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CENTER PROBLEM FOR A THIRD-ORDER ODES

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Abstract

We obtain necessary and sufficient conditions for the existence of a center on a local center manifold for three 4-parameter families of quadratic systems on $\mathbb{R}^3$. We also give a positive answer to an open question posed in [Dias & Mello(2010)] related to similar systems.

Key words and phrases

center-focus problem; center manifold; first integral

1. Introduction

The so-called *center* or the *center-focus* problem is one of the most famous problems in the qualitative theory of ordinary differential equations (ODEs) in dimension two. For a differential equation $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$ on $\mathbb{R}^2$ having an isolated fixed point at the origin, $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, at which the linear part $d\mathbf{f}(\mathbf{0})$ has two pure imaginary eigenvalues it consists of giving conditions on $\mathbf{f}(\mathbf{u})$ to distinguish between a center and a focus. It has turned out to be a very hard problem, which has been the subject of intensive research, see e.g. [Łojasiewicz(1994)], [Gasull *et al.*], [Briskin *et al.*], [Llibre & Valls(2011)], [Romanovski & Shafer(2009)], and [Christopher & Li(2007)]. Although in its full generality the problem remains unsolved, still for many important subclasses of planar polynomial vector fields the solution is known. For example, when $\mathbf{f}(\mathbf{u})$ is quadratic, i.e.

$$\begin{aligned}\dot{u} &= v + a_1 u^2 + a_2 uv + a_3 v^2 \\ \dot{v} &= -u + a_4 u^2 + a_5 uv + a_6 v^2,\end{aligned}\quad (1)$$

where $a_1, \dots, a_6 \in \mathbb{R}$, the center conditions have been established by Dulac [Dulac(1908)] and Kapteyn [Kapteyn(1912)]. It is well-known (see e.g. [Łojasiewicz(1994), Romanovski & Shafer(2009)]) that for system (1) the Bautin ideal $\mathcal{B}$ is generated by the first three focus quantities of this system. Moreover, the center variety $V(\mathcal{B}) \subset \mathbb{R}^6$ of the ideal $\mathcal{B}$ has four irreducible components, i.e.

$$V(\mathcal{B}) = V(I_{Ham}) \cup V(I_{sym}) \cup V(I_{\Delta}) \cup V(I_{con}),$$

where the four components correspond respectively to Hamiltonian systems, reversible systems, the Zariski closure of those systems having three invariant lines and finally Zariski closure of systems having an invariant conic and an invariant cubic. In other words for system (1) these are the four different mechanisms for creating a center.

Unlike the planar case, little is known about the center conditions in dimensions greater than two. This article is an attempt to partially fill this gap by studying the center-focus problem

for a quadratic three-dimensional system that can be written as a third-order differential equation:

$$\ddot{u} = \alpha \ddot{u} + \beta \dot{u} + \gamma u + f(u, \dot{u}, \ddot{u}), \quad (2)$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $f = f(u, \dot{u}, \ddot{u})$ is an analytic function starting with quadratic terms. Equivalently, we consider the associated system of differential equations

$$\dot{u} = v, \quad \dot{v} = w, \quad \dot{w} = \alpha w + \beta v + \gamma u + f(u, v, w). \quad (3)$$

Assume that $\alpha\beta\gamma \neq 0$. If $\beta < 0$ and $\gamma = -\alpha\beta$, then as a simple consequence of the Center Manifold Theorem (see [Carr(1981)]) system (3) admits a local center manifold at the origin. In order to make the problem computationally tractable we make the standing assumption that $\alpha = \beta = -1$. Thus we consider the following system

$$\dot{u} = v, \quad \dot{v} = w, \quad \dot{w} = -u - v - w + f(u, v, w), \quad (4)$$

where $f(u, v, w)$ starts with terms of degree two. We note that using the change of variables $u \rightarrow -u, v \rightarrow v, w \rightarrow -w$ and immediately dropping the bar we can write it as

$$\begin{aligned} \dot{u} &= -v + g(u, v, w) \\ \dot{v} &= u + g(u, v, w) \\ \dot{w} &= -w + g(u, v, w), \end{aligned} \quad (5)$$

where $g(u, v, w) = \frac{1}{2}f(-u+w, v-w, u+w)$. We shall call system (5) the *standard form* of system (4).

In this article we study the center problem on a local center manifold of system (4) having *quadratic* nonlinearities, i.e.

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= w \\ \dot{w} &= -u - v - w + a_1 u^2 + a_2 v^2 + a_3 w^2 + a_4 uv + a_5 uw + a_6 vw, \end{aligned} \quad (6)$$

where $a_1, \dots, a_6 \in \mathbb{R}$. We were able to obtain necessary and sufficient conditions for the existence of a center on the local center manifold for the following three 4-parameter families of system (6):

- $a_5 = a_6 = 0$ (see Theorem 8),
- $a_4 = a_6 = 0$ (see Theorem 9),
- $a_4 = a_5 = 0$ (see Theorem 10).

