$\sup_{\lambda \in A} \sum_{j=0}^m \|D_x^j w(\lambda, \cdot)\|_{C^\alpha[0, T]} \leq C_1(K)T^{1-\alpha}, \quad (4.10)$$

$$\sum_{j=0}^{m-1} \|D_x^j w_t(\cdot, t)\|_{C^1(A)} \leq C_1(K)T + K \quad \text{if } m \geq 1, \quad (4.11)$$

where $C_1(K)$ depends only on K .

Proof: The proof can be carried out in a same way as in [5, Lemma 2.2], integrating along the characteristics to obtain (4.9), and making use of the special zero initial condition (4.8). Then, using the equation (4.7) and the estimates (4.9), we immediately obtain (4.11). Next, using (4.7) and the estimates (4.11), we obtain the estimates (4.10) for $j = m - 1$. Finally, the estimate for $j = m$ in (4.10) is obtained by estimating first in the direction of characteristics and then using (4.9), similarly to the argument in [5].

For any $h \in W_{T,M}$, we define $\mathbf{X}$ by (4.4). We know from Section 3 that the static reduced problem has a unique solution satisfying the Schauder estimates (3.47) with $C = C(M)$ which depends on $M = \|h\|_T$. In particular, for $0 \leq t \leq T$,

$$\|\mathbf{v}_n(\cdot, t)\|_{C^{2+\alpha}(\bar{\Omega}(t))} \leq C(M), \quad (4.12)$$

where $C(M)$ depends on $M = \|h\|_T$.

We next extend the function $\mathbf{v}_n(\mathbf{x}, t)$ into $\Omega(t) \cup \Gamma \cup D(t)$ such that

$$\|\mathbf{v}_n(\cdot, t)\|_{C^{2+\alpha}(\bar{D}(t) \cup \bar{Q}(t))} \leq c_0 C(M), \quad c_0 > 0. \quad (4.13)$$

For simplicity, we denote the extended function also by $\mathbf{v}_n$. We shall need to use the same extension procedure for any $\mathbf{v}_n$ corresponding to any h in $W_{T,M}$. We can achieve such an extension by first extending $\mathbf{v}_n$ along each normal to the surface $\mathbf{x} = \mathbf{X}(t, \lambda)$ by a polynomial of degree two in the distance along this normal (to achieve smooth $C^{2+\alpha}$ extension) and then multiply this extension by a fixed C^3 cutoff function ψ which equals to 1 in $\Omega(t)$ and vanishes outside a small neighborhood of $\Omega(t)$, independent of t , $0 \leq t \leq T$ (T small).

We define a function $h(\lambda, t)$ by

$$\begin{cases} \frac{\partial \tilde{h}(\lambda, t)}{\partial t} + \sum_{i=1}^2 \frac{\mathbf{v}_n(\mathbf{X}(\lambda, t), t) \cdot \mathbf{e}_i}{H_i} \frac{\partial \tilde{h}(\lambda, t)}{\partial \lambda_i} = \mathbf{v}_n(\mathbf{X}(\lambda, t), t) \cdot \mathbf{e}_n - \sum_{i=1}^2 \left(\frac{\partial \mathbf{X}^0(\lambda)}{\partial \lambda_i} \cdot \mathbf{e}_n \right) \frac{\mathbf{v}_n(\mathbf{X}(\lambda, t), t) \cdot \mathbf{e}_i}{H_i} \\ \tilde{h}(\lambda, 0) = 0, \end{cases}$$

where $\mathbf{v}_n$ is the extension defined above. We note that the assumption (4.2) will be needed in order to prove that $h(\lambda, t)$ is in $C^{2+\alpha}$. We introduce the mapping

$$S: h \rightarrow \tilde{h}.$$

Clearly h is a fixed point of S if and only if the corresponding $\mathbf{v}_n$ and $\mathbf{X}(t, \lambda)$ form a solution of the reduced free boundary problem.

Applying Lemma 2 to h , we get

$$\|\tilde{h}(\cdot, t)\|_{C^{2+\alpha}(\Lambda)} \leq C_1(M)T,$$

and

$$\|\tilde{h}(\lambda, \cdot)\|_{C^\alpha[0, T]} + \|D_x \tilde{h}(\lambda, \cdot)\|_{C^\alpha[0, T]} + \|D_x^2 \tilde{h}(\lambda, \cdot)\|_{C^\alpha[0, T]} \leq C_2(M)T^{1-\alpha};$$

here the $C_i(M)$ are constants depending only on M . Taking T such that

$$C_1(M)T + C_2(M)T^{1-\alpha} < M,$$

we conclude that S maps $W_{T,M}$ into itself.

