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Assumptions regarding right censoring in the presence of left truncation

Jing Qian^{a,*} and Rebecca A. Betensky^b

Rebecca A. Betensky: betensky@hsph.harvard.edu

^aDivision of Biostatistics and Epidemiology, University of Massachusetts, Amherst, Massachusetts 01003, U.S.A.

^bDepartment of Biostatistics, Harvard School of Public Health, Boston, Massachusetts 02115, U.S.A.

Abstract

Clinical studies using complex sampling often involve both truncation and censoring, where there are options for the assumptions of independence of censoring and event and for the relationship between censoring and truncation. In this paper, we clarify these choices, show certain equivalences, and provide examples.

Keywords

Censoring; Cross-sectional sampling; Quasi-independence; Truncation

1. Introduction

An increasing number of time-to-event studies are assembled through various complex sampling schemes that involve left truncation, i.e., event times that are observed conditional on their exceeding associated truncation times. In addition to truncation, time-to-event data are often subject to right censoring. Taking onset of disease as the time origin of interest, let random variables T denote the time to study entry (i.e., truncation time), X denote the time to death (i.e., event time or survival time), and C denote the time until end of study or loss to follow-up (i.e., censoring time). Defining features of these data are that $Y = \min(X, C)$ is observed, but not both X and C , and that (T, Y) is observed only if $T \leq Y$. In the absence of truncation, analyses of such survival data typically assume independence of event and censoring times. In the presence of truncation, however, there are options for the assumptions of independence of censoring and event and for the relationship between censoring and truncation. Additionally, independence is typically assumed between event and truncation, which can be tested. For example, Tsai (1990), and later Martin and Betensky (2005) developed tests for quasi-independence between X and T , i.e. whether $T \perp X|T \leq X$. According to Tsai (1990), a rigorous definition of quasi-independence can be stated as follows:

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*Corresponding author. Tel: +1 413 545 1347 qian@schoolph.umass.edu (Jing Qian).

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$$\Pr(X \leq x, T \leq t | T \leq X) = \int \int_{\Delta(x,t)} d\Pr(X \leq u) d\Pr(T \leq v) / \alpha, \quad (1)$$

where $\alpha = \Pr(T \leq X) = \int \int_{u \leq v} d\Pr(X \leq u) d\Pr(T \leq v)$, $\Delta(x, t) = \{(u, v) | u \leq x, v \leq t, u \leq v\}$. If we assume that the random variables are absolutely continuous with respect to the Lebesgue measure and $\Pr(X \leq x)$, $\Pr(T \leq t)$ and $\Pr(X \leq x, T \leq t | T \leq X)$ have probability density functions (PDF) $f_X(x)$, $f_T(t)$ and $h(x, t)$, respectively, we will have an equivalent definition of quasi-independence as (1) in terms of the PDF's:

$$h(x, t) = \begin{cases} f_X(x)f_T(t)/\alpha, & t \leq x \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

There are two major components to a censoring model in the presence of truncation. One is whether or not censoring can occur for subjects who have not entered the study. That is, is the meaningful time origin for censoring the onset of disease or entry to the study? The second and related component is the set of independence assumptions among T , X , and C . The handful of papers that have addressed analysis of left truncated and right censored data (e.g., Tsai et al. 1987; Wang 1991; Lai and Ying 1991; Asgharian et al. 2002; Mandel and Betensky 2007; Shen et al. 2009; Tsai 2009) have introduced variations on independence assumptions among T , X , and C . In the application of these methods to analysis of such data, it is important to fully understand and justify the underlying assumptions, which we have listed below for future reference.

Assumption A1 : $\Pr(T \leq C) = 1$

Assumption A2 : $\Pr(T \leq C) < 1$

Assumption B1 : $C \perp T | T \leq C$

Assumption B2 : $(C - T) \perp (X - T, T) | T \leq X$

Assumption B3 : $(C - T) \perp (X, T) | T \leq X$

Assumption C1 : $X \perp (T, C) | T \leq X$

Assumption C2 : $X \perp (T, C) | T \leq Y$, where $Y = \min(X, C)$

In the presence of truncation and censoring, follow-up studies can be categorized into two types, according to whether or not censoring happens solely after study entry. In the first type, a well-defined population of subjects are recruited into a study only if they are event-free at the time of study enrollment, i.e., $T \leq X$. The disease onset times for these recruited subjects are retrospectively ascertained. The subjects are then followed prospectively in time, and their lifetimes, X , are observed or subject to right censoring. Thus, by definition, subjects cannot be censored until they have entered the study, i.e., $\Pr(T \leq C) = 1$. This type of censoring is present in the Channing House study (Hyde 1977).

