Dynamic behavior analysis of fractional-order Hindmarsh–Rose neuronal model

Dong Jun · Zhang Guang-jun · Xie Yong ·
Yao Hong · Wang Jue

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Abstract Previous experimental work has shown that the firing rate of multiple time-scales of adaptation for single rat neocortical pyramidal neurons is consistent with fractional-order differentiation, and the fractional-order neuronal models depict the firing rate of neurons more verifiably than other models do. For this reason, the dynamic characteristics of the fractional-order Hindmarsh–Rose (HR) neuronal model were here investigated. The results showed several obvious differences in dynamic characteristic between the fractional-order HR neuronal model and an integer-ordered model. First, the fractional-order HR neuronal model displayed different firing modes (chaotic firing and periodic firing) as the fractional order changed when other parameters remained the same as in the integer-order model. However, only one firing mode is displayed in integer-order models with the same parameters. The fractional order is the key to determining the firing mode. Second, the Hopf bifurcation point of this fractional-order model, from the resting state to periodic firing, was found to be larger than that of the integer-order

model. Third, for the state of periodically firing of fractional-order and integer-order HR neuron model, the firing frequency of the fractional-order neuronal model was greater than that of the integer-order model, and when the fractional order of the model decreased, the firing frequency increased.

Keywords Fractional-order · Hopf bifurcation · HR model · Transition of firing mode

Introduction

Fractional-order calculus and integer-order calculus share a similar history. Integer-order calculus is a special type of fractional-order calculus, and integer-order chaotic systems are idealized to actual dynamic systems (Ivo Petráš 2011; Magin 2004). Recently, fractional-order differential equations, especially fractional-order dynamic equations, have drawn considerable interest from researchers. One reason for this is the self-development of the theory of fractional-order calculus. Another is that the application of fractional-order differential equations on various subjects is expanding. Fractional-order differential equations are applied to many fields, including physics (Ahmad and Sprott 2003), chaos theory (Mandelbrot 1967), signal processes (Podlubny 1999), and control systems (Yong and Yong 2010; Liu 2010). Research shows that all physical phenomena occur in the form of fractional-order differential equations, specifically integer-order differential equations constitute a special case of fractional-order differential equations, and that fractional-order differential equations are a generalization of integer-order differential equations. Fractional-order differential equations are developing into a research hotspot because of their universal significance (Ahmad and

D. Jun · Z. Guang-jun (✉) · Y. Hong
College of Science, Air Force Engineering University,
Xi'an 710051, China
e-mail: zhanggj3@126.com

D. Jun
The First Aeronautical Institute of Air Force,
Xinyang 464000, Henan, China

Z. Guang-jun · W. Jue
School of Life Science and technology, Xi'an Jiao tong
University, Xi'an 710049, China

X. Yong
School of Aerospace, Xi'an Jiao tong University,
Xi'an 710049, China

Sprott 2003; Mandelbrot 1967; Podlubny 1999; Yong and Yong 2010; Liu 2010).