We next show that S is a contraction, and hence, it has a unique fixed point in $W_{T,M}$. Take $h_1, h_2 \in W_{T,M}$ and the corresponding functions $\mathbf{X}_1, \mathbf{v}_n, \mathbf{v}_s, P_1, P_2, h_1$ and $\mathbf{X}_2, \mathbf{w}_n, \mathbf{w}_s, Q_1, Q_2, h_2$, and set

$$\delta = \|h_1 - h_2\|_T.$$

We use the transformation

$$\tilde{\theta} = \theta, \quad \tilde{\phi} = \phi, \quad \tilde{r} = r - (h_1(\lambda, t) - h_2(\lambda, t))\psi(r, \theta, \phi)$$

in order to estimate the differences between the two solutions; here ψ is a cut off function which equals to 1 in a small neighborhood of the free boundary. By the same argument used in the proof of (4.12), we can estimate the $C_x^{2+\alpha}$ norm of $\mathbf{v}_n - \mathbf{w}_n$ in the transformed domain

$$\|\mathbf{v}_n - \mathbf{w}_n\|_{C_x^{2+\alpha}} \leq C(M)\|h_1 - h_2\|_T \leq C(M)\delta. \quad (4.14)$$

Using the equations for $\tilde{h}_i$ and Lemma 2 (after dividing by δ), we obtain,

$$\|(\tilde{h}_1 - \tilde{h}_2)(\cdot, t)\|_{C^{2+\alpha}(\Lambda)} + \sum_{j=0}^2 \|D_x^j(\tilde{h}_1 - \tilde{h}_2)(\lambda, \cdot)\|_{C^\alpha[0, T]} \leq C(M)(T + T^{1-\alpha})\delta < \frac{\delta}{2}, \quad (4.15)$$

if T is small enough. Hence S is a contraction, and it has a unique fixed point.

We summarize:

Theorem 3: Consider the free boundary problem (3.1)–(3.9), (4.1). If the conditions (4.2), (4.3) hold then there exists a unique solution $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, P_1, P_2, h)$, for a small time interval $0 \leq t \leq T$, satisfying the estimates

$$\sup_{0 \leq t \leq T} \|h(\cdot, t)\|_{C^{2+\alpha}(\Lambda)} + \sup_{\lambda \in \Lambda} \sum_{j=1}^2 \|D_x^j h\|_{C^\alpha([0, T])} < \infty, \quad (4.16)$$

$$\|\mathbf{v}_1\|_{C^{2+\alpha}(\overline{D}(t))} + \sum_{j=2}^3 \|\mathbf{v}_j\|_{C^{2+\alpha}(\overline{D}(t))} + \|\nabla P_1\|_{C^\alpha(\overline{D}(t))} + \|\nabla P_2\|_{C^\alpha(\overline{D}(t))} \leq \infty \quad (4.17)$$

Remark 4: Since the $C^{3+\alpha}$ Schauder estimates (3.48) are also valid, existence can be obtained also in the framework of $C^{3+\alpha}$ estimates, assuming that $X_0 \in C^{4+\alpha}$, $\theta_n \in C^{3+\alpha}$, $\mathbf{f}_j \in C^{1+\alpha}$, $\mathbf{g}_j, h_j \in C^{2+\alpha}$, i.e., the estimates in (4.17) hold with $C^{2+\alpha}$ replaced by $C^{3+\alpha}$ for $\mathbf{v}_i$, and C^α replaced by $C^{1+\alpha}$ for ∇P_i .

5 Existence and uniqueness of solution for the full problem

In this section we assume that

$$\begin{aligned}
& \|\mathbf{f}_j\|_{C^{1+\alpha, \alpha/2}(E \times [0, T])}, \|\mathbf{f}_j\|_{C^{\alpha, (1+\alpha)/2}(E \times [0, T])} < C, \quad j=1, 2, 3, \\
& \|\mathbf{g}_j\|_{C^{2+\alpha, \alpha/2}(E \times [0, T])}, \|\mathbf{g}_j\|_{C^{1+\alpha, (1+\alpha)/2}(E \times [0, T])} < C, \quad j=1, 2, \\
& \|h_j\|_{C^{2+\alpha, \alpha/2}(E \times [0, T])}, \|h_j\|_{C^{1+\alpha, (1+\alpha)/2}(E \times [0, T])} < C, \quad j=1, 2, \\
& \|G\|_{C^{1+\alpha, \alpha/2}(E \times [0, T])} < C
\end{aligned} \quad (5.1)$$

where E is a neighborhood of $\overline{D(0)\Omega(0)}$, and that the initial free boundary satisfies:

$$\mathbf{X}^0(\cdot) = \mathbf{X}(0, \cdot) \in C^{4+\alpha}(\Lambda). \quad (5.2)$$

In the previous sections 3 and 4, the function θ_n was a given function. In this section, we shall solve the full free boundary problem by taking θ_n to be coupled to $\mathbf{v}_i$ and the free boundary (defined in terms of $h(\lambda, t)$), where θ_n satisfies the parabolic equation

$$\frac{\partial \theta_n}{\partial t} + \nabla \cdot (\theta_n \mathbf{v}_1) = \varepsilon \Delta \theta_n + G(\mathbf{x}, t) \quad \text{in } \Omega(t), 0 \leq t \leq T \quad (\varepsilon > 0) \quad (5.3)$$

with boundary conditions

$$\frac{\partial \theta_n}{\partial n} = 0 \quad \text{on } \partial \Omega(t), 0 \leq t \leq T, \quad (5.4)$$

and initial conditions

$$\theta_n(\mathbf{x}, 0) = \theta_0(\mathbf{x}) \text{ for } \mathbf{x} \in \Omega, \quad 0 < \inf \theta_0(\mathbf{x}) \leq \sup \theta_0(\mathbf{x}) < 1. \quad (5.5)$$