Channing House study

The Channing House is a retirement center located in Palo Alto, California. All of the Channing House residents were covered by a health care program that provided direct access to medical care without any additional financial burden to the residents. The endpoint of interest is the lifetime of the residents, with time origin for each resident equal to date of birth. Channing House resident lifetimes were subject to left truncation since residents had to survive to their age at entry to the Channing House in order to be in the study. The lifetimes were subject also to right censoring since residents may have been alive at the end of the study or they have departed the Channing House prior to the end of the study. Thus,

censoring could occur only for residents of Channing house who were included in the study. The study included 462 individuals who were in residence during the period from January 1964 to July 1975.

In the second type of follow-up study with truncation and censoring, the target population fully identified at the time origin of disease onset, and is subsequently subject to right censoring. With cross-sectional sampling of this population, a random sample of subjects who are alive and not censored are recruited into the study and then followed for a period of time. The truncation time is the time between the onset of disease and study entry. Thus, censoring can precede study entry, i.e., $\Pr(T > C) > 0$, or equivalently, $\Pr(T - C) < 1$. This type of censoring occurs often in competing risk cross-sectional studies, such as the Stanford heart transplant study (Crowley and Hu 1977).

Competing risk cross-sectional sampling study

De Una-Alvarez (2004) described a lifetime testing problem in industry, which can be restated as a competing risk cross-sectional sampling study in the context of survival analysis. Of primary interest is the lifetime distribution of the subjects, identified at the time origin, who die due to cause A . Death due to cause B is a censoring event in the analysis of time to death due to cause A . A cross-sectional sample is taken, and subjects who are alive at that time and who remain under observation, are enrolled into the study. Subjects who die before the time of sampling due to cause A (i.e., $X < T$), or who die before the time of sampling due to cause B (i.e., $C < T$), or who depart from the cohort prior to the time of sampling (i.e., $C < T$), are not included in the study. Those who are enrolled at the cross-sectional sampling remain subject to right censoring. This implies that censoring may occur prior to study entry, i.e., $\Pr(T - C) < 1$.

Stanford heart transplant study

Crowley and Hu (1977) described 103 potential heart transplant recipients enrolled in the Stanford heart transplant program from October 1967 to April 1974. Among the 103 potential heart transplant recipients, 69 patients received a heart transplant. The time origin of interest was enrollment in the transplant program and the endpoint of interest was survival from the time of enrollment. Left truncation occurred due to death prior to transplant for patients enrolled in the transplant program. Right censoring occurred due to causes other than death that removed patients from the transplant program prior to receiving a transplant. The subpopulation of interest for analysis are the patients who received a transplant; this sampling scheme introduces left truncation and right censoring due to the events described above and is an example of a study with $\Pr(T - C) < 1$.

All studies that involve truncation and censoring require certain assumptions of independence among (T, X, C) or models for dependence, which may or may not be identifiable given the nature of the data. For the first type of study, with $\Pr(T < C) = 1$, assumptions of independence are most natural on the residual time scale, i.e., with regard to censoring measured from study entry, $C - T$. However, they may also be made on the original time scale, i.e., with regard to censoring measured from disease onset, C . We note in Section 2 that independence of the residual censoring time, $C - T$, and the truncation time, T , in the observable region, is necessary for valid application of standard tests and estimators. For example, in the Channing House study, if individuals who entered the Channing House at older ages had longer times from entry to departure, then assumptions based on the residual time scale would not be appropriate. For the second type of study, with $\Pr(T - C) < 1$, assumptions of independence are most natural, and in fact, necessary, on the original time scale, i.e., with regard to censoring measured from disease onset, C . Tsai (2009) also differentiated censoring on the residual scale from censoring on the original

scale, and stated that most applications can be classified as one of these two types of censoring. With Assumption C1, Tsai (2009) proposed two different pseudo-partial likelihood inference approaches for estimation of the regression coefficients in the Cox proportional hazard model for the two assumptions about censoring: Assumption B1 for censoring on the original scale and Assumption B2 for censoring on the residual scale.