With the consolidation of nonlinear science and neuroscience, more and more attention has been paid to the intricate spiking rhythms in neuronal models. Previous reports have revealed the mechanism for different modes of neuronal bursting or spiking from the point of view of dynamics. These investigators discovered nonlinear dynamic characteristics involved in bursting or spiking modes of neurons (Yong and Yong 2010; Liu 2010; Guang-jun and Jian-xue 2005; Perc 2005; Perc and Marhl 2005; Yang et al. 2006; Sun et al. 2011; Perc and Marhl 2007; Wang et al. 2011; Duan et al. 2013; Zhang et al. 2013; Wang et al. 2013; Yamada and Kashimori 2013). Yang et al. evaluated various bursting and spiking modes through the analysis of fast and slow dynamics and bifurcation, and they observed a series of ISI bifurcation modes (Yang et al. 2006). Most previous theoretical studies on the neuronal firing rhythm and modes have been in the form of integer-order neuronal models. Many useful conclusions consistent with the results of biological experiments can be drawn (Guang-jun and Jian-xue 2005; Perc 2005; Perc and Marhl 2005; Yang et al. 2006; Sun et al. 2011; Perc and Marhl 2007; Wang et al. 2011; Duan et al. 2013; Zhang et al. 2013; Wang et al. 2013; Yamada and Kashimori 2013). Brian et al. (REF) analyzed the dynamics of firing rate with a range of stimulus dynamics in their neurophysiology experiment (Lundstrom et al. 2008). Their results showed that single rat neocortical pyramidal neurons can adapt along a time-scale that depends on the changes in stimulus statistics. This multiple time-scale adaptation is consistent with fractional-order differentiation, such that the neuron's firing rate is a fractional-order derivative of slowly varying stimulus parameters (Lundstrom et al. 2008). Fractional-order differentiation is a fundamental and general computation that can contribute to efficient information processing, stimulus anticipation, and assessment of frequency-independent phase shifts of oscillatory neuronal firing. In this way, fractional-order differentiation has advantages over integer-order differentiation in the depiction of the firing characteristics of some types of neurons. It can usually be used to depict the neurons' dynamic characteristics more correctly than other types of differentiation can. For this reason, investigation of these dynamic characteristics of the fractional-order neuronal model should be performed. Although the firing characteristics of the integer-order neuronal model have been studied extensively (Guang-jun and Jian-xue 2005; Perc 2005; Perc and Marhl 2005; Yang et al. 2006; Sun et al. 2011; Perc and Marhl 2007; Wang et al. 2011; Duan et al. 2013; Zhang et al. 2013; Wang et al. 2013; Yamada and Kashimori 2013), there have been few reports on fractional-order neuronal models. Xie and Liu analyzed the dynamic characteristics of the fractional-order FitzHugh-Nagumo neuronal model (Yong and Yong 2010),

but their study only dealt with the case of a change in external current intensity and not with the transition of firing mode with a change in the order of the fractional-order model. Because the firing rate of multiple time-scales of adaptation for single rat neocortical pyramidal neurons is consistent with fractional-order differentiation (Yamada and Kashimori 2013), and the inter-order HR neuronal model is used to depict bursting behaviors successfully, it is more significant to investigate the dynamic characteristic of the fractional-order HR neuronal model than the fractional-order FHN neuronal model. For this reason, in the present study, the dynamic characteristics of the fractional-order HR neuronal model were studied with respect to changes in the intensity of the external current, where the order of the model was a constant fraction. The dynamic characteristics of the fractional-order HR neuronal model with changes in the order of fractional-order model are researched, where the external current intensity is a constant. In this way, the transitions in the firing mode with changes in the order of the fractional-order model were analyzed. The later work on the dynamic characteristics of the fractional-order HR neuronal model with changes in the order of fractional-order model is markedly different from the analysis of the dynamic characteristic of the fractional-order FHN neuronal model by Xie and Liu.

From this perspective, the present study was organized as follows. First, fractional-order calculus was evaluated; then the Hopf bifurcation behavior of the fractional-order HR neuronal model with external current intensity serving as the bifurcation parameter was evaluated, and the difference between it and an integer-order HR neuronal model was determined; third, the firing frequency of the fractional-order HR neuronal model and the difference between it and an integer-order HR neuronal model was determined; fourth, the bifurcation characteristics of the fractional-order HR neuronal model incorporating the order of the fractional-order neuronal model as bifurcation parameter were determined, and the transition in the firing mode with changes in the order of the fractional-order model was assessed. Several practical conclusions were drawn.

Fractional-order calculus and the methods of approximate calculation

Definition of fractional-order calculus

There are several definitions of fractional-order calculus (Ivo Petráš 2011; Magin 2004). Regarding practical applications, two definitions of fractional-order calculus are usually used: the Riemann–Liouville definition and the Caputo definition. Because the physical meaning of the

Fig. 1 Stable area of the q -order linear system

initial condition of the Caputo definition was clear, it was used in this paper (Ivo Petráš 2011; Magin 2004).