The precise statements and proofs are given in section 3. For the unrestricted system (6) it was possible to compute the first five nonzero focus quantities $g_{220}, \dots, g_{660}$ (see subsection 2.1), but even using a specialize computer algebra package (see subsection 2.4) it was not possible to decompose the radical of ideal $\langle g_{220}, \dots, g_{660} \rangle$ over $\mathbb{R}$ into primes (for more details see section 2).

As a simple corollary from Theorem 8 is that the center variety $\mathbf{V}$ of system (6) for $a_5 = a_6 = 0$ consists of the following four irreducible components:

$$\mathbf{V} = \mathbf{V}^a \cup \mathbf{V}^b \cup \mathbf{V}^c \cup \mathbf{V}^d,$$

where

$$\begin{aligned}\mathbf{V}^a &= \mathbf{V}(a_2, a_4, a_1 + a_3), & \mathbf{V}^b &= \mathbf{V}(a_1, a_4, a_2 - a_3), \\ \mathbf{V}^c &= \mathbf{V}(a_2, a_3, 2a_1 - a_4), & \mathbf{V}^d &= \mathbf{V}(a_3, 2a_2 - a_4, 2a_1 - a_4).\end{aligned}$$

As usual we denote by $\mathbf{V}(a_2, a_4, a_1 + a_3)$ the variety of the polynomial ideal $\langle a_2, a_4, a_1 + a_3 \rangle$. Note that in this case each component of the center variety of system (3) restricted to $a_5 = a_6 = 0$ is a *vector subspace* of the parameter space $\mathbb{R}^6$ of this system. From Theorems 8 and 9 we get that the same is true for system (3) restricted to $a_4 = a_6 = 0$ or $a_5 = a_6 = 0$. Based on this we have the following conjecture.

Conjecture—The irreducible components of the center variety of system (6) are vector subspaces of its parameter space $\mathbb{R}^6$.

We note that the above conjecture does not hold for more general quadratic systems on $\mathbb{R}^3$, which are not equivalent to the third-order ODE. For example, one can prove [Edneral *et al.* (2012)] that the center variety $\mathbf{V}_{ex}$ of the following system

$$\begin{aligned}\dot{u} &= -v + cuw + dvw \\ \dot{v} &= u + euw - cvw \\ \dot{w} &= -w + u^2 + v^2 + Tuw + Uvw,\end{aligned}\tag{7}$$

consists of two irreducible components $\mathbf{V}_{ex} = \mathbf{V}^A \cup \mathbf{V}^B$, where

$$\mathbf{V}^A = \mathbf{V}(8c + T^2 - U^2, 4e - 4d - T^2 - U^2, 2e + 2d + TU), \quad \text{and} \quad \mathbf{V}^B = \mathbf{V}(c, d + e).$$

It is easy to see that $\mathbf{V}^A$ is not a vector subspace of the parameter space $\mathbb{R}^5$ of system (7).

Our last result deals with an open question posed in [Dias & Mello(2010), Question 4.6] related to the following system

$$\dot{u} = v, \quad \dot{v} = w, \quad \dot{w} = -[(a_1 u + a_0)w + (b_1 u + b_0)v + u(u + 1)], \tag{8}$$

where $(a_0, a_1, b_0, b_1) \in \mathbb{R}^4$. The authors asked whether for $(a_0, a_1, b_0, b_1) \in \mathcal{L}_2$, where $\mathcal{L}_2 = \{a_1 = 0, a_0 = 1/b_0, b_1 = 2b_0\}$ the origin of system (8) is a center for the associated flow restricted to the center manifold. In the next theorem we provide a positive answer to this question. For a proof see section 3.

Theorem 1—If $(a_0, a_1, b_0, b_1) \in \mathcal{L}_2$, then system (8) admits a center at the origin on the local center manifold.

Although the center problem in the multidimensional case has not been fully explored, some of the recent contribution to this field include [Buic *et al.*(2011), Wang *et al.*(2010), Wang *et al.* (2011a), Wang *et al.*(2011b), Edneral *et al.*(2012)] and [Mahdi *et al.*(2011)].

The article is organized as follows. In section 2 we present general techniques and definitions necessary for the proof of our main results, which are given in section 3. Finally, concluding remarks and numerical simulations are presented in section 4.

2. Focus quantities, center variety and symbolic computations

2.1. Focus Quantities

Following [Edneral *et al.*(2012)], we briefly describe here an approach to study the center problem on a center manifold for the following three dimensional system

$$\begin{aligned}\dot{u} &= -v + P(u, v, w) = \tilde{P}(u, v, w) \\ \dot{v} &= u + Q(u, v, w) = \tilde{Q}(u, v, w) \\ \dot{w} &= -\lambda w + R(u, v, w) = \tilde{R}(u, v, w),\end{aligned}\quad (9)$$

where $\lambda \in (0, +\infty)$ and P, Q , and R are real analytic on a neighborhood of the origin. It is well-known that system (9) admits a local center manifold W_{loc}^c at the origin. We will let $\mathfrak{X}$ denote the corresponding vector field

$$\mathfrak{X} = \tilde{P} \frac{\partial}{\partial u} + \tilde{Q} \frac{\partial}{\partial v} + \tilde{R} \frac{\partial}{\partial w} \quad (10)$$

on a neighborhood of the origin.