The goal of this paper is to clarify the plausible censoring models to use with truncation, explain the implications of different censoring models, and show the equivalence between certain censoring models.

2. Residual time scale assumptions

In studies in which censoring can occur only after study entry, such as the Channing House Study, it is natural to specify assumptions with respect to the residual time scale, i.e., times subsequent to study entry. This includes the residual lifetime $X - T$ and the residual censoring time $C - T$. Nonetheless, the censoring model on the original scale is still applicable (see Section 3).

To facilitate likelihood or pseudo-likelihood inference for $S_X(\cdot)$, researchers have assumed Assumption B2 or equivalently, Assumption B3, in conjunction with the quasi-independence assumption $X \perp T | T < X$. The equivalence of the two independence relationships for $C - T$ follows from the one-to-one mapping between $(X - T, T)$ and (X, T) . Furthermore, Assumption B2 implies that $X \perp C | T, T < X$ (see the Appendix for the proof), which in turn renders the quasi-independence assumption $X \perp T | T < X$ testable using the conditional Kendall's tau approach in the presence of right censoring, as developed by Tsai (1990) and Martin and Betensky (2005). Upon establishment of quasi-independence, given the implied standard independence condition for X and C , the standard nonparametric maximum likelihood estimator (NPMLE) for left truncated and right censored data is thus valid under these residual time scale assumptions. Without the quasi-independence between truncation time and survival time (e.g., in length-biased sampling), there will be an induced dependent censoring problem in estimating $S_X(\cdot)$.

There are at least four papers that have made assumptions on the residual scale. Wang (1991) required Assumption B2 for estimation of the bivariate distribution of C and T and of the distribution of $C - T$. Asgharian, M'LAN and Wolfson (2002) assumed Assumption B2 for their development of an unconditional NPMLE of $S_X(\cdot)$ in the presence of length-biased sampling. Mandel and Betensky (2007) required Assumption B3 in their formulation of goodness-of-fit tests for the truncation distribution. Tsai (2009) assumed Assumption B2 for his formulation of a pseudo-partial likelihood estimator for Cox proportional hazards models with biased sampling data.

3. Original time scale assumptions

Whether or not censoring can occur prior to study entry, the censoring time can always be measured with the reference date set at the time origin (e.g., the onset of disease). Those subjects who are observable satisfy $T < Y$, where $Y = \min(X, C)$, and T , X , and C are measured the time origin. A typical censoring model in this scenario is Assumption C2, which means that the survival time is independent of the random vector consisting of the truncation and censoring times in the observable region. This is the typical necessary condition for estimation of the marginal distribution of survival time and of the regression coefficients from a survival model. Tsai (2009) required an additional assumption that C and T are quasi-independent in the region $T < C$ to enable his novel pseudo-partial likelihood based inference under biased sampling. For testing quasi-independence (Tsai 1990) between X and T , the assumption that $X \perp C | T < Y$ is necessary. However, in the literature, many

papers additionally assume Assumption A1 for ease of notation. In fact, this assumption is not necessary.

Tsai (1990) assumed Assumption A1 and $X \perp C|T$ and developed tests for quasi-independence between X and T , i.e., for $T \perp X|T \leq X$. Since conditional independence between X and C only matters in the observable region, the assumption $X \perp C|T$ actually can be relaxed to a weaker version $X \perp C|T, T \leq X$, which was assumed later in Martin and Betensky (2005) together with Assumption A1. Without the assumption of conditional independence between X and C , the conditional density of $X|T$ can not be separated from the joint conditional density $(X, C)|T$. Thus, the problem of testing quasi-independence between X and T would be unidentifiable. However, Assumption A1 is not necessary here. Under Assumption A1 and $X \perp C|T, T \leq X$, the quasi-independence definition (1) from Tsai (1990) can be extended to the left truncation and right censoring setting as

$$\Pr(X \leq x, T \leq t, C \leq c | T \leq X) = \iint \int \Delta(x, t, c) d\Pr(X \leq u) d\Pr(T \leq v, C \leq w) / \alpha, \quad (3)$$