The q -order differentiation of mono-variant function is defined as follows:

$${}_0^C D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q-n+1}} d\tau \quad (n-1 < q \leq n) \quad (1)$$

Here n is the least integer that is not less than q ($n-1 < q \leq n$); 0 and t are the upper and lower limits of the integral, respectively. $\Gamma(n-q)$ is the gamma function. $f^{(n)}(\tau)$ is the n th derivative of the function $f(\tau)$.

Properties of fractional differentiation

Ahmed et al. presented a sufficient and necessary condition for the fractional-order linear system to be stable (Ahmed et al. 2007).

Lemma considers the linear autonomous system (Ahmed et al. 2007).

$$D_*^q X = AX \quad X(0) = X_0 \quad (2)$$

Here $X \in R^n$ ($n \in N$), $A \in R^{n \times n}$. 1. System (2) is asymptotically stable when and only when $|\arg(\lambda)| > \frac{q\pi}{2}$ works for an arbitrary eigenvalue λ . 2. System (2) is stable when and only when $|\arg(\lambda)| \geq \frac{q\pi}{2}$ works for an arbitrary eigenvalue λ .

The stable area of the q -order linear system is shown in Fig. 1. If all eigenvalues of the Jacobia matrix of a system in equilibrium are outside of the fan-shaped area, then the fractional-order differential system is stable. Figure 1 shows that the stable area of the fractional-order linear system is larger than that of the inter-order linear system when $0 < q < 1$, but the stable area of the fractional-order linear system is smaller than that of the inter-order linear system when $q > 1$ (Ivo Petráš 2011).

Application of the approximate computation method to the fractional-order differential equation

There are usually two kinds of solutions to fractional-order differential equations, analytical and numerical. Generally, the analytical solution has considerable limitations, and only simple differential equations can be solved using it. For this reason, numerical simulation is widely adopted in practical applications. The numerical simulation solutions to differential equations of fractional-order include frequency-domain approximation solutions (Charef et al. 1992), time discrete predictor–corrector algorithms (Diethelm et al. 2002; Li and Peng 2004), and Adomian decomposition methods (Adomian 1990). The frequency domain solution sees widespread use in engineering, but this method has considerable limitations because it can disturb the qualitative property of system. Tavazoei et al. compared the approximate system in the frequency domain to the original system and reported the shortcomings of the frequency-domain approximation solution in detail (Tavazoei and Haeri 2007). Adomian decomposition methods (ADM), which were first presented by Adomian in 1990, are used to solve nonlinear differential equations (Adomian 1990). Recently, they have seen increasing use in the solution of fractional-order differential equations. Liu compared the three algorithms mentioned above and concluded that the predictor–corrector algorithm is time-discrete; its computational precision is higher than the frequency-domain approximation solution. The ADM algorithm is an excellent means of solving fractional-order differential equations. Its computational efficiency is greater than that of the predictor–corrector algorithm, and its computational precision is equivalent to that of the predictor–corrector algorithm (Liu 2010). The predictor–corrector algorithm and the ADM algorithm were used in the present paper because of the moderate demand for computational efficiency and the relatively high demand for computational precision.

Bifurcation behavior of the fractional-order HR neuronal model

The HR neuronal model qualitatively depicted the firing behavior of neurons using a group of simple differential equations. Using the Caputo definition of fractional-order differentials and the definition of the fractional-order system (Ivo Petráš 2011; Magin 2004; Ahmed et al. 2007), the HR neuronal model of fractional-order was found to be as follows:

Fig. 2 HR neuronal model response over time when $q = 1$ and $I = 1.2$

Fig. 3 HR neuronal model response over time when $q = 1$ and $I = 1.3$

$$\begin{cases} D_t^q x = y - ax^3 + bx^2 - z + I \\ D_t^q y = c - dx^2 - y \\ D_t^q z = r[S(x - \bar{x}) - z] \end{cases} \quad (3)$$