A *local first integral* of system (9) is a non-constant differentiable function Ψ from a neighborhood of the origin in $\mathbb{R}^3$ into $\mathbb{R}$ that is constant on trajectories of (9), equivalently, such that

$$\mathfrak{X}\Psi = \tilde{P}\Psi_u + \tilde{Q}\Psi_v + \tilde{R}\Psi_w \equiv 0. \quad (11)$$

A *formal first integral* for system (9) is a non-constant formal power series Ψ in u, v , and w such that when P, Q , and R are expanded in power series about the origin, every coefficient in the formal power series in (11) is zero.

One of the main tools for detecting a center on a center manifold is the following theorem. For a proof see [Edneral *et al.*(2012)].

Theorem 2—The following statements are equivalent:

- a. The origin is a center for $\mathfrak{X}|_{W_{loc}^c}$.
- b. System (9) admits a local analytic first integral.
- c. System (9) admits a formal first integral.

The equivalence of the statement (a) and (b) is called the *Lyapunov Center Theorem*; for a proof see also e.g. [Bibikov(1979)]. Moreover, a real analytic local first integral from statement (b) can always be chosen to be of the form

$$\Phi(u, v, w) = u^2 + v^2 + \dots$$

in a neighborhood of the origin in $\mathbb{R}^3$.

From now on we assume that P , Q and R in (9) are polynomials. We show how to compute the so-called focus quantities, the most important object in order to determine necessary and sufficient conditions for the existence of a center on the center manifold of system (9). We begin by introducing the complex variable $x = u + iv$. Then the first two equations in (9) are equivalent to a single equation $\dot{x} = ix + X(x, x, \bar{w})$, where X is a sum of homogeneous polynomials of degrees between 2 and N . Adjoining to this equation its complex conjugate, replacing x everywhere by y , regarding y as an independent complex variable, and replacing w by z simply as a notational convenience we obtain the complexification of family (9),

$$\begin{aligned}\dot{x} &= ix + \sum_{p+q+r=2}^N a_{pqr} x^p y^q z^r \\ \dot{y} &= -iy + \sum_{p+q+r=2}^N b_{pqr} x^p y^q z^r \\ \dot{z} &= -\lambda z + \sum_{p+q+r=2}^N c_{pqr} x^p y^q z^r,\end{aligned}\quad (12)$$

where $b_{pqr} = \overline{a_{pqr}}$ and the c_{pqr} are such that $\sum_{p+q+r=2}^N c_{pqr} x^p \bar{x}^q w^r$ is real for all $x \in \mathbb{C}$, for all $w \in \mathbb{R}$. Let $\mathfrak{F}$ denote the corresponding vector field on $\mathbb{C}^3$. Existence of a first integral $\Phi(u, v, w) = u^2 + v^2 + \dots$ for a system in family (9) is equivalent to existence of a first integral

$$\Psi(x, y, z) = xy + \sum_{j+k+\ell=3} v_{jkl} x^j y^k z^\ell \quad (13)$$

for the corresponding system in family (12).

We now investigate the existence of a first integral Ψ for a system in family (12) by computing the coefficients of $\mathfrak{F}\Psi$ and equating them to zero. When Ψ has the form (13) the coefficient $g_{k_1 k_2 k_3}$ of $x^{k_1} y^{k_2} z^{k_3}$ in $\mathfrak{F}\Psi$ can be calculated explicitly (see [Edneral *et al.* (2012)]). Except when $(k_1, k_2, k_3) = (K, K, 0)$ for $K \in \mathbb{N}$, the equation $g_{k_1 k_2 k_3} = 0$ can be solved uniquely for $v_{k_1 k_2 k_3}$ in terms of the known quantities $v_{\alpha\beta\gamma}$. A formal first integral Ψ thus exists if $g_{kk0} = 0$ for all $k \in \mathbb{N}$. An obstruction to the existence of the formal series Ψ occurs when the coefficient g_{KK0} is nonzero. This coefficient is the K th *focus quantity*.

The focus quantity $g_{110} = 0$ and g_{220} is uniquely determined, but the remaining ones depend on the choices made for v_{KK0} , $K \in \mathbb{N}$, $K \geq 2$. Once such an assignment is made Ψ is determined and satisfies

$$\mathfrak{F}\Psi(x, y, z) = g_{220}(xy)^2 + g_{330}(xy)^3 + \dots \quad (14)$$

It is known [Edneral *et al.* (2012)] that if for one choice of the v_{KK0} at least one focus quantity is non-zero, then the same is true for every other choice of the v_{KK0} .

2.2. Necessary conditions

Vanishing of all the focus quantities is sufficient for existence of the formal first integral. The following theorem establishes that it is also *necessary*. To shorten the notation we let (a, b, c) stand for the coefficient string

$$(a_{200}, \dots, a_{00N}, b_{200}, \dots, b_{00N}, c_{200}, \dots, c_{00N}).$$

Recall that the focus quantities $g_{220}, g_{330}, \dots$ are polynomials in (a, b, c) that satisfy (14) with respect to the system (12). For a proof the following two theorems see [Edneral *et al.* (2012)].