where $\alpha = \Pr(T \leq X) = \int \int \int \Delta(x, t, c) d\Pr(X \leq u) d\Pr(T \leq v, C \leq w) / \alpha$, $\Delta(x, t, c) = \{(u, v, w) | u \leq x, v \leq t, w \leq c, u \leq v, w \leq v\}$. Assuming that $\Pr(X \leq x)$, $\Pr(T \leq t, C \leq c)$ and $\Pr(X \leq x, T \leq t, C \leq c | T \leq X)$ have PDF's $f_X(x)$, $f_{T,C}(t, c)$ and $h(x, t, c)$, respectively, we will have an equivalent definition of quasi-independence as (3) in terms of PDF:

$$h(x, t, c) = \begin{cases} f_X(x) f_{T,C}(t, c) / \alpha, & t \leq x, t \leq c \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

Under Assumption A2, the observable region can be redefined as $T \leq Y$ where $Y = \min(X, C)$, and the definition of quasi-independence is the same as (4) except that the constant α becomes $\Pr(T \leq Y)$. Note that T and C are not symmetric in the definition due to the sampling scheme. Quasi-independence testing is based on the conditional Kendall's tau presented by Tsai (1990). In the presence of censoring with Assumption A1, Martin and Betensky (2005) showed that the test statistic has an expectation of 0 under the null hypothesis. It is straightforward to verify through similar arguments that this result holds as well under Assumption A2.

To estimate the marginal distribution of X , the seminal paper of Kaplan and Meier (1958) discussed how to adjust the product-limit estimator for randomly right-censored data when observations are censored on the right and truncated on the left. Some early papers on truncation do not include censoring, such as Lynden-Bell (1971) and Woodroffe (1985). Tsai, Jewell and Wang (1987) and Wang (1991) did consider censoring with the Assumptions A1 and C1, and developed the product-limit estimator based on the conditional likelihood given the data (T_i, Y_i, δ_i) ($i = 1, \dots, n$):

$$L_C = \prod_{i=1}^n \frac{f_X(Y_i)^{\delta_i} S_X(Y_i)^{1-\delta_i}}{S_X(T_i)},$$

where $Y = \min(X, C)$ and $\delta = I(X \leq C)$. These assumptions ensure independent left truncation and independent right censoring for nonparametric estimation of the marginal distribution of X . The product-limit estimator remains valid under Assumptions A2 and C2, which is verified through extension of the conditional likelihood argument in Section 3 of Wang (1991). The only difference is that the probability of being untruncated in the likelihood

function becomes $\Pr(T = Y)$ instead of $\Pr(T = X)$. However, estimation of the distribution of truncation as in Wang (1991), does require the Assumption A1.

To estimate the regression coefficients of a survival model with left-truncated and right-censored data, Lawless (1982) and Cox and Oakes (1984) described how to modify a standard Cox proportional hazards model for right-censored data through adjustments to the risk sets. The Assumption C2 is necessary to ensure that the adjusted at-risk set approach is valid. Lai and Ying (1991) developed a rank-based regression method for left-truncated and right-censored data by assuming that $X \perp (T, C)$, given covariates $\mathbf{Z}$. Their assumption can be relaxed to Assumption C2, given $\mathbf{Z}$. Tsai (2009) proposed pseudo-partial likelihood inference procedure under Cox model for left-truncated and right-censored data, requiring the Assumptions B1 and C2, given $\mathbf{Z}$. Mandel and Betensky (2007) described how their testing procedure for uniform truncation, formulated through a censoring model on the residual scale, can be extended to a censoring model on a stronger variation of the original scale model that makes the additional assumption that $C \perp T$, in conjunction with the Assumption C2.

4. Connection between censoring models on different scales

Since both the censoring model on the residual scale in Section 2 and the censoring model on the original scale in Section 3 are applicable to data in which censoring happens only after truncation, it is helpful to understand the relationship between these models and their costs and benefits. In Lemma 1, we show that the censoring model on the residual scale implies the censoring model on the original scale.