Here, x is the membrane action potential, y is a recovery variable, z is a slow adaptation current, D_t^q is the differential operator defined by Caputo, $0 < q \leq 1$ is the order of fractional-order. The order q in this paper is a variable. And a, b, c, d, r, S , and $\bar{x}$ are system parameters. Throughout the paper, these values were fixed as $a = 1.0$, $b = 3.0$, $c = 1.0$, $d = 5.0$, $S = 4.0$, $r = 0.0006$, $\bar{x} = 1.56$. I is the external current intensity, and it is a variable in this paper. The variables in this HR neuronal model were all nondimensional.

When $q = 1$, the fractional-order HR neuronal model degenerates to an ordinary HR neuronal model of integer-order (hereafter, the model of integer-order refers to the case of $q = 1$ except where otherwise specified). The Hopf bifurcation in the integer-order HR neuronal model occurs when $I = 1.2$, as indicated by both theoretical analysis and numerical calculations, and the HR neuronal model enters a periodic firing mode from the resting state (Hong-jie and Jian-hua 2005). When $I < 1.2$ the integer-order HR neuronal model is in a state of rest, and when $I \geq 1.2$ the HR neuronal model is in a state of firing. The responses of the integer-order HR neuronal model in two cases of the value

of I are shown in Figs. 2 and 3. The HR neuronal model of integer-order is in periodic firing mode when $q = 1.0$, $I \geq 1.2$, and the other parameters of Eq. (3) are chosen as the normal value mentioned above, and $I = 1.2$ is the Hopf bifurcation point of integer-order HR neuronal model (Hong-jie and Jian-hua 2005).

Next, the bifurcation characteristics of the fractional-order HR neuronal model were explored and compared to those of the integer-order HR neuronal model order. Assuming that the equilibrium (x^*, y^*, z^*) of the HR neuronal model satisfies the Eq. (4),

$$\begin{cases} y^* - ax^{*3} + bx^{*2} - z^* + I = 0 \\ c - dx^{*2} - y^* = 0 \\ r[S(x^* - \bar{x}) - z^*] = 0 \end{cases} \quad (4)$$

The Jacobia matrix of a system (3) in equilibrium (x^*, y^*, z^*) is as follows:

$$J = \begin{pmatrix} -3ax^{*2} + 2bx^* & 1 & -1 \\ -2dx^* & -1 & 0 \\ rS & 0 & -r \end{pmatrix} \quad (5)$$

When the other parameters of Eq. (3) are set at normal values, as given above, and when $I = 1.3$, the equilibrium of system (3) is as follows: $(-1.2799, -7.1908, 1.1204)$, according to Eq. (4). In this way, the eigenvalues of the Jacobia matrix in the equilibrium $(-1.2799, -7.1908, 1.1204)$ are $\lambda_1 = -13.6073$, $\lambda_2 = 0.0037 + 0.0407i$, and $\lambda_3 = 0.0037 - 0.0407i$, respectively. If $|\arg(\lambda_{2,3})| = \frac{q_{cr}\pi}{2}$, then $q_{cr} = 0.9424$. q_{cr} is the critical value of order when the fractional-order HR neuronal model fires periodically. That is, when $q < q_{cr} = 0.9424$, the equilibrium $(-1.2799, -7.1908, 1.1204)$ is stable, and the system (3) is in a resting state.

Next, the cases of the order $q = 1.0, 0.9$ and 0.5 are delineated as follows:

When $q = 1.0$ and $|\arg(\lambda_{2,3})| = 1.4802 < 1 \times \frac{\pi}{2} = 1.5708$, then the equilibrium $(-1.2799, -7.1908, 1.1204)$ is unstable; That is, the system (3) fires periodically.

When $q = 0.9$ and $|\arg(\lambda_{2,3})| = 1.4802 > 0.9 \times \frac{\pi}{2} = 1.4137$, the equilibrium $(-1.2799, -7.1908, 1.1204)$ is stable; That is, the system (3) is in a resting state.