Theorem 3—System (12) with $(a, b, c) = (a^*, b^*, c^*)$ admits a formal first integral of the form (13) if and only if $g_{kk0}(a^*, b^*, c^*) = 0$ for all $k \in \mathbb{N}$.

As a simple corollary we get that the conditions for the existence of a center on the local center manifold of a polynomial vector field in $\mathbb{R}^3$ can always be expressed as zeros of polynomials.

Theorem 4—There exists a variety $\mathbf{V}_C$ in the space of admissible coefficients such that the origin is a center for $\mathfrak{X}|_{W_{loc}^c}$ if and only if the coefficients of $\mathfrak{X}$ lie in $\mathbf{V}_C$.

2.3. Sufficient conditions

The vanishing of the first, say $j \in \mathbb{N}$, focus quantities gives only necessary conditions for the existence of a center, since we do not know how many focus quantities suffice to generate the full ideal $\mathcal{B} := \langle g_{kk0} : k \in \mathbb{N} \rangle$. Denote by $\mathcal{B}_j$ the ideal generated only by the first j focus quantities, i.e. $\mathcal{B}_j = \langle g_{220}, \dots, g_{jj0} \rangle$. Note that requiring to find $k \in \mathbb{N}$ such that $\mathcal{B}_k = \mathcal{B}$ is too much as by Theorems 3 and 4 one seeks only the following equality to hold

$$\mathbf{V}(\mathcal{B}_j) = \mathbf{V}(\mathcal{B}). \quad (15)$$

It is well-known [Cox *et al.* (1997)] that if I, J are two polynomial ideals over $\mathbb{C}$, then $\mathbf{V}(I) = \mathbf{V}(J)$ if and only if $\sqrt{I} = \sqrt{J}$. Thus, to check if we have enough focus quantities for the

equality (15) to hold, we decompose the radical $\sqrt{\mathcal{B}_j}$ into an intersection of prime ideals, to which correspond the irreducible components of $\mathbf{V}(\sqrt{\mathcal{B}_j})$. Generators of these prime ideals give finite sets of conditions which, if they can be proved to be sufficient for existence a center on the local center manifold W_{loc}^c , are then a collection of conditions that together characterize existence of a center on W_{loc}^c .

2.4. Symbolic computations

Both, the computation of the focus quantities as well as the decomposition of the radical ideal into primes, described above require the use of computer algebra packages. The focus quantities can be computed following, for example, the ideas outlined in subsection 2.1 and be implemented in any general symbolic computation package such as mathematica or Maple.

The next computational step is the decomposition of the radical $\sqrt{\mathcal{B}_k}$ of the ideal $\mathcal{B}_k$ generated by the first $k \in \mathbb{N}$ focus quantities $g_{220}, \dots, g_{kk0}$ into the prime ideals. Because the focus quantities tend to be very large and contain several hundreds (or even thousands) of monomials, the decomposition procedure requires more specialized computer algebra package such as Singular [Greuel *et al.* (2005)], CALI [Gräbe (1995)] or Macaulay [Grayson & Stillman()]. We performed our computation with minAssGTZ of Singular (library *primedec.lib* [Decker *et al.* (2001)]) in order to find the minimal associated primes of a polynomial ideal by means of the Gianni-Trager-Zacharias method [Gianni *et al.* (1988)].

From the library *primedec.lib* one can also use *minAssChar*, which computes the minimal associated primes using characteristic sets; as well as *primdecSY*, *primdecGTZ*, which compute the complete primary decomposition via Gianni-Trager-Zacharias [Gianni *et al.* (1988)] and Shimoyama-Yokoyama [Shimoyama & Yokoyama(1996)] method, respectively.

3. Quadratic case

In this section we obtain necessary and sufficient conditions for the existence of a center at the origin of system (6) restricted to $a_5 = a_6 = 0$, $a_4 = a_6 = 0$ and $a_4 = a_5 = 0$. Equivalently, instead of system (6), we can study its standard form (5), where

$$g(u, v, w) = \frac{1}{2} \left[a_1(u-w)^2 + a_2(v-w)^2 + a_4(v-w)(w-u) + a_6(v-w)(u+w) + a_5(w-u)(u+w) + a_3(u+w)^2 \right].$$

We start by computing the first five nonzero focus quantities $g_{220}, \dots, g_{660}$. Because of the size of the expressions we provide the explicit formula only for the first one

$$g_{220} = \left[-18a_1^2 - 18a_1a_2 - 4a_2^2 - 10a_1a_3 - 4a_2a_3 + 8a_3^2 + 11a_1a_4 + 11a_2a_4 + 13a_3a_4 - a_4^2 + 23a_1a_5 + 11a_2a_5 - 3a_3a_5 - 12a_4a_5 - 5a_5^2 - 2a_2a_6 - 2a_3a_6 - a_4a_6 + a_5a_6 + 2a_6^2 \right] / 10.$$

The other four, which have respectively 126, 462, 1287 and 3003 different monomials, can be computed using the ideas of subsection 2.1. We would also like to highlight that the computations have been performed *without* any restrictions on the parameters $a_1, \dots, a_6$.

Before we prove our main results we shall need the following theorem. It was first proven by Dulac [Dulac(1908)] and Kapteyn [Kapteyn(1912)], but we present the version given in [Coppel(1966)].