Lemma 1. *The censoring model on the residual scale, which assumes Assumptions A1 and B2, and the quasi-independence assumption between X and T , i.e., $X \perp T|T = X$, implies the censoring model on the original scale, which assumes Assumption C1.*

Proof of Lemma 1. Under Assumptions A1, the observable region is defined as $T = X$. Given the Assumption B2 for the censoring model on the residual scale, it follows that $(C - T) \perp T|T = X$ and $(C - T) \perp X|T = X$. Note that $X \perp T|T = X$ by the quasi-independence assumption, which, combined with $(C - T) \perp X|T = X$ and the independence between $C - T$ and T in the observable region, implies that X is independent of the random vector (T, C) in the observable region, i.e., Assumption C1.

Remark 1. Furthermore, the Assumption C1 on the original scale is weaker than that on the residual scale, which assumes Assumption B2 and the quasi-independence between X and T , as stated in Lemma 1. By writing out the Jacobian matrix of transformation from (X, C, T) to $(X, C - T, T)$, we can see that the matrix's determinant is 1 and, further, that the Assumption C1 is equivalent to $X \perp (C - T, T)|T = X$, which does not assume the independence between $C - T$ and T in the observable region. Thus, the assumption of the censoring model on the original scale is weaker than that of the censoring model on the residual scale.

Remark 2. Wang (1991) recognized and used the result of Lemma 1 in Section 6 of her paper, though she did not prove it. To estimate the bivariate distribution of C and T or the distribution of $C - T$, Wang (1991) imposed the assumption $T \perp (C - T)$ in addition to the assumption $X \perp (T, C)$, and stated that these two assumptions together imply the independence of X , T and $C - T$.

Based on Lemma 1, studies with $\Pr(T = C) = 1$ that satisfy the independence assumptions on the residual scale, can be analyzed according to either censoring model. The choice of model depends on the nature of the censoring and the goals of the analysis. For example, the stronger assumption of independence on the residual scale is necessary for testing the

uniformity of the distribution of left truncation time, T , as shown in Mandel and Betensky (2007). Their method, which tests the equivalence of the conditional distributions of T and the residual lifetime, $X - T$, in the observable region using weighted log-rank tests for paired censored data (Jung 1999) requires Assumption B3. When $\Pr(T = C) = 1$, if the residual censoring time, $C - T$, and truncation time, T , are not independent in the observable region due to a feature of the sampling, then the censoring model based on the residual scale (with typical assumptions) is not applicable, but the censoring model on the original scale may be assumed.

The censoring model on the residual scale also allows for a “prevalence model” (Wang, Brookmeyer and Jewell 1993), which removes the problem of truncation and transforms the analysis to a regular right-censoring problem on the residual scale. This, in effect sets the time origin to be study entry, which depending on the study design, may or may not yield a meaningful analysis.

5. Discussion

In this paper, we clarify the plausible censoring models that may accompany truncation, explain the implications of different censoring models, and show the relationship among censoring models. By setting the censoring time origin to be the onset of disease, the general censoring model on the original scale is applicable regardless of when censoring may occur. Nonetheless, if censoring may occur only after truncation, it is natural to specify the censoring model on the residual scale.

Although the typical assumption of the censoring model on the residual scale is stronger than that on the original scale, it is necessary for certain testing and estimation procedures, such as the estimation of the joint distribution of C and T or the distribution of $C - T$ (Wang 1991) and the test for a uniform truncation distribution (Mandel and Betensky 2007). Thus, choice of the censoring model should primarily be guided by the particular sampling scheme of the study, but secondarily might be guided by the goals of the analysis. Given the wide selection of estimators available for left truncated and right censored data, it is imperative upon investigators to identify their assumptions and test them where possible.

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Appendix A. Proof

In Section 2, we state that $(C - T) \perp (X - T, T) | T = X$ implies that $X \perp C | T, T = X$. To prove this, we show that $(C - T) \perp (X - T, T)$ implies $X \perp C | T$ in the following. Our proof assumes that probability density functions (PDF) exist, but similar arguments can be followed in terms of more general cumulative distribution functions.

