An Invariant for Certain Operator Algebras

RICHARD W. CAREY* AND JOEL D. PINCUS

Department of Mathematics, State University of New York, Stony Brook, N.Y. 11790; and *University of Kentucky at Lexington, Ky.

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ABSTRACT It is shown that the so-called principal function invariant, which is associated in a unitarily invariant way to operators with trace class self commutator $TT^* - T^*T$, is invariant under trace class perturbations of T and is an extension of the index of T - z to the whole plane. The connection of the principal function, under additional hypothesis, with the determination of the maximal ideal space of the C^* algebra generated by T is discussed, and it is shown that the principal function, even when it takes noninteger values, plays a role in establishing the existence of invariant subspaces for T and in determining the point spectrum of T .

This note continues the investigation of the properties of the so-called principal function of an almost normal operator. When T is a bounded completely non-normal operator on a Hilbert space H with self commutator $TT^* - T^*T = 2C/\pi$ in trace class, it was shown in a series of papers (1, 2, 4, 10, 13, 14) that a summable function $G(\delta, \gamma)$, the principal function, could be associated with T in a unitarily invariant way. This function, which we think of as an invariant associated with the C^* algebra generated by T , is explicitly and simply computable for many operators. It is known to be a complete unitary invariant for T when C has one dimensional range. It is also invariant under trace class perturbations of T .

Several years ago J. D. Pincus conjectured that $G(\delta, \gamma) = \text{Index } T - (\delta + i\gamma)$ when $\delta + i\gamma$ is not in $\sigma_{\text{ess}}(T)$, and established the result, as a completely elementary consequence of classical index results for singular integral operators, provided that strong continuity assumptions were imposed upon the coefficients that enter into a singular integral representation for T .

The general conjecture has now been affirmatively settled in two different ways. First J. W. Helton and R. Howe (11) established that a certain representing measure coming from a bilinear functional was a constant multiple of the index times planar Lebesgue measure on the components of the complement of the essential spectrum of T , and J. Pincus (5) showed that the representing measure was actually equal to the principal function times Lebesgue measure. (The precise statements are given in the remarks following Theorems 1 and 4.) Then we showed how the conjecture could be established on the basis of earlier results (4). We give this derivation below.

It is known that the map $T \rightarrow G$ is onto (19). To any summable function there corresponds a completely non-normal operator with trace class self commutator and G as principal function. Thus the typical situation is the one in which G rarely if at all assumes integer values.

Accordingly, while it is indeed possible and useful to adapt the two-complex variable approach which originally gave rise

to the principal function to the purpose of proving mild index results, by extracting only the relevant bilinear functional and bypassing the rest of the theory, we present results in this paper that seem to require stronger methods.

Elementary identities

The principal function G was defined first by means of the so-called determining function of $T = U + iV$. If $VU - UV = (1/\pi i)C$, we factor C through an auxiliary coefficient Hilbert space h by setting $K^*SgnCK = C$, $J^2 = Sgn C$, with $K^*: h \rightarrow H$, $J: h \rightarrow h$, and $K: H \rightarrow h$.

The determining function is defined as an operator on h by setting.

$$E(l, z) = 1_h + (1/\pi i)JK(V - l)^{-1}(U - z)^{-1}K^*J$$

when (l, z) is not in $\sigma(V) \times \sigma(U)$. E is the solution of a certain Riemann Hilbert problem associated with T (1), and G can be so defined that

$$\det E(l, z) = \exp \frac{1}{2\pi i} \iint G(\delta, \gamma) \frac{d\delta}{\delta - l} \frac{d\gamma}{\gamma - z}.$$

Because $\det(1 + AB) = \det(1 + BA)$, we see at once that $\det E(l, z) = \det (V - l)(U - z)(V - l)^{-1}(U - z)^{-1}$, and it becomes natural to try to evaluate $\det e^{WQ}e^{-W}e^{-Q}$ in terms of G for arbitrary polynomials W and Q in the operators U and V . With the observation (5) that $\det e^{WQ}e^{-W}e^{-Q} = e^{Tr(W, Q)}$, the following theorem does this in a simple way.

THEOREM 1. (5) Let W and Q be polynomials in U and V , then

$$\det e^{WQ}e^{-W}e^{-Q} = \exp (i/2\pi) \iint \{W, Q\} G(\delta, \gamma) d\delta d\gamma, \quad [1]$$

and

$$\text{Trace}[W, Q] = (i/2\pi) \iint \{W, Q\} G(\delta, \gamma) d\delta d\gamma \quad [2]$$

where $\{W, Q\} = \frac{\partial W}{\partial \delta} \frac{\partial Q}{\partial \gamma} - \frac{\partial W}{\partial \gamma} \frac{\partial Q}{\partial \delta}$ is the ordinary Poisson bracket of the polynomials W and Q considered as scalar functions of the scalar variables δ and γ , and $[W, Q] = WQ - QW$.