Theorem 5 (Quadratic Center Theorem)—The system

$$\begin{aligned} \dot{u} &= -v - bu^2 - (B+2c)uv - dv^2 \\ \dot{v} &= u + au^2 + (A+2b)uv + cv^2 \end{aligned}$$

has a center at the origin if and only if at least one of the following three conditions is satisfied:

- I. $a + c = b + d$;
- II. $A(a+c) = B(b+d)$ and $aA^3 - (3b+A)A^2B + (3c+B)AB^2 - dB^3 = 0$;
- III. $A + 5b + 5d = B + 5a + 5c = ac + bd + 2a^2 + 2d^2 = 0$.

Also an observation made in [Buic *et al.*(2011)] can sometimes help to simplify the proof of the existence of a center on the center manifold in some concrete cases. Before we reproduce it here, we introduce some definitions, restricting them to the three-dimensional vector field

$$\mathfrak{X} = f_1 \frac{\partial}{\partial u} + f_2 \frac{\partial}{\partial v} + f_3 \frac{\partial}{\partial w},$$

where $f_j = f_j(u, v, w)$ are analytic functions on an open set $U \subset \mathbb{R}^3$. Following [Berrone & Giacomini(2003)] we say that a C^1 function $\mathcal{F} = \mathcal{F}(u, v, w)$ is an *inverse Jacobi multiplier* if it satisfies the equation

$$\mathfrak{X}\mathcal{F} := f_1 \frac{\partial \mathcal{F}}{\partial u} + f_2 \frac{\partial \mathcal{F}}{\partial v} + f_3 \frac{\partial \mathcal{F}}{\partial w} = \mathcal{F} \operatorname{div} \mathfrak{X}.$$

Note that if $\mathcal{F}$ is defined in an analogous way for planar vector fields, then it is called an *inverse integrating factor* (for more details see e.g. [García & Grau(2010), Llibre *et al.* (2007)]). Note also that if $\mathfrak{X}$ is a polynomial vector field and $\mathcal{F} \in \mathbb{R}[u, v, w]$, then by definition $\mathcal{F} = 0$ is an invariant algebraic surface with the cofactor $\mathfrak{K} = \operatorname{div} \mathfrak{X}$.

In the following theorem and proposition we assume that a three dimensional vector field $\mathfrak{X}$ possesses a singularity at the origin having two pure imaginary and one nonzero eigenvalue. The results are due to Buic , García and Maza [Buc *et al.*(2011)].

Theorem 6—The vector field $\mathfrak{X}$ has a center at the origin if and only if it admits a local analytic inverse Jacobi multiplier $\mathcal{F}$ at the origin such that $\nabla \mathcal{F}(u^*) = 0$.

Proposition 7—Assume the existence of a smooth and nonflat inverse Jacobi multiplier $\mathcal{F}(u, v, w)$ of $\mathfrak{X}$ near the origin. If one has the following Taylor expansion

$$\mathcal{F}(v, v, v) = a_1 v + \mathcal{O}(v^2), \quad \text{with } a_1 \neq 0,$$

then the origin is a center for $\mathfrak{X}$.

Now we shall study the existence of the center on a local center manifold for three subfamilies of system (6). Our first result deals with system (6) restricted to $a_5 = a_6 = 0$.

Theorem 8—If $a_5 = a_6 = 0$, then system (6) admits a center on the local center manifold if and only if one of the following four conditions holds:

- a. $a_2 = a_4 = a_1 + a_3 = 0$
- b. $a_1 = a_4 = a_2 - a_3 = 0$
- c. $a_2 = a_3 = 2a_1 - a_4 = 0$
- d. $a_3 = 2a_2 - a_4 = 2a_1 - a_4 = 0$.

Proof: To prove that the conditions (a)–(d) are necessary, we follow the ideas described in section 2. To be more precise we computed the first five nonzero focus quantities $g_{220}, \dots, g_{660}$ and then using Gianni-Trager-Zacharias method we found the minimal associated primes of the polynomial ideal $I = \langle g_{220}, \dots, g_{660} \rangle$. It has turned out that I consist of the following five components:

$$I_1 = \langle a_2, a_4, a_1 + a_3 \rangle$$

$$I_2 = \langle a_1, a_4, a_2 - a_3 \rangle$$

$$I_3 = \langle a_2, a_3, 2a_1 - a_4 \rangle$$

$$I_4 = \langle a_3, 2a_1 - a_4, 2a_2 - a_4 \rangle$$

$$I_5 = \langle 4a_2^2 + a_4^2, a_3, a_1 + a_2 \rangle$$

Necessary conditions for the existence of a center is that all the generators in at least one of these ideals be zero. Since the parameters $a_1, \dots, a_4$ are real we get conditions (a)–(d), that correspond to the nontrivial varieties of the ideals $I_1, \dots, I_4$. Now we shall show the sufficiency.

