On the construction of nonequatorial minimal hyperspheres in $S^n(1)$ with stable cones in $\mathbb{R}^{n+1}$

(spherical Bernstein problem/local structure of singularities/equivariant differential geometry)

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ABSTRACT Within the framework of equivariant differential geometry, we outline the construction of some imbedded minimal hyperspheres of $S^n(1)$ and show that many of them have a stable cone in $\mathbb{R}^{n+1}$. The statement of these and related results are given.

Introduction

In differential geometry and geometric measure theory, the study of closed minimal submanifolds of $S^n(1)$ is closely related to that of the local structure of singularities of minimal submanifolds in the general Riemannian setting. The spherical Bernstein problem of Chern (1) asks whether a minimal imbedding of the differentiable $(n - 1)$ -sphere into $S^n(1)$ is necessarily an equator. It is equivalent to the basic problem of whether the local topology around an isolated singularity of an n -dimensional minimal hypersurface [in an arbitrary Riemannian $(n + 1)$ -space] is necessarily different from that of an n -disc. The beginning case of $n = 3$ was proved to be affirmative (2, 3) before the above problem was proposed. In two recent papers of the first author (4, 5), 18 equivariant families of infinitely many, mutually noncongruent examples of minimal imbeddings of S^{n-1} into $S^n(1)$ were constructed, with their dimensions ranging over $n = 4, 5, 6, 7, 8, 10, 12$, and 14. However, in the study of parametric variational theory, there is an important distinction between a stable singularity and an unstable singularity. Therefore, it is important to settle whether there exists a nonequatorial, imbedded, minimal hypersphere, Σ^{n-1} , in $S^n(1)$ whose cone, $C\Sigma^{n-1}$, has a stable singularity at the origin. The following is a brief outline of a method of construction of such examples.

An Outline of the Method of Construction

Step I: Following the setting of refs. 4 and 5, we use the Jacobi–Morse theory to establish the existence of many more new examples of imbedded, minimal hyperspheres of cohomogeneity 1 (cf. *Theorem 1*). Suppose that (G, Σ^{n-1}) is a cohomogeneity 1, imbedded, minimal hypersphere in $S^n(1)$. Then it is easy to see that $\dim S^n(1)/G = 2$ and $F(G, S^n(1)) = S^0$. Such equivariant geometric systems, $(G, S^n(1))$, are classified in ref. 6 and their orbital geometries are systematically analyzed in ref. 5. It is not difficult to show that the particular 18 such systems singled out in ref. 5 are the only ones that can accommodate infinitely many (distinct) imbedded, minimal hyperspheres of cohomogeneity 1. Of course, there may still exist many other equivariant systems that still accommodate at least one nonequatorial, imbedded, minimal hypersphere of cohomogeneity 1. Let $(G, S^n(1))$ be a given system; according to ref. 5, the orbit space, $S^n(1)/G$, is

a spherical lune, and it is convenient to parametrize it by $\{(r, \theta); 0 \leq r \leq \pi, 0 \leq \theta \leq \pi/d\}$ such that $ds^2 = dr^2 + \sin^2 r d\theta^2$. Then the image curve of an invariant, minimal hypersurface can be characterized by the following ordinary differential equation:

$$\frac{d\alpha}{ds} + \cos r \frac{d\theta}{ds} + \sin r \frac{d\theta}{ds} \frac{\partial}{\partial r} \ln f(r, \theta) - \frac{1}{\sin r} \frac{dr}{ds} \frac{\partial}{\partial \theta} \ln f(r, \theta) = 0, \quad [1]$$

where $f(r, \theta)$ is the volume function of principal orbits and α is the angle between the radial direction and the curve. The function $f(r, \theta)$ has a unique maximal point $(\pi/2, \theta_0)$, and the line $\theta = \theta_0$ is an obvious solution curve of Eq. 1. As it was clearly demonstrated in refs. 4 and 5, the intersection pattern of a given solution curve, γ , of Eq. 1 with the line $\theta = \theta_0$ plays an important role in the construction of certain desired global solution curves of Eq. 1. The crucial starting point of our construction of new examples is to determine the distribution of conjugate points along the line $\theta = \theta_0$, thus establishing the existence of some nearby solution curves with a certain desired intersection pattern.