We shall use the contraction fixed point theorem establish the existence of a unique solution for a small time interval $0 \leq t \leq T$. We shall define a set W_{T, M_1} of functions $h(\lambda, t)$, $\lambda \in \Lambda$, $0 \leq t \leq T$, and a set Z_{T, M_2} of vectors $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Then for any $h^{old} \in W_{T, M_1}$ and $(\mathbf{v}_1^{old}, \mathbf{v}_2^{old}, \mathbf{v}_3^{old}) \in Z_{T, M_2}$, we shall solve the parabolic equation (5.3) with (5.4) and (5.5) to obtain θ^{new} , and then apply Theorem 3 to the reduced free boundary problem with θ_n^{new} , to obtain new functions $(\mathbf{v}_1^{new}, \mathbf{v}_2^{new}, \mathbf{v}_3^{new})$ and h^{new} . We define a mapping S by

$$S(h^{old}, \mathbf{v}_1^{old}, \mathbf{v}_2^{old}, \mathbf{v}_3^{old}) = (h^{new}, \mathbf{v}_1^{new}, \mathbf{v}_2^{new}, \mathbf{v}_3^{new}), \quad (5.6)$$

and shall prove that S is a contraction mapping on (W_{T, M_1}, Z_{T, M_2}) , if T is sufficiently small. But in order to accomplish this, we need θ_n to have sufficient regularity in t . We are unable to accomplish this if we take $h(\lambda, t)$ to be in one of the standard Schauder spaces. Instead, we shall require the function $h(\lambda, t)$ to have a rather unusual type of regularity in λ and t , namely, to belong to the class

$$W_{T, M_1} = \{h(\lambda, t); h(\lambda, 0) = 0, \|h\|_T \leq 1, \|h_t\|_{L^\infty} + \|D_x h_t\|_{L^\infty} + \|D_x^2 h_t\|_{L^\infty} \leq M_1\}, \quad M_1 > 0, \quad (5.7)$$

where

$$\|h\|_T = \sup_{0 \leq t \leq T} \|h(\cdot, t)\|_{C^{3+\alpha}(\Lambda)} + \sup_{\lambda \in \Lambda} \|D_x^3 h(\lambda, \cdot)\|_{C^\alpha[0, T]} + \sum_{j=0}^1 \sup_{\lambda \in \Lambda} \sup_{0 < t_1 < t_2 < T} \frac{|D_x^j [h_t(\lambda, t_1) - h_t(\lambda, t_2)]|}{|t_1 - t_2|^{(1+\alpha-\varepsilon_1)/2}}, \quad (5.8)$$

and $\varepsilon_1 < \alpha$ is a small positive constant. Notice that the last term is roughly a bound on $D_x D_t^{(3+\alpha-\varepsilon)/2} h$. The reason for introducing ε_1 will become clear later on when we show that it gives rise to a small factor $T^{\varepsilon_1/4}$ in a crucial estimate for h .

We introduce the sets $\mathcal{N}_T = \bigcup_{t \in [0, T]} \Omega_t^- \times \{t\}$ and $\mathcal{D}_T = \bigcup_{t \in [0, T]} D_t^- \times \{t\}$, and the variables $\mathbf{v}_n$ in $\mathcal{N}_T$ and $\mathbf{v}_s$ in $\mathcal{N}_T \cup \mathcal{D}_T$ by

$$\begin{aligned} \mathbf{v}_n(\mathbf{x}) &= \mathbf{v}_2(\mathbf{x}), & \mathbf{x} \in \mathcal{N}_T, \\ \mathbf{v}_s(\mathbf{x}) &= \begin{cases} \mathbf{v}_1(\mathbf{x}), & \mathbf{x} \in D_T \\ \mathbf{v}_3(\mathbf{x}), & \mathbf{x} \in N_T \end{cases} \end{aligned}$$

We take $\mathbf{v}_n$ and $\mathbf{v}_s$ in the spaces

$$D_x^j \mathbf{v}_n \in C^{\alpha, \alpha/2}(\mathcal{N}_T), \quad j=0, 1, 2, \quad D_x^j \mathbf{v}_s \in C^{\alpha, \alpha/2}(\mathcal{N}_T) \cup C^{\alpha, \alpha/2}(\mathcal{D}_T), \quad j=0, 1, 2,$$

and we define the set Z_{T, M_2} is given in terms of $(\mathbf{v}_n, \mathbf{v}_s)$ by restricting them as follows:

$$Z_{T, M_2} = \{(\mathbf{v}_n, \mathbf{v}_s); \|D_x^j \mathbf{v}_n\|_{C^{\alpha, \alpha/2}(\mathcal{N}_T)} \leq M_2, \|D_x^j \mathbf{v}_s\|_{C^{\alpha, \alpha/2}(\mathcal{N}_T)} \leq M_2, \|D_x^j \mathbf{v}_s\|_{C^{\alpha, \alpha/2}(\mathcal{D}_T)} \leq M_2, j=0, 1, 2, \}$$

We also assume that

$$\begin{aligned} \theta_0 &\text{ is in } C^{3+\alpha}(\overline{\Omega}_0), \quad \frac{\partial \theta_0}{\partial n} = 0 \quad \text{on } \partial \Omega_0, \\ &\text{and the second order compatibility condition is satisfied on } \partial \Omega_0 \times \{t=0\} \end{aligned} \quad (5.9)$$

This compatibility condition, on the free boundary $\mathbf{x} = \mathbf{X}(\lambda, t)$ at $t = 0$, is the condition that

$$\lim_{t \rightarrow 0} \frac{d}{dt} \left[\left(\frac{\partial \theta_n}{\partial n} \right) (\mathbf{X}(\lambda, t), t) \right] = 0,$$

where $\frac{\partial \theta_n}{\partial t}$ is computed from the differential equation (5.3) and all the derivatives $D_x^j \theta_n$ are replaced, as $t \rightarrow 0$, by $D_x^j \theta_0$. Similarly, one defines the compatibility condition on $G \times \{t = 0\}$.

