Axial Presentations of Regular Arcs on M_n

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Contributed by Marston Morse, September 22, 1972

ABSTRACT **THEOREM 1.** *Let M_n be a Riemannian manifold of class C^m , $m > 0$. On M_n let g be a simple compact, sensed, regular arc whose local coordinates are functions of class C^m of the algebraic arc length s , measured along g from a prescribed point of g . There then exists a presentation $(F:U, X) \in \mathcal{DM}_n$ such that $g \subset X$, and each point $p(s)$ of g is represented in the euclidean domain U by coordinates $(x^1, \dots, x^n) = (s, 0, \dots, 0)$.*

When the conditions and conclusion of *Theorem 1* hold, we say that g is *axially presented** by F . *Theorem 1* is a variant of a theorem known to differential geometers. The object of this note is to give a direct proof that will be available *a priori* in both differential and variational topology. Here $m \geq 1$. Most proofs assume $m > 1$.

Theorem 1 is a consequence of the following two statements: (i) If C_ρ is a sufficiently small positive constant, there exists an "axial presentation" of any closed subarc of g of length at most C_ρ . (ii) If each of two overlapping closed subarcs of g admits an "axial presentation," then the union of these subarcs admits an axial presentation.

The proof of (i) is left to the reader. I shall prove (ii) by proving *Theorem 2*. In *Theorem 2*, U and V are open subsets of the n -plane $E_n = R^n$ with rectangular coordinates $x^1, \dots, x^n$. X and Y are open subsets of M_n .

THEOREM 2. *Let $g_{a,b}$ and $g_{c,d}$ be two closed overlapping subarcs of g on which $a \leq s \leq b$, and $c \leq s \leq d$, respectively, with $a < c < b < d$. If $g_{a,b}$ and $g_{c,d}$ are axially presented, respectively, by presentations*

$$(\varphi:U,X); (\psi:V,Y) \quad (1)$$

in $\mathcal{DM}_n$, there exists a presentation in $\mathcal{DM}_n$ that axially presents the subarc $g_{a,d}$ of g on which $a \leq s \leq d$.

Our proof consists in restricting and modifying the presentations φ and ψ so that the modified presentations φ and ψ are identical on the intersection of their domains and serve to define a presentation in $\mathcal{DM}_n$ that axially presents $g_{a,d}$. A presentation $(F:U,Z)$ in $\mathcal{DM}_n$ that axially presents a subarc $g_{\alpha,\beta}$ of g can be *admissibly* modified as follows. Let $\bar{\alpha}, \bar{\beta}$ be the closed subarc of the x^1 -axis on which $\alpha \leq x^1 \leq \beta$. Let $\hat{U}$ be an open subset of U that includes $\bar{\alpha}, \bar{\beta}$. If $\oplus$ is a diff of $\hat{U}$ into U that leaves $\bar{\alpha}, \bar{\beta}$ pointwise invariant, the composite presentation $\dagger (F \circ \oplus: \hat{U}, Z)$ is in $\mathcal{DM}_n$ and axially presents $g_{\alpha,\beta}$. To continue, further notation is required.

* See page 92 of ref. 1 for $\mathcal{DM}_n$. Mappings in $\mathcal{DM}_n$ are termed *presentations*; their inverses are termed *charts*.

$\dagger F \circ \oplus$ is defined, since $\oplus(\hat{U}) \subset U$ by hypothesis. All diffs in this paper are of class C^m , $m \geq 1$.

Notation. Corresponding to the interval $[\alpha, \beta]$, let $[\alpha, \beta]_e$ denote the open subset of E_n of points within a distance $e > 0$ of α, β . Set

$$\Pi_\alpha^\beta = \{x \in E_n | \alpha \leq x^1 \leq \beta\} \quad (2)$$

Let δ be a positive constant and set

$$a_0 = a - \delta, b_0 = b + \delta, c_0 = c - \delta, d_0 = d + \delta \quad (3)$$

We suppose δ so small that $\overline{a_0, b_0} \subset U$ and $\overline{c_0, d_0} \subset V$. With δ so chosen and *fixed*, let $e > 0$ be so small that $[a_0, b_0]_e \subset U$ and $[c_0, d_0]_e \subset V$. Other conditions will be put on the magnitude of e . The constant δ is held fast.

Transition diffs. Let two diffs in $\mathcal{DM}_n$ be given, of the form of φ and ψ in (1), together with open subsets $\hat{U} \subset U$ and $\hat{V} \subset V$, such that $\varphi(\hat{U}) = \psi(\hat{V})$. Then $\hat{U}$ and $\hat{V}$ are termed *mated* by φ and ψ . Diffs $(\psi^{-1}\varphi)|\hat{U}$ and $(\varphi^{-1}\psi)|\hat{V}$ are well defined and termed *transition diffs*, respectively, of $\hat{U}$ onto $\hat{V}$ and $\hat{V}$ onto $\hat{U}$.

The fact that $g_{c,b}$ is axially presented by both φ and ψ implies *Lemma 1*. *Lemma 1* is the principal aid in proving *Theorem 2*. We continue with the above notation.