This theorem can be extended beyond the case where W and Q are polynomials. A Mikhlin functional calculus that associates with $p(\delta, \gamma)$ in $C^2(\Sigma)$, where Σ is a neighborhood of $\sigma(U) \times \sigma(V)$, an operator $\mathfrak{M}(p)$ so that $[\mathfrak{M}(p_1), \mathfrak{M}(p_2)]$, $\mathfrak{M}(p_1)\mathfrak{M}(p_2) - \mathfrak{M}(p_2)\mathfrak{M}(p_1)$, and $\mathfrak{M}(\bar{p}) - \mathfrak{M}(p)^*$ are all in trace class can easily be constructed. The theorem remains true when W and Q are chosen in $C^2(\Sigma)$.

Remark: It has been known for years that $Tr[u^j v^k, v] = j(i/2\pi) \iint x^{j-1} y^k G(x, y) dx dy$. The new ingredient that made a proof

of the present theorem possible is a purely algebraic result of N. Wallach: If B is a bilinear form on polynomials in two commuting variables x and y such that $B(P_1(M), P_2(M)) = 0$ when P_1 and P_2 are single variable polynomials and M is a polynomial in the two variables x and y , then $B[R(x, y), S(x, y)] = L\{R(x, y), S(x, y)\}$, where L is a linear functional.

The bilinear form $B(R, S) = \text{Trace}[R, S]$ satisfies the hypothesis of Wallach's lemma, and Helton and Howe (11) verified that the functional L is represented by a measure. That is, they proved that $\text{Trace}[R, S] = \iint \{R, S\} dP$ for some planar measure dP . They did not identify dP with G nor prove the absolute continuity of dP .

Theorem 1 implies that G is invariant under trace class perturbations of T .

L. Brown (12) has extended the validity of relation 1 above.

We turn now to a study of the polar decomposition of T and its relation to the principal function.

Dilations and the polar decomposition

Let us define $T_z = T - zI$. T_z admits a representation in the form $W_z Q_z$ where $(T_z^* T_z)^{1/2} = Q_z$ and W_z is a partial isometry with initial and final spaces equal, respectively, to $T_z^* H$ and $T_z H$. With $D = T_z T_z^* - T_z^* T_z$, we have $W_z P_z = (P_z + D)W_z$ with $P_z = T_z^* T_z$.

Observe also that the space $[T_z^* H]^\perp = H \ominus [T_z^* H]$ is invariant for P_z while the space $[T_z H]^\perp = H \ominus [T_z H]$ is invariant for $P_z + D$. For fixed z , define the Hilbert space H_z by setting

$$H_z = \dots [T_z H]^\perp \oplus [T_z H]^\perp \\ \oplus H \oplus [T_z^* H]^\perp \oplus [T_z^* H]^\perp + \dots$$

Let $\tilde{W}_z$ be defined on H_z as transforming

$$\tilde{x} = [\dots x_{-2}, x_{-1}, x_0, x_1, x_2, \dots]$$

into

$$\tilde{W}_z \tilde{x} = [\dots x_{-3}, x_{-2}, x_{-1} + W_z x_0, P_R x_0, x_1, \dots]$$

where P_R is the projection onto $[T_z^*]^\perp$. Further, let

$$\tilde{P}_z \tilde{x} = [\dots (P_z + D)x_{-2}, (P_z + D)x_{-1}, P_z x_0, P_z x_1, P_z x_2, \dots]$$

and

$$\tilde{D} \tilde{x} = [\dots 0, 0, D x_0, 0, 0, \dots].$$

It is clear that $\tilde{W}_z \tilde{P}_z = (\tilde{P}_z + \tilde{D})\tilde{W}_z$, and that $\tilde{W}_z$ is the minimal unitary dilation of W_z (see ref. 4 for the proof).

Let $\tilde{\varphi}_z(l, w): h \rightarrow h$ be defined by setting

$$\tilde{\varphi}_z(l, w) = 1_h + J \tilde{K} \tilde{W}_z (\tilde{W}_z - w)^{-1} (\tilde{P}_z - l)^{-1} \tilde{K}^* J,$$

where we have factored $\tilde{D}$ in the form $\tilde{D} = 1/\pi \tilde{K}^* \text{Sgn } D \tilde{K}$, just as we factored C before.

This operator valued function is closely related to the determining function introduced above and has been extensively studied (see refs. 4, 10, 18).

It is a consequence of the cited results that there exists a summable function, $G_z^p(\lambda, \tau)$, such that

$$\det \tilde{\varphi}_z(l, w) = \exp \left\{ \frac{1}{2\pi i} \int_{|\tau|=1} \int_0^\infty G_z^p(\lambda, \tau) \frac{d\lambda}{\lambda - l} \frac{d\tau}{\tau - w} \right\}.$$

The polar invariant, $G_z^p(\lambda, \tau)$, is actually related in a very simple way to our original principal function.