When $q = 0.5$ and $|\arg(\lambda_{2,3})| = 1.4802 > 0.5 \times \frac{\pi}{2} = 0.7857$, the equilibrium $(-1.2799, -7.1908, 1.1204)$ is stable. That is, the system (3) is in a resting state.

Hence, when $I = 1.3$ the HR neuronal model of integer-order is in a state of firing periodically; whereas, the HR neuronal model of fractional-order when the orders are $0.9, 0.5$, respectively, is in a state of rest. In this way, the intensity of the external current of the fractional-order HR neuronal model when Hopf bifurcation occurs is greater than that of the integer-order HR neuronal model. Using

Fig. 4 HR neuronal model response over time when $q = 0.9$ and $I = 1.3$

Fig. 5 HR neuronal model response over time when $q = 0.5$ and $I = 1.3$

the predictor–corrector algorithm, the HR neuronal model in the three cases of the orders of 1.0, 0.9, and 0.5 can be calculated numerically when $I = 1.3$ (Figs. 3, 4, 5).

As shown in Fig. 3, when $I = 1.3$ and the other parameters are the set at the normal values given above, the HR neuronal model of integer-order is in a state of periodic oscillation. The phase diagram of the system is in this case a stable limit cycle, and the equilibrium $(-1.2799, -7.1908, 1.1204)$ is unstable. As shown in Figs. 4 and 5, when $I = 1.3$ and the other parameters are set at the normal values mentioned above, the fractional-order HR neuronal model of the orders, which are 0.9 and 0.5, respectively, are all in a state of rest. The equilibrium $(-1.2799, -7.1908, 1.1204)$ of the system in this case is stable. Specifically, the HR neuronal model is in this case in a state of rest. Therefore, when $I = 1.3$ the Hopf bifurcation has still not taken place in fractional-order HR neuronal model, but when $I = 1.2$ the Hopf bifurcation has already occurred in integer-order HR neuronal model.

As shown in the numerical calculations in Figs. 2, 3, 4, and 5, the external current intensity of the fractional-order HR neuronal model when Hopf bifurcation takes place is larger than that of the integer-order HR neuronal model. That is, the value of the bifurcation parameter of the fractional-order system when Hopf bifurcation takes place is larger than that of the bifurcation parameter of the

Fig. 6 Integer-order HR neuronal model over time when $I = 2.0$ and $q = 1.0$

integer-order system. These numerical results are consistent with the results of quantitative analysis.

Firing frequency of the fractional-order neuronal model

According to the results of the Sect. 3, the figures show that the external current intensity of the fractional-order HR neuronal model when Hopf bifurcation occurs is larger than that of the integer-order HR neuronal model. However, using the predictor–corrector algorithm, the results of numerical simulation also show that the periodically firing frequency of the fractional-order HR neuronal model is greater than that of the integer-order HR neuronal model when the two systems have the same parameter values. Therefore, the periodically firing frequency of the fractional-order HR neuronal model is investigated. When $I = 2.0$ and the other parameters are set as the normal values given above, the membrane potential of the fractional-order HR neuronal models over time gives orders of 1.0, 0.95, 0.9, and 0.85, respectively (Figs. 6, 7, 8, 9). In this condition, the bifurcation characteristic of ISI (integer-spike interval) of the fractional-order HR neuronal model is calculated numerically using the order q as bifurcation parameter by predictor–corrector algorithm (Fig. 10). Accordingly, the periodically firing frequency of the fractional-order HR neuronal model in the four figures (Figs. 6, 7, 8, 9) are respectively 0.1316, 0.1500, 0.2129 and 0.2763. As shown in Fig. 10, the fractional-order HR neuron model is all in a state of periodically firing when $I = 2.0$ and $0.5 < q < 1$. As shown in Figs. 6, 7, 8, 9, 10, two conclusions are drawn as follows. First, for the state of periodically firing of fractional-order and integer-order HR neuron model, the firing frequency of the fractional-order neuronal model was greater than that of the integer-order model. Second, with the order of the model decreases, the periodically firing frequency of the fractional-order HR neuronal model increases markedly when the other parameters