Choosing M_1 : Under our assumption (5.8), the function h^{old} is clearly $C^{3+\alpha-\varepsilon_1, (3+\alpha-\varepsilon_1)/2}$, with a bound depending only on M_1 . From the theory for parabolic equations [26], $\theta_n \in C^{3+\alpha-\varepsilon_1, (3+\alpha-\varepsilon_1)/2}$. In particular, the parabolic theory asserts that $D_x^j \theta_n$, ($j = 0, 1$) are Lipschitz continuous in t , $D_x^2 \theta_n$ is $C^{(1+\alpha-\varepsilon_1)/2}$ Hölder continuous in t , and $D_x^3 \theta_n$ is $C^{(\alpha-\varepsilon_1)/2}$ Hölder continuous in t . Here we have not used the “full norm” in (5.8), i.e., the bound on the term with $j = 1$ in (5.8) was not used.

With the $j = 1$ term in (5.8), the solution θ_n is more regular in spatial direction. After flattening the boundary locally as we did in section 3, the coefficients of the equations for θ_n involve h_t^{old} and $D_x^j h^{old}$ ($j=0, 1, 2$). Furthermore, we can choose the flattening map such that the homogeneous Neumann boundary condition remains intact. The $j = 1$ term in (5.8) makes it possible for differentiating in x -direction: when differentiating in tangential direction we get a Neumann boundary value problem; and when differentiating in normal direction we get a Dirichlet boundary value problem. Since, under our assumption, $D_x(h^{old}, h_t^{old}, D_x h^{old}, D_x^2 h^{old})$ are all Hölder continuous in t with Hölder exponent equal to $\min[\alpha, (1 + \alpha - \varepsilon_1)/2] > \alpha/2$ (by (5.8) and the definition of W_{T,M_1}); these functions are obviously also in C_x^α . Thus, we actually obtain from parabolic Schauder estimates that

$$D_x \theta_n \in C_{\mathbf{x},t}^{2+\alpha,1+\alpha/2}, \quad \|D_x \theta_n\|_{C_{\mathbf{x},t}^{2+\alpha,1+\alpha/2}} \leq C(M_1, M_2);$$

the last estimate depends also on M_2 since the equation for θ_n also involves $\mathbf{v}_n, \mathbf{v}_s$.

We conclude that the parabolic equation for θ_n produces enough regularity, i.e.,

$$\theta_n \in C_{\mathbf{x},t}^{3+\alpha,(3+\alpha-\varepsilon_1)/2}(\mathcal{N}_T), \quad D_x \theta_n \in C_{\mathbf{x},t}^{2+\alpha,1+\alpha/2}(\mathcal{N}_T),$$

with

$$\|\theta_n\|_{C_{\mathbf{x},t}^{3+\alpha,(3+\alpha-\varepsilon_1)/2}(\mathcal{N}_T)} \leq C(M_1, M_2), \quad \|D_x \theta_n\|_{C_{\mathbf{x},t}^{2+\alpha,1+\alpha/2}(\mathcal{N}_T)} \leq C(M_1, M_2),$$

Since

$$\sup_t \|D_x^j \theta_n(\cdot, t)\|_{C_x^{1+\alpha}} \leq C(M_1, M_2),$$

for $\beta \in (0, 1)$ and $j = 0, 1, 2$, the C_x^β Hölder norm is $C_t^{\frac{1+\alpha-\beta}{2}}$ continuous, by interpolation [26], namely

$$\begin{aligned} \|D_x^j[|\theta_n(\cdot, t_1) - \theta_n(\cdot, t_2)|]\|_{C_x^\beta} &\leq \|D_x^j[|\theta_n(\cdot, t_1) - \theta_n(\cdot, t_2)|]\|_{C_x^{1+\alpha}}^{\frac{\beta}{1+\alpha}} \cdot \|D_x^j[|\theta_n(\cdot, t_1) - \theta_n(\cdot, t_2)|]\|_{L^\infty}^{\frac{1+\alpha-\beta}{1+\alpha}} \\ &\leq C(M_1, M_2) |t_1 - t_2|^{\frac{1+\alpha}{2} \cdot \frac{1+\alpha-\beta}{1+\alpha}} \\ &= C(M_1, M_2) |t_1 - t_2|^{(1+\alpha-\beta)/2}, \quad j=0, 1, 2. \end{aligned} \quad (5.10)$$

Using (5.8) and interpolation, as in the proof of (5.10), we obtain, for any $\alpha_1 < \alpha - \varepsilon_1$,

$$\|\theta_n(\cdot, t)\|_{C^{3+\alpha_1}} \leq 1 + \|\theta_n(\cdot, 0)\|_{C^{3+\alpha_1}} \leq C^*, \text{ if } T \ll 1, \quad (5.11)$$

for a universal constant C^* . Using the equations for $\mathbf{v}_n, \mathbf{v}_s$, we can apply Schauder $C^{3+\alpha_1}$ estimates (3.48) to obtain

$$\|\mathbf{v}_n(\cdot, t)\|_{C^{3+\alpha_1}}, \|\mathbf{v}_s(\cdot, t)\|_{C^{3+\alpha_1}} \leq C^{**}, \quad (5.12)$$

for some universal constant C^{**} .