Step II: Geometrically, it is not difficult to see that a higher dimensional, closed, minimal hypersurface, Σ^{n-1} , in $S^n(1)$ has a much better chance of having a stable cone, $C\Sigma^{n-1}$, in $\mathbb{R}^{n+1}$ than a lower dimensional one does. Analytically, it is rather straightforward to deduce the following sufficient condition of stability from the work of Simons (7):

LEMMA 1: Let Σ^{n-1} be a closed, minimal hypersurface in $S^n(1)$ and A be the second fundamental form of Σ^{n-1} . If $\|A\|^2$ is uniformly bounded by $(n - 2/2)^2$, then $C\Sigma^{n-1}$ is a stable cone in $\mathbb{R}^{n+1}$.

Since the proofs of existence of those specific global solution curves of Eq. 1 in *Theorems 1* and *4* are explicitly constructive, they are readily accessible to numerical computations, up to a high degree of accuracy, with the help of computers, thus enabling us to establish a uniform bound of $\|A\|^2$ for each new example.

Statement of Results

We announce here the following results, along the direction of the stable spherical Bernstein problem, obtained by the above method of equivariant differential geometry.

THEOREM 1: Let $(G, S^n(1))$ be one of those equivariant geometric systems listed in Table 1 of ref. 5. If $d = 3$ or 4 or in the case $(SO(2) \times SO(6), S^8(1))$, then there exists at least one nonequatorial, imbedded, invariant, minimal hypersphere in $(G, S^n(1))$.

Remark: Per Tomter also obtained a proof of the above theorem for the case $(SO(2) \times SO(m), S^{2m}(1))$ along a different approach.

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THEOREM 2: In the following cases, namely,

$d = 3$, $G = \text{Sp}(3)$ or F_4 ,

$$d = 4, G = \begin{cases} \text{U}(5), \text{U}(1) \times \text{Spin}(10), \\ \text{SO}(2) \times \text{SO}(m); & 12 \leq m \leq 50 \\ & \text{or } m = 100, 200, \\ \text{S}(\text{U}(2) \times \text{U}(m)); & 6 \leq m \leq 20 \\ & \text{or } m = 50, 100, \\ \text{Sp}(2) \times \text{Sp}(m); & 3 \leq m \leq 20 \\ & \text{or } m = 25, 50, \end{cases}$$

there exists a nonequatorial, imbedded, invariant, minimal hypersphere Σ^{n-1} , whose cone, $C\Sigma^{n-1}$, is stable.

Remark: We list only those cases that have been carefully checked by the above mentioned method of numerical computation. Actually, the evidences clearly indicate that all of those cases with $d = 4$, $m \geq 20$ should also have stable cones.

THEOREM 3: For each equivariant system $(G, S^n(1))$ listed in Table 1 of ref. 5, the corresponding ordinary differential Eq. 1 has global solution curves γ_i , $i = 1, 2$, which start and end at the boundary β_i [$\beta_1 = \{(r, 0); 0 < r < \pi\}$ and $\beta_2 = \{(r,$

$\pi/d), 0 < r < \pi\}$] and have exactly two points of intersection with the line $\theta = \theta_0$.

Remark: In the case $d = 2$ —i.e., $(O(p) \times O(q), S^{p+q}(1))$ —the corresponding minimal hypersurfaces are exactly the generalized Clifford tori of the types $S^p \times S^{q-1}$ and $S^{p-1} \times S^q$. Therefore, it is in the cases of $d = 3, 4$, and 6 that the existence of such examples needs a nontrivial proof. One may consider the inverse images of γ_i , $i = 1, 2$, as new types of generalized Clifford tori. It is also interesting to note that most of them have stable cones.

THEOREM 4: For all dimensions $n \geq 4$, there always exist nonequatorial, immersed, minimal hyperspheres in $S^n(1)$.

THEOREM 5: For all dimension $18 \leq n \leq 100$, there always exists at least one nonequatorial, immersed, minimal hypersphere, Σ^{n-1} , in $S^n(1)$ whose cone, $C\Sigma^{n-1}$ is stable in $\mathbb{R}^{n+1}$.

Remark: Again, we believe Theorem 5 holds for all dimensions $n \geq 18$. We only list those cases that already have been carefully checked by numerical computation.

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