Assume condition (a) holds. In this case system (6) is

$$\dot{u} = v \quad \dot{v} = w \quad \dot{w} = -u - v - w + a_1 u^2 - a_1 w^2,$$

Applying the change of coordinates $u \rightarrow u + w, \bar{v} \rightarrow v + w, \bar{w} \rightarrow w$ and immediately dropping the bar, i.e. transforming this system into its standard form (5) we get

$$\dot{u} = -v - 2a_1 u w, \quad \dot{v} = u + 2a_1 u w, \quad \dot{w} = -w + 2a_1 u w. \quad (16)$$

It is clear that the plane $w = 0$ is invariant and, in fact, is a center manifold for this system. Moreover, the restriction of the associated vector field to $w = 0$ gives rise to a linear center.

Assume condition (b) holds. Writing the system in the standard form, we get

$$\begin{aligned} \dot{u} &= -v + \frac{a_2}{2} [(v-w)^2 + (u+w)^2], \\ \dot{v} &= u + \frac{a_2}{2} [(v-w)^2 + (u+w)^2], \\ \dot{w} &= -w + \frac{a_2}{2} [(v-w)^2 + (u+w)^2]. \end{aligned} \quad (17)$$

It is easy to check by direct computation that

$$\mathcal{F} := 2w - a_2 [(v-w)^2 + (u+w)^2] = 0$$

is an invariant algebraic surface for this system having the cofactor $\mathcal{K} = -1 + 2a_2(u + w)$. Since $\mathcal{F} = 0$ is tangent to $w = 0$, it is a center manifold for this system. To determine the dynamics on it first we use the change of coordinates $u \rightarrow V - w, v \rightarrow U + w$, that transforms (17) into

$$\dot{U} = V, \quad \dot{V} = U(-1 + a_2 U) + a_2 V^2 - 2w, \quad \dot{W} = \frac{a_2}{2}(U^2 + V^2) - W. \quad (18)$$

The center manifold $\mathcal{F} = 0$ in the new variables writes $\bar{\mathcal{F}} = -a_2(U^2 + V^2) + 2w = 0$. It is easy to see that the restriction of system (18) to $\bar{\mathcal{F}} = 0$ is a linear center.

Assume condition (c) holds. Thus system (6) is

$$\dot{u} = v, \quad \dot{v} = w, \quad \dot{w} = -u - v - w + a_1 u(u + 2v), \quad (19)$$

where $a_1 \in \mathbb{R}$. In this case we found the following invariant algebraic surface

$$\mathcal{F} := -u - w + a_1 u^2 = 0.$$

Using a similar technique as in the previous case, we prove that $\mathcal{F} = 0$ is, in reality, a center manifold for our system. Thus, to study the dynamics on it we solve $\mathcal{F} = 0$ for w and substitute into (19) and we get

$$\dot{u} = v, \quad \dot{v} = -u + a_1 u^2. \quad (20)$$

Since this is a Hamiltonian system it admits a center. In fact, for $a_1 \neq 0$ it is a non-global center, whose period annulus is surrounded by a homoclinic orbit.

Assume condition (d) holds. Now we study the system

$$\dot{u} = v, \quad \dot{v} = w, \quad \dot{w} = -u - v - w + a_1 (u + v)^2,$$

where $a_1 \in \mathbb{R}$. In the standard form it is

$$\dot{u} = -v + \frac{a_1}{2}(u - v)^2, \quad \dot{v} = u + \frac{a_1}{2}(u - v)^2, \quad \dot{w} = -w + \frac{a_1}{2}(u - v)^2.$$

Note that by Theorem 2 it is enough to show that it admits a local analytic first integral at the origin. Since the first two components are decoupled from the third we only need to show that

$$\dot{u} = -v + \frac{a_1}{2}(u - v)^2, \quad \dot{v} = u + \frac{a_1}{2}(u - v)^2,$$

admits a local analytic first integral $\mathcal{H}(u, v)$. Since it is a non-degenerate planar divergent free vector field, it admits a center at the origin or equivalently there exists an analytic first integral defined at the origin.

The following theorem is our main result in the case $a_4 = a_6 = 0$.

Theorem 9—If $a_4 = a_6 = 0$, then system (6) admits a center on the local center manifold if and only if one of the following three conditions holds

- a. $2a_2 - 2a_3 + a_5 = 2a_1 - a_5 = 0$
- b. $a_2 = a_1 + a_3 - a_5 = 0$
- c. $a_3 = a_2 - a_5 = 2a_1 - a_5 = 0$.

Proof: First we shall show that the conditions (a)–(c) are necessary. Computations show that in the case $a_4 = a_6 = 0$ the ideal $I = \langle g_{220}, \dots, g_{660} \rangle$ consists of three components:

$$I_1 = \langle 2a_2 - 2a_3 + a_5, 2a_1 - a_5 \rangle$$

$$I_2 = \langle a_2, a_1 + a_3 - a_5 \rangle$$

$$I_3 = \langle a_3, a_2 - a_5, 2a_1 - a_5 \rangle.$$

Now we shall prove sufficiency.