In order to solve for new h , we extend the function $\mathbf{v}_n$ into a small neighborhood of the free boundary as we did in Section 4. Then, from the equation for h and Lemma 2, we obtain the uniform estimates

$$h_t, D_x h_t, D_x^2 h_t \in L^\infty, \quad \|h_t, D_x h_t, D_x^2 h_t\|_{L^\infty} \leq C^{***}$$

where C^{***} is a universal constant, and therefore we can choose M_1 in (5.7) to depend only on C^{***} .

Choosing M_2 : To choose M_2 , we first take care of the Hölder continuity in t . For fixed $t_2 > t_1 > 0$, we shall make a change of variables to reduce the problems to the same domain by the transformation $(r, \theta, \phi) \rightarrow (r, \tilde{\theta}, \tilde{\phi})$, where

$$\tilde{\theta} = \theta, \quad \tilde{\phi} = \phi, \quad \tilde{r} = r - (h^{old}(\lambda, t_1) - h^{old}(\lambda, t_2))\psi(r, \theta, \phi),$$

where ψ is a C^∞ function such that $\psi = 1$ in a neighborhood of the initial free boundary and $\psi = 0$ near the external boundary (fixed boundary) of the domain. Clearly, this transformation pulls the domains $(\mathcal{A}(t_2), D(t_2))$ to the domains $(\mathcal{A}(t_1), D(t_1))$, and the regularity of this transformation is the same as the regularity of the function $h^{old}(\lambda, t_1) -$

$h^{old}(\lambda, t_2)$. Since, $\|h^{old}(\cdot, t)\|_{C_x^{3+\alpha}} \leq C$, by interpolation, for any $\beta \in (0, 1)$,

$$\begin{aligned} \|h^{old}(\cdot, t_1) - h^{old}(\cdot, t_2)\|_{C_x^{2+\beta}} &\leq \|h^{old}(\cdot, t_1) - h^{old}(\cdot, t_2)\|_{C_x^{3+\alpha}}^{\beta/(1+\alpha)} \|h^{old}(\cdot, t_1) - h^{old}(\cdot, t_2)\|_{C_x^2}^{(1+\alpha-\beta)/(1+\alpha)} \\ &\leq CT^{\beta/(1+\alpha)}(t_2 - t_1)^{\frac{1+\alpha-\beta}{1+\alpha}} \end{aligned}$$

Applying Schauder estimates (and using (5.1)), we can obtain the Hölder continuity of $\mathbf{v}_n$,

$$\begin{aligned} &\|\mathbf{v}_n(\cdot, t_2) - \mathbf{v}_n(\cdot, t_1)\|_{C_x^{2+\beta}} \\ &\leq CT^{\beta/(1+\alpha)}(t_2 - t_1)^{\frac{1+\alpha-\beta}{1+\alpha}} + C_{f,g}(t_2 - t_1)^{(1+\alpha-\beta)/2} + C_\theta(t_2 - t_1)^{(1+\alpha-\beta)/2}; \end{aligned} \quad (5.13)$$

the corresponding estimates for $\mathbf{v}_s, P_1, P_2$ are obtained in a similar manner. Note that the last term in (5.13) comes from the $C_{\mathbf{x},t}^{3+\alpha, \frac{3+\alpha}{2}}$ estimate on $\mathbf{v}_n$.

Since $\frac{1+\alpha-\beta}{1+\alpha} > \frac{1+\alpha-\beta}{2} > \frac{\alpha}{2}$, we get from (5.13),

$$\|\mathbf{v}_n(\cdot, t_2) - \mathbf{v}_n(\cdot, t_1)\|_{C_x^{2+\beta}} \leq (t_2 - t_1)^{\alpha/2} \quad \text{if } T \ll 1, \quad (5.14)$$

and a similar estimate holds for $\mathbf{v}_s$. Thus we can choose M_2 to map the set Z_{T,M_2} back to itself.

The rest of the estimates to get back to W_{T,M_1} : Now M_1 and M_2 are fixed, and therefore θ_n is uniformly in $C^{3+\alpha}$ in x -direction. It follows, by Remark 4 that the $C^{3+\alpha}$ estimates for $\mathbf{v}_n, \mathbf{v}_s$ in x -direction are uniformly valid. Using the equation (4.6) and Lemma 2, we find, upon recalling the equation (5.14), that

$$\sup_{0 \leq t \leq T} \|h(\cdot, t)\|_{C^{3+\alpha}(A)} + \sup_{\lambda \in A} \|D_x^3 h(\lambda, \cdot)\|_{C^\alpha[0,T]} \leq \frac{1}{2}.$$

From (5.13) and the equation (4.6), we also derive uniform estimates on

$$h_t, D_x h_t \in C_t^\gamma, \quad \text{where } \gamma = \frac{1+\alpha-\beta}{2}.$$

Choosing $\beta = \varepsilon_1/2$, we then obtain, by interpolation,

$$\sup_{\lambda} [D_x^j h_t]_{C_t^{(1+\alpha-\varepsilon_1)/2}} \leq T^{\varepsilon_1/4} \|D_x^j h_t\|_{C^\gamma}, \quad j=0, 1.$$

Therefore, if T is sufficiently small,

$$\sum_{j=0}^1 \sup_{\lambda \in A} \sup_{0 < t_1 < t_2 < T} \frac{|D_x^j [h_t(\lambda, t_1) - h_t(\lambda, t_2)]|}{|t_1 - t_2|^{(1+\alpha-\varepsilon_1)/2}} \leq \frac{1}{2}.$$

Hence h belongs to W_{T,M_1} . We have thus proved that the mapping S defined in (5.6) maps (W_{T,M_1}, Z_{T,M_2}) into itself. The assertion that S is a contraction mapping can be proved as in the previous section.

