No-splash theorems for fluid interfaces

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The article (1) by Gancedo and Strain in PNAS studies how singularities may develop in the initially smooth interfaces separating two or more incompressible fluids. The fluids and interfaces are assumed to evolve by either of the two standard systems of equations from fluid mechanics, namely the surface quasi-geostrophic (SQG) sharp front equation (2) or the Muskat equation (3). Gancedo and Strain prove that initially smooth fluid interfaces evolving by either of those two equations cannot form a splash singularity (4).

Let us first describe the SQG sharp front equation. We consider a temperature $\theta(x, t)$ depending on time t and position $x = (x_1, x_2)$ in the plane. The temperature is carried along by an incompressible fluid whose velocity at position x and time t is $u(x, t)$. That is

$$(\partial_t + u \cdot \nabla_x) \theta(x, t) = 0.$$

Because the fluid is incompressible, the velocity $u(x, t)$ arises from a stream function ψ by the standard formula

$$u(x, t) = \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1} \right).$$

We take the stream function ψ to arise from the temperature θ by the formula

$$\psi(x, t) = \int_{\mathbb{R}^2} |x - y|^{-1} \theta(y, t) dy.$$

The above system for (θ, u, ψ) is called the SQG equation. Originally proposed in ref. 5

Fig. 1. Muskat interface.

as a model for weather phenomena, the SQG equation serves as an excellent 2D model for the 3D Euler equations (6).

Suppose we take the temperature $\theta(x, t)$ equal to 1 if x lies inside a closed contour $\Gamma(t)$ and equal to 0 if x lies outside $\Gamma(t)$. Then the fluid velocity $u(x, t)$ diverges logarithmically as x approaches the contour $\Gamma(t)$. Nevertheless, it is possible to make sense of the SQG equation in this case. This leads to an evolution equation for the contour $\Gamma(t)$, called the SQG sharp front equation (2).

The SQG sharp front equation crudely models the evolution of a cold front on the earth's surface. It also serves as a simplified model of vortex filaments for the 3D Euler equation.

Suppose that the initial contour $\Gamma(0)$ is smooth. We ask whether the contour $\Gamma(t)$ may develop a singularity at some positive time. If so, then we ask how that singularity might look.

Rodrigo (2) proved that an initially smooth contour $\Gamma(0)$, evolving by the SQG sharp front equation, remains smooth up to some positive time τ [that may depend on $\Gamma(0)$] (7). Numerical simulations (8) suggest that an asymptotically self-similar singularity arises in finite time; however, no mathematical rigorous proof of this is known.

Next, we describe the Muskat equation in two space dimensions. We consider two or more incompressible, immiscible fluids in a porous medium (e.g., oil and water in sand). The fluids are separated by one or more interfaces. The same equations that govern the behavior of two or more fluids trapped between two closely spaced parallel vertical plates (a "Hele Shaw cell") (9).

Each fluid has its own density and viscosity. We assume here that the fluids all have the same viscosity, but their densities are different.

The fluid velocity $u(x, t)$ and pressure $p(x, t)$ are assumed to satisfy Darcy's law (10), which, in suitable units, asserts that

$$u(x, t) = -\nabla p(x, t) - (0, \rho),$$

where ρ denotes the density of the fluid at position x and time t . We assume also that

Fig. 2. Splash singularity.

the pressure $p(x, t)$ is continuous across the interface, and the jump in the velocity as we cross any interface is tangential.

Under the above assumptions, the interfaces separating the fluids above evolve according to a contour dynamics equation called the Muskat equation (11).

Suppose that at time 0, the interfaces are all smooth, and no two interfaces touch. We ask whether the Muskat equation has a smooth solution, at least up to a small positive time (depending on the initial conditions). If such a solution exists, can the Muskat equation be solved for all positive times or does a singularity arise in finite time?

To address these questions, we first consider different types of interfaces that may arise. In Fig. 1, fluids I, II, and III are separated by two interfaces A and B. Interface A is the graph of a function, $x_2 = f(x_1)$, and consequently, fluid I lies entirely above fluid II. On the other hand, interface B is not a graph. Fluid II lies above fluid III near the point y , but fluid II lies below fluid III near the point x .

If fluid I has a lower density than fluid II, then we say that interface A satisfies the "Rayleigh–Taylor condition" (see ref. 11 and recall that we assume that fluids I and II have the same viscosity).

Suppose that at time 0, each interface is smooth and satisfies the Rayleigh–Taylor condition. Suppose also that at time 0, no two interfaces touch. Then it is known that

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a smooth solution to the Muskat equation exists, at least up to some small positive time depending on the initial conditions. On the other hand, if at time 0 the Rayleigh–Taylor condition fails, then one expects that no solution exists up to any positive time (11).

Suppose a short-time solution of the Muskat equation exists. Under certain assumptions on the initial data, it is known that a smooth solution exists for all positive time. However, there exist smooth initial conditions for which the Rayleigh–Taylor condition holds for small positive time, after which the interface turns over (i.e., it becomes a nongraph), and finally a singularity arises in that interface. For the above results, see refs. 12–14.

It is of great interest to know what kind of singularities may arise in interfaces evolving by the Muskat equation. For instance, we do not know whether a singularity can form without a turnover.

Next, we describe “splash singularities.” Consider a system of two or more fluids, separated by interfaces. Suppose that the interfaces look initially like the solid lines in Fig. 2. In particular, suppose that the interfaces are initially smooth and that no two interfaces touch. After a finite time, suppose that the interfaces look like the dotted lines in Fig. 2. More precisely, suppose that the interfaces remain smooth

but that the two interfaces touch at a single point. Then we say that a splash singularity occurs. We also allow the possibility that the two arcs depicted in Fig. 2 are parts of a single interface.

Splash singularities are known to arise in initially smooth solutions of the water wave equation (4, 15, 16). It is natural to ask whether splash singularities may arise in any given fluid-mechanics problem involving interfaces.

In PNAS, Gancedo and Strain announce a rigorous proof that splash singularities

cannot occur for solutions of the Muskat and SQG sharp front equations.

Repeatedly in the fluid dynamics literature, numerical simulations have indicated the formation of singularities in finite time for solutions of important equations (17). It is notoriously difficult to decide whether the scenarios indicated by such simulations actually occur in the exact solutions of the relevant equations. Rigorous proofs are therefore very valuable. This applies in particular to the proofs shown by Gancedo and Strain. Their proofs are simple and ingenious.

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