Fig. 7 Fractional-order HR neuronal model over time when $I = 2.0$ and $q = 0.95$

Fig. 8 Fractional-order HR neuronal model over time when $I = 2.0$ and $q = 0.9$

Fig. 9 Fractional-order HR neuronal model over time when $I = 2.0$ and $q = 0.85$

are the set at the normal values given above and the external current intensity is at a certain value. That is to say under these conditions the period of firing of the fractional-order HR neuronal model decreases.

Nonlinear dynamical characteristic of the fractional-order HR neuronal model with a bifurcation parameter of a given integer order or fractional order

Peng et al. analyzed the dynamic characteristic of the integer-order HR neuronal model. The integer-order HR neuronal model was found to display chaotic bursting at

Fig. 10 Bifurcation diagram of the fractional-order HR neuronal model with a bifurcation parameter of order q when $I = 2.0$

Fig. 11 Phase diagram in plane x - y of the response of the integer-order HR neuronal model when $I = 3.0$

$I = 3.0$ and when the other parameters are the same as the values given in this paper (Hong-jie and Jian-hua 2005). The phase diagram of integer-order HR neuronal model is shown in Fig. 11. The dynamic characteristics of the fractional-order HR neuronal model for different orders when the parameters of system are the same as those reported of Peng were also determined. In order to show the nonlinear dynamical characteristics of the fractional-order HR neuronal model under these parameters, when I was fixed at 3.0 the bifurcation characteristic of ISI (integer-spike interval) of the fractional-order HR neuronal model were calculated numerically using the order q as bifurcation parameter by predictor–corrector algorithm. The bifurcation diagram is shown in Fig. 12. As shown, when $I = 3.0$ and the order is $0.55 < q < 0.66$ or $0.725 < q < 0.925$, the fractional-order HR neuronal model displays periodic bursting. When $I = 3.0$, for the different order of fractional-order of

Fig. 12 Bifurcation diagram of the fractional-order HR neuronal model with a bifurcation parameter of order q when $I = 3.0$. **a** $q = 0.85$ **b** $q = 0.95$

neuronal model, the firing mode of neuronal model is different. When $q = 0.85$, the fractional-order HR neuronal model displays periodic-5 bursting (Fig. 13a). When $q = 0.95$, the fractional-order HR neuronal model displays chaotic bursting (Fig. 13b). The dynamic characteristics of the fractional-order HR neuronal model were found to change qualitatively with the gradually changing of the system order q . That is, the change in the order of the fractional-order HR neuronal model can induce a transition in the firing mode of the neuronal model.

When the integer-order HR neuronal model was at chaotic firing; the corresponding fractional-order model with the same parameters may also be undergoing chaotic firing, but it could also be undergoing periodic firing. However, if the integer-order HR neuronal model is at periodic firing, then the firing mode of the corresponding fractional-order model using parameters other than order q is open to question. The firing mode of the fractional-order HR neuronal model was analyzed when $I = 3.5$. When $I = 3.5$, according to Eq. (4), the equilibrium of the

Fig. 14 Phase diagram of the integer-order HR neuronal model response in plane x - y

system (3) is $(-0.5421, -0.4694, 4.0715)$. In this way, the eigenvalues of the Jacobia matrix of the system (3) at equilibrium $(-0.5421, -0.4694, 4.0715)$ are $\lambda_1 = -5.3702$, $\lambda_2 = -0.2159$, and $\lambda_3 = 0.0140$. This lemma shows that for arbitrary $0 < q < 1$, the equilibrium of system (3) is unstable, and the neuronal model is in a state of firing. But if the order q of the neuronal model is different, then its firing mode is also different. For example, as shown in Fig. 14, the integer-order HR neuronal model displays periodic spiking; the bifurcation diagram of the corresponding fractional-order model ISI, with the order q as a parameter is shown in Fig. 15. As shown in Fig. 15, when $q \in [0.55, 0.6167]$, the neuronal model is at chaotic bursting; and when $q \in [0.6167, 0.6271]$ the neuronal model is at spiking in period-4; and when $q \in [0.6167, 0.6271]$ the neuronal model is at spiking in period-2; and when $q \in [0.6667, 1]$ the neuronal model is at spiking in period-1. When $I = 3.5$, for the different order of fractional order of the neuronal model, the firing mode of the neuronal model is different. When $q = 0.80$, the fractional-order HR neuronal model displays periodic-1 spiking (Fig. 16(a)); when $q = 0.65$, the fractional-order HR