We have thus proved the following theorem:

Theorem 5: Assume that (5.1), (5.2), (5.5), and (5.9) hold. Then there exists a unique solution to the free boundary problem (2.1)–(2.13) for a small time interval $0 \leq t \leq T$, $T > 0$, where the free boundary is of the form (4.4), with h belonging to the regularity class W_{T,M_1} defined by (5.7), (5.8).

6 Application to biofilm growth

In this section, we show how Theorem 5 can be used to establish the existence and uniqueness of solution for a model of biofilm growth described below. The model is based on the physics and detailed biochemistry of biofilm growth, and has the potential to address the spatial distribution patterns of biofilms observed in experiments [28]. But to do so will

require detailed parameter estimation for the model and numerical simulations of the model with a free boundary which is beyond the scope of this paper.

6.1 The biofilm model

Geometry and variables: We consider a special geometry shown in Figure 1. The biofilm occupies a region $\Omega(t)$ with substrate G , and is surrounded by fluid occupying a region $D(t)$. The two regions share a common boundary $\Gamma(t)$. For simplicity, we assume that there is no external flow.

We model the biofilm as an immiscible mixture of three components, namely, the polymer network, the bacteria, and the solvent/nutrient. Biofilm polymer network usually consists of one or several extracellular polysaccharides (EPS) regulated by cellular genes, such as *alginate*, *pel*, and *psl*. Here we consider biofilms formed by bacterium *Pseudomonas aeruginosa* with polysaccharides mediated only by *psl* and, for simplicity, we denote this polymer network by Psl [4]. The protein CdrA has been shown to either crosslink EPS and/or tether EPS to cells [3]. Bacteria cells usually account for a small portion of the biofilm [11]. The distribution of bacterial cells within the biofilm is of great interest [27], thus we include bacteria distribution explicitly in contrast to previous models such as [6].

The following variables will be used in the model:

- θ_p - EPS (Psl) volume fraction
- θ_b - Bacteria volume fraction
- θ_n - Solid phase volume fraction, which is $\theta_p + \theta_b$
- θ_s - Fluid volume fraction, which is $1 - \theta_n$
- c - CdrA concentration
- w - Nutrient (oxygen) concentration
- $\mathbf{v}_n$ - Velocity of the EPS
- $\mathbf{v}_s$ - Velocity of the solvent
- P - Hydrostatic pressure of the mixture

Here P is the pressure of the mixture when there is no macroscopic movement of each component. It is introduced in order to enforce the incompressibility condition of the mixture in G [22].

Since bacteria are non-motile in biofilms, bacteria and EPS have the same velocity $\mathbf{v}_n$. We assume that all components are incompressible, i.e., the intrinsic densities of EPS, bacteria, and solvent (mass of the component per volume of the component) are constants equal to 1. Therefore the mass densities of bacteria, EPS, and solvent per unit mixture volume are associated with their volume fractions, given by θ_p , θ_b and θ_s , respectively.

Model equations

Fluid region mechanics: In the domain $D(t)$, we only have solvent. We model the solvent as an incompressible Newtonian fluid. Since the Reynolds number is small, we neglect inertia. Therefore the governing equations are

$$\nabla \cdot \mathbf{v}_s = 0 \quad \text{in } D(t), \quad (6.1)$$

$$\nabla \cdot \sigma_s = 0 \quad \text{in } D(t), \quad (6.2)$$

where

$$\sigma_s = -PI + \eta_s \frac{\nabla \mathbf{v}_s + \nabla \mathbf{v}_s^T}{2}, \quad (6.3)$$

and η_s is the viscosity of the solvent, Equation (6.1) is the incompressibility condition, and Equation (6.2) is the force balance equation under the assumption (6.1).

Biofilm region mechanics: In the domain $\Omega(t)$, we have both phases, with $\theta_p, \theta_b, \theta_s \in (0, 1)$. The velocity of bacteria is the same as the velocity of *psl*, as cells are convected by *psl*. If we ignore diffusion of the polymers and the bacteria, then we obtain

$$\frac{\partial \theta_p}{\partial t} + \nabla \cdot (\theta_p \mathbf{v}_n) = G_p \quad \text{in } \Omega(t), \quad (6.4)$$

$$\frac{\partial \theta_b}{\partial t} + \nabla \cdot (\theta_b \mathbf{v}_n) = G_b \quad \text{in } \Omega(t), \quad (6.5)$$

where

$$G_p = \lambda_p \theta_b \frac{((1-\theta_n)w - w_*)_+}{w + K_p}, \quad G_b = \lambda_b \theta_b \frac{(1-\theta_n)w - w_*}{w + K_b}. \quad (6.6)$$

As bacteria grow, they absorb water and nutrient from the solvent, and the right-hand-sides of the above equations model the mass transfer between different phases. Here we assumed that bacteria secrete *psl* and grow only when the average nutrient concentration is above a positive threshold w_* . When the nutrient concentration is smaller than w_* , bacteria may die.

