New Integral Representations for Solutions of Cauchy's Problem for Abstract Parabolic Equations

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ABSTRACT A study has been made of Cauchy's problem for a class of abstract parabolic differential equations. Our study is based upon a transformation that maps solutions of second-order abstract Cauchy problems into solutions of first-order. This is a preliminary report on some of the results obtained. These results include (1) new representations for solutions of abstract parabolic equations, (2) applications to the heat equation, (3) a discussion of solution representations for second-order abstract Cauchy problems, and (4) applications to non-well posed (in the sense of Hadamard) Cauchy problems.

In this short note the notion of a singular abstract Cauchy problem (SACP²), which was introduced and investigated by the author [2], is employed in a program where solutions are obtained for a certain class of regular abstract Cauchy problems for parabolic operator equations. The key step in this program is the derivation of an integral transformation that maps solutions of the SACP² into solutions of Cauchy's problem for abstract parabolic operator equations. It is shown that if A generates a strongly continuous group on a given Banach space, then the solution of the first-order abstract Cauchy problem can be represented in terms of the solution of the SACP². For a detailed discussion of strongly continuous semigroups (groups), see [3].

NOTATIONS

Let X denote a Banach space with generic element φ . Here, and in the sequel, A will denote the infinitesimal generator of a strongly continuous group, $T_A(t)$, of operators on X . Let $P(z)$ be a polynomial in z with real or complex coefficients, whose leading term is of the form $a_0 z^n$ where the real part of a_0 has sign $(-1)^{n+1}$. The symbol D_x will denote the differential operator $\partial/\partial x$ and B_ν will denote Bessel's differential operator of order ν , which is defined by

$$B_\nu U \equiv [D_{rr} + (1/r)D_r - (\nu^2/r^2)]U.$$

The operator B_ν^m , m a positive integer, is defined by the recurrence formula

$B_\nu^m U = B_\nu(B_\nu^{m-1}U)$; B_ν^0 is defined to be the identity operator.

Abbreviation: SACP², singular abstract Cauchy problem.

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REPRESENTATION THEOREM

In this section a representation theorem for the Cauchy problem for the first-order abstract parabolic equation

$$V'(t) = P(A^2)V(t) \quad (1)$$

satisfying the initial condition

$$V(0) = \varphi \quad (2)$$

where $\varphi \in \mathcal{D}(A^{2n})$, the domain of A^{2n} , is obtained. We associate with this problem the singular abstract Cauchy problem SACP²:

$$W''(s) + [(2\nu + 1)/s]W'(s) - A^2W(s) = 0, \quad (3)$$

$$W(0) = \varphi, \quad (4)$$

and

$$W'(0) = 0, \quad (5)$$

where $\varphi \in \mathcal{D}(A^{2n})$, and ν is a real parameter. For $\nu = -1/2$, the Cauchy problem (3-5) reduces to the abstract Cauchy problem as was formulated by Phillips [5]. The following results concerning the SACP² are needed.

Theorem 1

Let $\nu > -1/2$ and let A generate a group $T_A(t)$ strongly continuous at the origin with $T_A(0) = I$. Then the corresponding SACP² has a solution for each $\varphi \in \mathcal{D}(A^2)$, namely

$$W_{2\nu+1}(t; \varphi) = \int_{-1}^1 f(\nu; s) T_A(st) \varphi ds, \quad (6)$$

where

$$f(\nu; s) = C_\nu (1 - s^2)^{\nu-(1/2)}$$

and

$$C_\nu = \Gamma(\nu + 1) / [\Gamma((2\nu + 1)/2) \Gamma(1/2)].$$

This integral is divergent for $\nu \leq -1/2$. The following result provides a solution valid for $x \in N' = \{x: x \leq -1/2, \text{ and } x \text{ not a negative integer}\}$.

Theorem 2

Let $\varphi \in \mathcal{D}(A^{m+2})$ where m is the smallest positive integer such that $\alpha = 2m + 2\nu + 1 > 0$. Then

$$W_{2\nu+1}(t; \varphi) = Ct^{-2\nu} [(1/t)D_t]^m \left(t^{2m+2\nu} \int_{-1}^1 f(\alpha; s) T_A(st) \varphi ds \right) \quad (7)$$

is a solution of the SACP² valid for $\nu \in N'$. Here, $C = 1/[(2\nu+2)(2\nu+3)(2\nu+4)\cdots(2m+2\nu)]$. The proofs of these theorems, together with other properties of the SACP² may be found in [2].

We are now able to state our major result which indicates explicitly the connection between (1-2) and the SACP².

Theorem 3

Let $H_\nu(r, t)$ be the solution of the initial value problem

$$D_t[H_\nu(r, t)] = P(B_\nu)H_\nu(r, t), \quad (8)$$

$$H_\nu(r, 0) = [2^\nu \Gamma(\nu + 1)/r^{\nu+1}] \delta(r) \quad (9)$$

where $\delta(r)$ is the Dirac delta function. Then the solution of (1) satisfying the initial condition (2) is provided by the representation

$$V(t) = [1/2^\nu \Gamma(\nu + 1)] \int_0^\infty r^{\nu+1} W_{2\nu+1}(r; \varphi) H_\nu(r, t) dr, \quad (10)$$

valid for $\nu > -1$.

Sketch of the proof

The uniform convergence of the integral in (10) may be demonstrated by appealing to an asymptotic expansion for $H_\nu(r, t)$, together with an easily obtainable bound for the norm of $W_{2\nu+1}(r; \varphi)$. Differentiation under the integral sign in (10) may be justified by employing an asymptotic expansion for derivatives of $H_\nu(r, t)$. The assertion that $V(0) = \varphi$ can be shown by letting $t \searrow 0$ under the integral sign. The result then follows from (9), along with the strong continuity of the solution of the SACP² near the origin. A slightly involved computation, which will not be given here, demonstrates that $V'(t) = P(A^2)V(t)$.