Fig. 13 Phase diagram in x - z plane of bursting fractional-order HR neuronal model for the different order of fractional order when $I = 3.0$, $a = 1.0$, $b = 3.0$, $c = 1.0$, $d = 5.0$, $S = 4.0$, $r = -0.0006$, and $\bar{x} = 1.56$ **a** $q = 0.85$ **b** $q = 0.95$

Fig. 15 Bifurcation diagram of the fractional-order HR neuronal model with a bifurcation parameter of order q when $I = 3.5$

neuronal model displays periodic-2 spiking (Fig. 16b). When $q = 0.62$, the fractional-order HR neuronal model displays periodic-4 spiking (Fig. 16c). When $q = 0.59$, the fractional-order HR neuronal model displays chaotic bursting (Fig. 16d). The results shown in Fig. 15 demonstrate

that the decrease in the order q fractional-order HR neuronal model led to chaos by periodic-double bifurcation. In conclusion, by analysis of chaotic characteristics of the fractional-order HR neuronal model, the following results were collected: The order of the model is a key parameter that determines the mode of system motion, and the change of the order of the fractional-order HR neuronal model can modify the firing mode of the system when the parameters are otherwise the same. When the inter-order HR neuronal model displays periodic firing, the firing mode of the fractional-order HR neuronal model is possibly chaotic, and vice versa.

Conclusions

Fractional-order calculus provides a more cogent mathematical tool for scientific study than integer-order calculus does. The dynamic characteristic of our system depicted by fractional-order calculus is much more accurate than that depicted by integer-order calculus.

Previous experimental works have shown that the firing rate of multiple time-scale adaptations of single rat neocortical pyramidal neurons is consistent with fractional-order differentiation. Results demonstrated that the

Fig. 16 Phase diagram in x - z plane of firing fractional-order HR neuronal model for the different orders q when $I = 3.5$, $a = 1.0$, $b = 3.0$, $c = 1.0$, $d = 5.0$, $S = 4.0$, $r = -.0006$, and $\bar{x} = 1.56$ **a** $q = 0.80$ **b** $q = 0.65$ **c** $q = 0.62$ **d** $q = 0.59$

fractional-order neuronal model depicts the firing rate of neurons more accurately than other models do. For this reason, the dynamic characteristics of the fractional-order HR neuronal model were evaluated in this paper. The results of the present study showed the dynamic characteristics depicted by the fractional-order HR neuronal model to be different from those depicted in the integer-order HR neuronal model. First, the fractional-order HR neuronal model displayed a different firing mode (specifically chaotic firing and periodic firing) as the order of fractional order changed when the other parameters were the same as those of the integer-order model, but only a firing mode was displayed the integer-order model with the same parameters. The order of fractional-order of the neuronal model is a key parameter determining its firing mode. When the order of the fractional-order model changed, the firing mode of the fractional-order HR model transitioned from the chaotic firing mode to the periodic firing mode. Second, analysis of the bifurcation behavior of the fractional-order HR neuronal model provided the following results: the external current intensity of the fractional-order HR neuronal model during Hopf bifurcation was larger than that of the integer-order HR neuronal model, and the firing frequency of the fractional-order model was greater than that of the integer-order model. Firing frequency was continuously augmented in a manner commensurate with the decrease in the order of the model.

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