However, polymer molecules and bacteria may randomly disperse [9], and we incorporate this effect by introducing small diffusion terms into the equations for the polymer and bacteria. For simplicity, we assume that the diffusion rates for the polymer and bacteria are the same, and replace Equations (6.4) and (6.5) by

$$\frac{\partial \theta_p}{\partial t} + \nabla \cdot (\theta_p \mathbf{v}_n) = \varepsilon \Delta \theta_p + G_p \quad \text{in } \Omega(t), \quad (6.7)$$

$$\frac{\partial \theta_b}{\partial t} + \nabla \cdot (\theta_b \mathbf{v}_n) = \varepsilon \Delta \theta_b + G_b \quad \text{in } \Omega(t). \quad (6.8)$$

The incompressibility condition of the mixture is

$$\nabla \cdot (\theta_n \mathbf{v}_n + (1-\theta_n) \mathbf{v}_s) = 0 \quad \text{in } \Omega(t), \quad (6.9)$$

where $\theta_n = \theta_p + \theta_b$.

Note that $\theta_n < 1$ for all time. Indeed, adding (6.7) and (6.8) we obtain

$$\frac{\partial \theta_n}{\partial t} + \nabla \cdot (\theta_n \mathbf{v}_n) = \varepsilon \Delta \theta_n + \alpha(1 - \theta_n) - \beta \quad \text{in } \Omega(t),$$

with functions $\alpha \geq 0$ and $\beta > 0$ properly defined. By (6.9), we have

$$\nabla \cdot (\theta_n \mathbf{v}_n) = \mathbf{v}_s \cdot \nabla \theta_n - (1 - \theta_n) \nabla \cdot \mathbf{v}_s.$$

Substituting this into the above equation, we obtain

$$\frac{\partial \theta_n}{\partial t} - \varepsilon \Delta \theta_n - \mathbf{v}_s \cdot \nabla \theta_n = (\alpha + \nabla \cdot \mathbf{v}_s)(1 - \theta_n) - \beta \quad \text{in } \Omega(t). \quad (6.10)$$

Suppose θ_n increases to 1 at some point $(\mathbf{x}_0, t_0)$ with smallest t_0 . Then the RHS of (6.10) becomes strictly negative in a small neighborhood $\Omega(t) \cap \{|\mathbf{x} - \mathbf{x}_0| < \delta, t_0 - \delta < t < t_0\}$. Since $\theta_n|_{n=0} = 0$ on $\Omega(t)$, we get a contradiction to the maximum principle.

Biofilm growth occurs on a time scale of days to weeks, therefore the elastic response of the EPS is negligible and we ignore it. Hence the momentum equation for the EPS and the solvent are written as

$$\begin{aligned} \frac{\partial(\theta_n \mathbf{v}_n)}{\partial t} + \nabla \cdot (\theta_n \mathbf{v}_n \otimes \mathbf{v}_n) &= \nabla \cdot (\theta_n \tau_n) - \theta_n \nabla(P + \mu_n + K) + \mathbf{f}_n \quad \text{in } \Omega(t), \\ \frac{\partial((1 - \theta_n) \mathbf{v}_s)}{\partial t} + \nabla \cdot ((1 - \theta_n) \mathbf{v}_s \otimes \mathbf{v}_s) &= \nabla \cdot ((1 - \theta_n) \tau_s) - (1 - \theta_n) \nabla(P + \mu_s) + \mathbf{f}_s \quad \text{in } \Omega(t), \end{aligned}$$

where τ_n and τ_s are the deviatoric stress tensors of the EPS and the solvent due to the flow; P, μ_n, μ_s are the hydrostatic pressure, and chemical potential densities introduced by mixing the two components; $K = K(\theta_n, c)$ models crosslinking of Psl by CdrA, and $\mathbf{f}_n, \mathbf{f}_s$ are the forces due to interaction with other components. Here we neglected body forces (such as gravity) and the momentum generation term due to the source of mass [10]. We introduce the notation for the total stress tensors σ_n and σ_s as

$$\sigma_n = -(P + \mu_n + K)I + \tau_n, \quad (6.11)$$

$$\sigma_s = -(P + \mu_s)I + \tau_s, \quad (6.12)$$

neglect shear viscosity, and assume that

$$\tau_n = \eta_n \frac{\nabla \mathbf{v}_n + \nabla \mathbf{v}_n^T}{2}, \quad \tau_s = \eta_s \frac{\nabla \mathbf{v}_s + \nabla \mathbf{v}_s^T}{2}.$$

The formulas for the chemical potentials are given in terms of the free energy f [9],

$$\mu_s = f - \theta_n \frac{\partial f}{\partial \theta_n}, \quad \mu_n = f + (1 - \theta_n) \frac{\partial f}{\partial \theta_n}. \quad (6.13)$$

As in [32,22], we take

$$f = k_B T \left[\frac{\theta_n}{N} \log(\theta_n) + (1 - \theta_n) \log(1 - \theta_n) \right] + \chi \theta_n (1 - \theta_n) + (\mu_n^0 - \mu_s^0) \theta_n + \mu_s^0. \quad (6.14)$$