APPLICATIONS

Our first application involves a class of one-dimensional parabolic partial differential equations. Let $X = C(-\infty < x < \infty)$ and let $A = f(x)D_x$, where f is continuous and $0 < \delta \leq |f| \leq M$. This choice of A generates the strongly continuous group

$$T_A(t)\varphi(x) = \varphi(k^{-1}(t + k(x))), \text{ where}$$

$$k(x) = \int_0^x [1/f(z)] dz.$$

Therefore, $P(A^2) = a_0[f(x)]^{2n}(D_x)^{2n} + \text{lower-order terms}$. Now if $f(x)$ is $2n - 1$ times differentiable, $V'(t) = P(A^2)V(t)$ is a $2n$ th-order parabolic partial

differential equation with continuous coefficients. The expression in (10) yields an explicit solution to Cauchy's problem for this equation, namely

$$V(x, t) = G_\nu \int_0^\infty r^{\nu+1} \left\{ \int_{-1}^1 f(\nu; s) \varphi(k^{-1}(sr + k(x))) ds \right\} H_\nu(r, t) dr,$$

where here and in the sequel $G_\nu = 1/[2^\nu \Gamma(\nu + 1)]$. The above expression is valid for $\nu > -1/2$. A similar representation of the solution, valid for $-1 < \nu \leq -1/2$, can be obtained by employing the representation in (7).

Our next application involves the initial value problem for the heat equation: $U_t = U_{xx}$, $U(x, 0) = \varphi(x)$, where $\varphi \in \mathcal{D}(D_x^2)$. We let $X = C(-\infty < x < \infty)$ and let A be the operator D_x . The strongly continuous group generated by this choice of A is $T_A(t)\varphi(x) = \varphi(x + t)$. We obtain now the following class of representations for the solution of the Cauchy problem for the heat equation:

$$U(x, t) = G_\nu \int_0^\infty r^{\nu+1} W_{2\nu+1}(r; \varphi(x)) I_\nu(r, t) dr, \quad (11)$$

valid for $\nu > -1$, where $I_\nu(r, t)$ is the solution of the initial value problem $D_t[I_\nu(r, t)] = B_\nu I_\nu(r, t)$, $I_\nu(r, 0) = [1/(G_\nu r^{\nu+1})] \delta(r)$; and $W_{2\nu+1}(t; \varphi)$ is the solution of the Euler-Poisson-Darboux equation $W_{tt} + [(2\nu + 1)/t]W_t = W_{xx}$ satisfying the initial conditions $W(x, 0) = \varphi(x)$, and $W_t(x, 0) = 0$. Upon employing Hankel transform theory, it is readily seen that

$$I_\nu(r, t) = [r^\nu/(2t)^{\nu+1}] \exp(-r^2/4t), \quad (12)$$

valid for $\nu > -1$. This result, along with the expressions in (6) and (7), provides the following representations of the solution for the heat equation:

$$U(x, t) = \begin{cases} K_1 \int_0^\infty r^{2\nu+1} \left\{ \int_{-1}^1 f(\nu; s) \varphi(x + sr) ds \right\} \exp(-r^2/4t) dt, & \nu > -1/2; \\ K_2 \int_0^\infty D_r \left(r^{2\nu+2} \int_{-1}^1 (1 - s^2)^{\nu+(1/2)} \varphi(x + sr) ds \right) \exp(-r^2/4t) dr, & -1 < \nu \leq -1/2; \end{cases} \quad (13)$$

where

$$K_1 = 1/[2^{2\nu+1} \Gamma(\nu + 1) t^{\nu+1}]$$

and

$$K_2 = 1/[2^{2\nu+1} \Gamma(\nu + (3/2)) t^{\nu+1}].$$

When one sets $\nu = -1/2$, it is not difficult to show that the right side of (13) becomes

$$(1/\sqrt{4\pi t}) \int_{-\infty}^\infty \varphi(x + r) \exp(-r^2/4t) dr,$$

the well-known solution representation of the heat equation.

SECOND-ORDER ABSTRACT CAUCHY PROBLEMS

The extension to second-order abstract Cauchy problems can be made by using a standard argument. The procedure for carrying out this extension is a slight modification of that given by Hersh [4]. For this reason it will not be given here.

NON-WELL POSED PROBLEMS

The formula in (10) may be applicable in the study of non-well posed Cauchy problems for partial differential equations. We illustrate this remark with the following example which involves the initial value problem for the backward heat equation: $U_t + U_{xx} = 0$,

$$U(x,0) = \varphi(x).$$

Now formula (11) provides a solution to this problem, where now $W_{2\nu+1}(t)$ is the solution of the initial value problem for the one-dimensional generalized axially symmetric potential equation:

$$W_{tt} + [(2\nu + 1)/t]W_t + W_{xx} = 0,$$

$$W(x,0) = \varphi(x), W_t(x,0) = 0.$$

To be sure, more stringent conditions must now be placed on φ since the Cauchy problem for elliptic equations is not well posed in the sense of Hadamard.

CONCLUSION

In conclusion, we mention that the special case $P(A^2) = A^2$ was treated by Bragg and Dettman [1], where they showed that the Cauchy problem for a parabolic equation and the Cauchy problem for an associated hyperbolic equation could be transformed into the same Cauchy problem by making suitable transformations of the independent and dependent variables, and then by employing the Laplace transformation. We wish also to mention that Hersh [4] has obtained solutions of (1-2) by employing an abstract Plancherel formula.

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