Proof. Denote $U = C - T$ and $V = X - T$. Since $C - T \perp (X - T, T)$, we have

$$f_{U,V,T}(u, v, t) = f_U(u) f_{V,T}(v, t). \quad (\text{A.1})$$

Thus, the conditional joint PDF of (U, V) given $T = t$ is

$$f_{U,V|T}(u, v|T=t) = \frac{f_{U,V,T}(u, v, t)}{f_T(t)} = \frac{f_U(u)f_{V,T}(v, t)}{f_T(t)} = f_U(u)f_{V|T}(v|T=t) = f_{U|T}(u|T=t)f_{V|T}(v|T=t), \quad (\text{A.2})$$

where the second equality is based on equality (A.1), and the last equality is due to $U \perp T$. Therefore, the conditional joint PDF of (C, X) given $T = t$ is

$$\begin{aligned} f_{C,X|T}(c, x|T \\ &= t) = f_{C-T, X-T|T}(c \\ &\quad - t, x \\ &\quad - t|T \\ &= t) = f_{U,V|T}(c \\ &\quad - t, x \\ &\quad - t|T \\ &= t) = f_{U|T}(c - t|T=t)f_{V|T}(x \\ &\quad - t|T \\ &= t) = \frac{f_{U,T}(c - t, t)}{f_T(t)} \frac{f_{V,T}(x - t, t)}{f_T(t)} \\ &= \frac{f_{C,T}(c, t)}{f_T(t)} \frac{f_{X,T}(x, t)}{f_T(t)} \\ &= f_{C|T}(c|T \\ &\quad = t)f_{X|T}(x|T=t), \end{aligned}$$

where the third equality is due to equality (A.2). This demonstrates that $X \perp C|T$.

References

- Asgharian M, M'LAN CE, Wolfson DB. Length-biased sampling with right censoring: An unconditional approach. *Journal of the American Statistical Association*. 2002; 97:201–209.
- Cox, DR.; Oakes, D. *Analysis of Survival Data*. Chapman & Hall; 1984.
- Crowley J, Hu M. Covariance analysis of heart transplant survival data. *Journal of the American Statistical Association*. 1977; 72:27–36.
- De Una-Alvarez J. Nelson-aalen and product-limit estimation in selection bias models for censored populations. *Nonparametric Statistics*. 2004; 16:761–777.
- Hyde J. Testing survival under right censoring and left truncation. *Biometrika*. 1977; 64:225–230.
- Jung SH. Rank tests for matched survival data. *Lifetime Data Analysis*. 1999; 5:67–79. [PubMed: 10214003]
- Kaplan EL, Meier P. Nonparametric estimation from incomplete observations. *Journal of the American Statistical Association*. 1958; 53:457–481.
- Lai TL, Ying Z. Rank regression methods for left-truncated and right-censored data. *The Annals of Statistics*. 1991; 19:531–556.
- Lawless, J. *Statistical Models and Methods for Lifetime Data*. Wiley; 1982.
- Lynden-Bell D. A method of allowing for known observational selection in small samples applied to 3cr quasars. *Monthly Notices of the Royal Astronomical Society*. 1971; 155:95–118.
- Mandel M, Betensky RA. Testing Goodness of Fit of a Uniform Truncation Model. *Biometrics*. 2007; 63:405–412. [PubMed: 17688493]
- Martin EC, Betensky RA. Testing quasi-independence of failure and truncation times via conditional kendall's tau. *Journal of the American Statistical Association*. 2005; 100:484–492.

- Shen Y, Ning J, Qin J. Analyzing Length-Biased Data With Semiparametric Transformation and Accelerated Failure Time Models. *Journal of the American Statistical Association*. 2009; 104:1192–1202. [PubMed: 21057599]
- Tsai WY. Testing the assumption of independence of truncation time and failure time. *Biometrika*. 1990; 77:169–177.
- Tsai WY. Pseudo-partial likelihood for proportional hazards models with biased-sampling data. *Biometrika*. 2009; 96:601–615. [PubMed: 22422175]
- Tsai WY, Jewell NP, Wang MC. A note on the product-limit estimator under right censoring and left truncation. *Biometrika*. 1987; 74:883–886.
- Wang MC. Nonparametric estimation from cross-sectional survival data. *Journal of the American Statistical Association*. 1991; 86:130–143.
- Wang MC, Brookmeyer R, Jewell NP. Statistical models for prevalent cohort data. *Biometrics*. 1993; 49:1–11. [PubMed: 8513095]
- Woodroffe M. Estimating a distribution function with truncated data. *The Annals of Statistics*. 1985; 13:163–177.