Assume condition (a) holds. Thus we consider the system

$$\dot{u} = v, \quad \dot{v} = w, \quad \dot{w} = -u - v - w + a_1(u+w)^2 + a_2(v^2 + w^2), \quad (21)$$

where $a_1, a_2 \in \mathbb{R}$. We found the following invariant algebraic surface

$$\mathcal{F} = \mathcal{F}(u, v, w) := -w + u(a_1 u - 1) + a_2 v^2 + 2a_1 u w + (a_1 + a_2)w^2 = 0.$$

Since the cofactor $\mathcal{K} = -1 + 2a_2 w + 2a_1(u + w)$ satisfies $\mathcal{K} = \text{div } \mathfrak{X}$, where $\mathfrak{X}$ is the vector field associated to (21), we deduce that $\mathcal{F}$ is an inverse Jacobi multiplier. Moreover,

$$\mathcal{F}(v, v, v) = -2v + 2(2a_1 + a_2)v^2,$$

and by Proposition 7 it admits a center on the local center manifold.

Assume condition (b) holds. Thus we consider the system

$$\dot{u} = v, \quad \dot{v} = w, \quad \dot{w} = -u - v - w + (u + w)(a_1 u + a_3 w),$$

where $a_1, a_3 \in \mathbb{R}$. Writing it in the standard form we get

$$\dot{u} = -v + g(u, v, w), \quad \dot{v} = w + g(u, v, w), \quad \dot{w} = -w + g(u, v, w),$$

where $g(u, v, w) = w[a_1(-u + w) + a_3(u + w)]$. It is clear that $w = 0$ is the center manifold, on which the system has a center.

Assume condition (c) holds. In this case we study the system

$$\dot{u} = v, \quad \dot{v} = w, \quad \dot{w} = -u - v - w + a_1(u^2 + 2v^2 + 2uw),$$

where $a_1 \in \mathbb{R}$. We were able to find the following polynomial first integral of degree four

$$\mathcal{H} = (u - a_1 u^2 + v)^2 + (v - 2a_1 uv + w)^2.$$

Thus by Theorem 2 the system admits center on the local center manifold.

The following theorem is our main result in the case $a_4 = a_5 = 0$.

Theorem 10—If $a_4 = a_5 = 0$, then system (6) admits a center on the local center manifold if and only if one of the following four conditions holds

- a. $a_6 = a_2 = a_1 + a_3 = 0$
- b. $a_6 = a_2 - a_3 = a_1 = 0$
- c. $2a_3 - a_6 = 2a_2 - a_6 = a_1 = 0$
- d. $a_3 = 2a_2 - a_6 = a_1 = 0$.

Proof: First we shall show that the conditions (a)–(d) are necessary. Computations (again using e.g. Gianni-Trager-Zacharias method), show that the ideal $I = \langle g_{220}, \dots, g_{550} \rangle$ consists of five components:

$$I_1 = \langle a_6, a_2, a_1 + a_3 \rangle$$

$$I_2 = \langle a_6, a_2 - a_3, a_1 \rangle$$

$$I_3 = \langle 4a_3^2 + a_6^2, a_2 + a_3, a_1 \rangle$$

$$I_4 = \langle 2a_3 - a_6, 2a_2 - a_6, a_1 \rangle$$

$$I_5 = \langle a_3, 2a_2 - a_6, a_1 \rangle$$

Now we shall prove the sufficiency.

Assume condition (a) holds. Thus we consider

$$\dot{u} = v, \quad \dot{v} = w, \quad \dot{w} = -u - v - w + a_1(u^2 - w^2).$$

The standard form (5) in this case

$$\dot{u} = -v - 2a_1uw, \quad \dot{v} = u - 2a_1uw, \quad \dot{w} = -w + 2a_1uw.$$

It is clear that $w = 0$ is an invariant plane for this system as well as the center manifold on which the system has a global center.

Assume condition (b) holds. We get the same system as in Proposition 8(b) and it has been proved that it admits a center.

Assume condition (c) holds. Thus in standard form the system is

$$\dot{u} = -v + \frac{a_2}{2}(u+v)^2, \quad \dot{v} = u + \frac{a_2}{2}(u+v)^2, \quad \dot{w} = -w + \frac{a_2}{2}(u+v)^2,$$

where $a_2 \in \mathbb{R}$. Since the first two equations are decoupled, it is enough to show that they admit a local analytic first integral at the origin. Thus we consider the following system

$$\dot{u} = -v + \frac{a_2}{2}(u+v)^2, \quad \dot{v} = u + \frac{a_2}{2}(u+v)^2. \quad (22)$$

Using the notation of Theorem 5, we get that $a = c = a_2/2$, $b = d = -a_2/2$, $A = a_2$ and $B = -2a_2$. By inspection we check that condition II of Theorem 5 is fulfilled, and therefore system (22) admits a center at the origin. Thus by Poincaré Center Theorem there exists a local analytic first integral $\mathcal{H}(u, v)$ of system (22).

Assume condition (d) holds. Thus we consider the system

$$\dot{u} = v, \quad \dot{v} = w, \quad \dot{w} = -u - v - w + a_2 v(v + 2w),$$

where $a_2 \in \mathbb{R}$. It is a straightforward computation to check that it admits the following invariant algebraic surface

$$\mathcal{F} := u - a_2 v^2 + w = 0.$$

It is easy to check that it is also a center manifold for this system. Thus solving $\mathcal{F} = 0$ for w , we can consider the dynamics restricted to the system manifold $u \dot{=} v$, $v \dot{=} -u + a_2 v^2$. Since the system is reversible, it admits a center.