Here k_B is Boltzmann's constant, T is the temperature, χ is the Flory interaction energy parameter, μ_n^0 is the standard free energy for the EPS, μ_s^0 is the standard free energy of the solvent, μ_s^0 is a constant assumed to be zero, and N is the length of the polymer *psl*. Motivated by [24], we take

$$K(\theta_p, c) = \lambda_{cl} \frac{\theta_p c}{\theta_p + p_0}. \quad (6.15)$$

Finally, we assume that frictional force acted on the biofilm by the solvent is $-h_f \theta_n (1 - \theta_n) (\mathbf{v}_n - \mathbf{v}_s)$ as in [6, 22], and thus

$$\begin{aligned} \mathbf{f}_n &= -h_f \theta_n (1 - \theta_n) (\mathbf{v}_n - \mathbf{v}_s), \\ \mathbf{f}_s &= -h_f \theta_n (1 - \theta_n) (\mathbf{v}_s - \mathbf{v}_n). \end{aligned}$$

By neglecting inertia, the momentum equations become

$$\nabla \cdot (\theta_n \tau_n) - \theta_n \nabla (P + \mu_n + K) + \mathbf{f}_n = 0 \quad \text{in } \Omega(t), \quad (6.16)$$

$$\nabla \cdot ((1 - \theta_n) \tau_s) - (1 - \theta_n) \nabla (P + \mu_s) + \mathbf{f}_s = 0 \quad \text{in } \Omega(t). \quad (6.17)$$

In summary, the equations for the mixture of biofilm and solvent are (6.7), (6.8), (6.9), (6.16) and (6.17). Next we couple these equations with the nutrient and CdrA.

Nutrient: The nutrient diffuses everywhere, is convected by the solvent and consumed by bacteria; therefore its equation in $\Omega(t)$ is

$$\frac{\partial w}{\partial t} + \nabla \cdot (w \mathbf{v}_s) = D_w \Delta w - G_w \quad \text{in } D(t) \cup \Omega(t), \quad (6.18)$$

where

$$G_w = \chi_{\{(x,y) \in \Omega\}} \frac{\lambda_w \theta_b w}{w + K_w}.$$

CdrA: The adhesin CdrA is produced by cells. We assume that once it is secreted, it binds to *psl* immediately to form crosslinking sites, therefore the equation for CdrA is

$$\frac{\partial c}{\partial t} + \nabla \cdot (c \mathbf{v}_n) = G_c \quad \text{in } \Omega(t), \quad (6.19)$$

where

$$G_c = \lambda_c \theta_b \frac{(w - w_*)_+}{w + K_c}.$$

We assume that cells secrete CdrA only if they can access to enough nutrient, i.e., only if $w > w_*$.

Boundary conditions: Boundary conditions at the fixed boundary S are

$$\frac{\partial w}{\partial n} = 0, \quad \mathbf{v}_s = 0. \quad (6.20)$$

Here we assumed no external flow. Boundary conditions at the fixed boundary G are

$$\frac{\partial w}{\partial n} = 0, \quad \frac{\partial \theta_n}{\partial n} = 0, \quad \frac{\partial \theta_p}{\partial n} = 0, \quad \mathbf{v}_n = \mathbf{v}_s = 0. \quad (6.21)$$

Boundary conditions at the free boundary $\Gamma(t)$ are

$$(\sigma_n)_n \cdot \mathbf{n} = (\sigma_s)_n \cdot \mathbf{n} = (\sigma_s)_f \cdot \mathbf{n}, \quad (6.22)$$

$$(\theta_n \mathbf{v}_n + (1 - \theta_n) \mathbf{v}_s)_n = (\mathbf{v}_s)_f, \quad (6.23)$$

$$\frac{\partial \theta_n}{\partial n} = 0, \quad \frac{\partial \theta_p}{\partial n} = 0, \quad (6.24)$$

where the notation $(\cdot)_n, (\cdot)_f$ means taking the limit from the biofilm side and the fluid side respectively. Here we assumed that there is no surface tension on $\Gamma(t)$.

The equation for the free boundary $\Gamma(t)$ is

$$V_n = \mathbf{v}_n \cdot \mathbf{n},$$

where V_n is the velocity of $\Gamma(t)$ in the direction $\mathbf{n}$.

Initial conditions: Appropriate initial conditions are prescribed on w, θ_p, θ_n . We assume that the initial data are smooth enough, and for some constant $c > 0$,

$$c \leq \theta_p(x_1, x_2, x_3, 0) \leq 1 - c, \quad c \leq \theta_n(x_1, x_2, x_3, 0) \leq 1 - c.$$

6.2 Local existence and uniqueness of solution

It is clear that Theorem 5 can be extended to include the parabolic equations for w and c . On the other hand, in the biofilm model the inhomogeneous terms are not just functions of (x, t) ; they are nonlinear functions in the variables $\mathbf{v}_i, \theta_p, \theta_b$. However, the derivatives which appear in the nonlinear terms are of low order. This enables us to easily prove existence and uniqueness for a small time interval $0 \leq t \leq T, T > 0$. Thus, Theorem 5 can be extended to establish the existence and uniqueness for the full biofilm model, for a small time interval.

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Fig. 1.
The Geometry of the model.