LEMMA 1. *Let $\sigma \in R$ be such that $c < \sigma < b$. If e is a sufficiently small positive constant, the following is true.*

(i) *The set $[c, \sigma]_e$ is an open subset U_e of U which is mapped by a "transition diff," $\lambda_e = (\psi^{-1}\varphi)|U_e$, onto a mate $V_e \subset V$.*

(ii) *The diff λ_e is extendable by a diff $\oplus_e$ of $[c, d]_e$ onto an open neighborhood of c, d , a diff $\oplus_e$ that leaves points of c, d invariant.*

Proof of (i). If e is sufficiently small, (i) is clearly true.

My proof of (ii) makes use of a special diff T of $\Pi_{a_0}^{b_0}$ onto $\Pi_{c_0}^{d_0}$.

The diff T defined. Let $\tau \in R$ be such that $\sigma < \tau < b$. The following is readily proved. There exists a diff $t \rightarrow f(t): [a_0, b_0] \rightarrow R$ onto $[a_0, d_0]$ which reduces to the identity on $[a_0, \tau]$ and to a translation of $[b, b_0]$ onto $[d, d_0]$ for $t \in [b, b_0]$, with $f'(t) > 0$ for each t in the domain of f . One defines T by setting

$$T(x^1, \dots, x^n) = [f(x^1), x^2, \dots, x^n] \quad (x \in \Pi_{a_0}^{b_0}) \quad (4)$$

It is clear that T is a diff of $\Pi_{a_0}^{b_0}$ onto $\Pi_{c_0}^{d_0}$. Moreover $T(x) \equiv x$ for $x \in \Pi_{a_0}^\tau$. For e sufficiently small, $\psi^{-1}\varphi$ maps $[c, b]_e$ diffeomorphically onto an open neighborhood K_e of c, b . Let the restriction of $\psi^{-1}\varphi$ to $[c, b]_e$ be denoted by $(\psi^{-1}\varphi)_e$. Set

$$T_e = T|_{[a, b]_e}; \quad \hat{T}_e = T|_{K_e}, \quad (5)$$

taking e so small that $[a, b]_e$ and K_e are included in $\Pi_{a_0}^{b_0}$, the domain of T .

Proof of (ii). We suppose e conditioned as above. For a domain $[c, d]_e$, a diff

$$\mathbb{H}_e = \hat{T}_e \circ (\psi^{-1}\varphi)_e \circ T_e^{-1} \quad (6)$$

is then well defined. The composition factors of $\mathbb{H}_e$, applied successively, map $[c, d]_e$ first onto $[c, b]_e$, then onto K_e , and finally onto an open neighborhood of $\overline{c, d}$, as affirmed in Lemma 1. Moreover if e is sufficiently small

$$\mathbb{H}_e(x) \equiv \lambda_e(x), \quad (x \in [c, \sigma]_e) \quad (7)$$

since both T_e^{-1} and $\hat{T}_e$ reduce to the identity on $[c, \sigma]_e$, if e is sufficiently small. Finally $\mathbb{H}_e$ leaves points of $\overline{c, d}$ pointwise invariant.

Thus Lemma 1 is true.

Proof of Theorem 2. It is sufficient to define a presentation $(\Lambda: W, Z)$ in $\mathfrak{DM}_n$, which is an axial presentation of $g_{a, d}$. The domain W of Λ shall be $[a, d]_e = [a, \sigma]_e \cup [c, d]_e$, where e is conditioned as in the preceding paragraphs. One sets

$$\begin{aligned} \Lambda(x) &= \varphi(x) & (x \in [a, \sigma]_e) \\ \Lambda(x) &= (\psi \circ \mathbb{H}_e)(x) & (x \in [c, d]_e) \end{aligned} \quad (8)$$

Λ is thereby twice defined for $x \in [c, \sigma]_e$, and for no other points in its domain, if e is sufficiently small, since g is simple and regular. Moreover

$$\varphi(x) \equiv (\psi \circ \lambda_e)(x) \equiv (\psi \circ \mathbb{H}_e)(x) \quad (x \in [c, \sigma]_e) \quad (9)$$

in accord with the identity (7) and the definition of the transition diff λ_e . Thus, Λ is in $\mathfrak{DM}_n$. It remains to verify the following

(μ) Λ maps $x \in \overline{a, d}$ into $s = x^1$ on $g_{a, d}$.

Proof of (μ). Since $c < \sigma < b$, φ in (8) maps a point $x \in \overline{a, \sigma}$ into the point $s = x^1$ on $g_{a, b}$. Moreover $\mathbb{H}_e$ leaves $\overline{c, d}$ pointwise invariant, so that by (8) and (9) Λ is an axial presentation of $g_{c, d}$ and, hence, of $g_{a, d}$.

This research was supported in part by Army Research Office-Durham Grant DA-ARO-D-31-124-72-G130.

1. Morse, M. & Cairns, S. S. (1969) *Critical Point Theory in Global Analysis and Differential Topology* (Academic Press, New York and London).