Proof of Theorem 1: For $(a_0, a_1, b_0, b_1) \in \mathcal{L}_2$ system (8) is

$$\dot{u} = v, \quad \dot{v} = w, \quad \dot{w} = -u(1+u) - (b_0 + 2b_0 u)v - w/b_0, \quad (23)$$

where $b_0 \in \mathbb{R}$. It is a straightforward computation to check that

$$\mathcal{F} := b_0 u(1+u) + w = 0$$

is an invariant surface for the flow associated with system (23). We claim that it is, in fact, the center manifold for this system. By the change of variables

$$u \rightarrow -(U/b_0) + b_0^2 W, \quad v \rightarrow V/\sqrt{b_0} - b_0 W, \quad w \rightarrow U + W,$$

we transform system (23), so that the center eigenspace of its linear part coincides with the plane $w = 0$. The function $\mathcal{F}$, in the new variables (U, V, W) becomes

$$\overline{\mathcal{F}} = \mathcal{F}(U, V, W) = (1 + b_0^3)W + (U - b_0^3 W)^2 / b_0.$$

Now it is clear that $W = 0$ is the tangent plane to $\overline{\mathcal{F}} = 0$ at $(0, 0, 0)$, which concludes the prove of the claim.

The knowledge of the analytic expression of the center manifold (i.e. $\mathcal{F} = 0$), permits us to write explicitly the dynamics of system (23) restricted it. Substituting $z = -b_0u(1 + u)$ in the first two equations of (23) we get

$$\dot{u} = v \quad \dot{v} = -b_0u - b_0u^2.$$

This is a nondegenerate Hamiltonian system with the Hamiltonian function

$$\mathcal{H} = \frac{1}{2}b_0v^2 + \frac{1}{3}u^3 + v^2.$$

Therefore it admits a center at the origin.

We proved that the three dimensional system (23) admits a center on the local center manifold about the origin. In Figure 3 we plot the center manifold $\mathcal{F} = 0$ and three trajectories of system (23) for $b_0 = 1$.

4. Numerical simulations and concluding remarks

We can illustrate Theorem 8 by performing numerical simulations. We consider the following system

$$\dot{u} = v, \quad \dot{v} = w, \quad \dot{w} = -u - v - w + a_1\sigma u(u + 2v), \quad (24)$$

where $a_1, \sigma \in \mathbb{R}$. For $\sigma = 1$ we are in the case of Theorem 8(c). For $a_1 = 0.2$ and $\sigma = 1$ in Figure 1(a) we plot the center manifold $\mathcal{F} := -u - w + a_1u^2 = 0$ and three orbits of system (24). The local center manifold is attracting, on which as predicted by the Theorem 8 we have a center. The origin is a stable but *not* asymptotically stable singular point. On the other hand if $\sigma = 1$, we expect a focus on the center manifold. In Figure 1(b) we plot the center manifold and three orbits of system (24) for $\sigma = 1.4$. Again, as expected, the center has been “destroyed” and we have a focus. In this case the origin is asymptotically stable.

Similarly, we can illustrate Theorem 9 by considering the system

$$\dot{u} = v, \quad \dot{v} = w, \quad \dot{w} = -u - v - w + a_1(\sigma u + w)^2 + a_2(v^2 + w^2), \quad (25)$$

where $a_1, a_2, \sigma \in \mathbb{R}$. Note that for $\sigma = 1$ we are in the case of Theorem 9(a). Setting $a_1 = 0.1$, $a_2 = 0.3$ and $\sigma = 1$ in Figure 2(a) we plot the center manifold $\mathcal{F} := -w - u + a_1u^2 + a_2v^2 + 2a_1uw + (a_1 + a_2)w^2 = 0$ and three orbits. As predicted by Theorem 9 the local center manifold, on which we have a center. Now we set $\sigma = 1.3$, and in Figure 2(b) we plot the orbits of system (25) for the same values of parameters $a_1 = 0.1$, $a_2 = 0.3$. As expected, now we have focus.

Looking at the common features of systems (9) corresponding to the same irreducible variety helps to identify and classify different “mechanisms” for creating a center on the local center manifold. For restricted system (6) (see Theorem 8, 9 and 10) we have identified the following common features of systems belonging to the same irreducible center variety:

- Existence of a polynomial first integral.

- Existence of an algebraic center manifold of degree 1 or 2.
- Decoupling of the recurrent part, i.e. for system (9) the property $P(x, y, z) = P(x, y)$ and $Q(x, y, z) = Q(x, y)$.

Can the unrestricted system (6) have irreducible components of the center variety whose nature is different then the ones described above?

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Figure 1. The center manifold and three trajectories of system (24) for $a_1 = 0.2$ and $\sigma = 1$ (left) $\sigma = 1.4$ (right).

Figure 2. The center manifold and three trajectories of system (25) for $a_1 = 0.1$, $a_2 = 0.3$ and $\sigma = 1$ (left); $\sigma = 1.4$ (right).

Figure 3.
The center manifold $\mathcal{F}: = b_0 u(1+u) + w = 0$ and three trajectories of system (23), for $b_0 = 1$.