THEOREM 2. Let $\delta + i\gamma - z = \lambda^2 e^{i\theta}$ for fixed $z = x + iy$. Then $G_z^p(\lambda, e^{i\theta}) = G(\delta, \gamma)$.

Although G is not analytic, the combination $\delta + i\gamma$ occurs often enough so that we define $G^c(\delta + i\gamma) = G(\delta, \gamma)$.

The connection between the principal function and index will be approached through the study of the perturbation problem $P_z \rightarrow P_z + D$.

By a well-known result of M. G. Krein (16), there exists a summable function $\delta_z(\lambda)$ such that

$$\det (1_h + D(P_z - l)^{-1}) = \exp \int_0^\infty \frac{\delta_z(\lambda)}{\lambda - l} d\lambda.$$

The following result (which we simply quote from ref. 4) expresses the known connection between the polar principal function G_z^p and the phase shift $\delta_z(\lambda)$.

$$\text{THEOREM 3. } \delta_z(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} G_z^p(\lambda, e^{i\theta}) d\theta.$$

Our proof of the index conjecture stated in the introductory section will follow very simply from the fact that $\delta_z(\lambda)$ is the index of $T - z$ when λ is restricted to a small-enough positive neighborhood of zero.

Proof of the index conjecture—remarks on the essential spectrum

If z is not in $\sigma_{\text{ess}}(T)$, then T_z and T_z^* are Fredholm operators and both $T_z T_z^*$ and $T_z^* T_z$ have closed range. It follows that $N(T_z) = R(T_z^*)^\perp = R(T_z^* T_z)^\perp = N(T_z^* T_z)$ and $N(T_z^*) = R(T_z)^\perp = R(T_z T_z^*)^\perp = N(T_z T_z^*)$ are both finite dimensional.

Thus, zero can be at most an isolated eigenvalue of finite multiplicity for one or both of $T_z T_z^*$ and $T_z^* T_z$.

For a densely defined closed operator A in H and $R(l, A) = (l - A)^{-1}$, let $\rho(A)$ be the resolvent set of A , and let $\bar{\rho}(A) = \rho(A)$ plus isolated singularities of $R(l, A)$. For each fixed l define

$$\alpha(l, A) = \begin{cases} \dim N(l, A) & \text{if } l \text{ is in } \bar{\rho}(A) \\ \infty & \text{if } l \text{ is not in } \bar{\rho}(A) \end{cases}$$

where $N(l, A)$ is the range of the projection

$$P(l, A) = \frac{1}{2\pi i} \int_{@} R(\zeta, A) d\zeta$$

where $@$ is a sufficiently small circle with center l . If l is in $\rho(A)$, then $\alpha(l, A) = 0$. If l is in $\bar{\rho}(A) \setminus \rho(A)$, then $\alpha(l, A)$ is exactly the algebraic multiplicity of the eigenvalue l .

If we take $A = T_z^* T_z$, then our remarks above imply that $\det (1 + D(P_z - l)^{-1})$ is meromorphic at $l = 0$ when z is not in the essential spectrum of T .

$$\text{Then } \det (1_h + D(P_z - l)^{-1}) = \exp \int_0^\infty \frac{\delta_z(\lambda)}{\lambda - l} d\lambda \text{ has the}$$

Laurent expansion $\sum_{j=-r}^\infty a_j l^j$ in some neighborhood of zero.

It follows in an entirely elementary way that $\delta_z(\lambda)$ must be equal (almost everywhere) to the constant r in some positive interval containing zero. Thus by a generalization of the Aronszajn-Weinstein formula due to Kuroda (8), $\delta_z(\lambda) = r = \alpha(0, T_z^* T_z) - \alpha(0, T_z T_z^*) = \text{Index } (T - z)$ for λ sufficiently small.

In fact, we can show very easily that $\delta_z(\lambda) = G_z^p(\lambda, e^{i\theta})$ for λ sufficiently small. Recall that $\text{Index } (T - z)$ is constant on the components of the complement of the essential spectrum of T . Now select a component, say Ω . With z in Ω , there exists

a number $b(z) > 0$ such that $\delta_z(\lambda) = \text{Index}(T - z)$ for almost all λ in the interval $[0, b(z)]$.

Let $B_z(R)$ denote the ball centered at z with radius R less than $b(z)$. With area measure denoted by dA we have

$$\begin{aligned} \iint_{B_z(R)} G^c(\zeta) dA &= \iint_{B_0(R)} G^c(\zeta + z) dA \\ &= \int_0^R \lambda d\lambda \int_0^{2\pi} G_z^p(\sqrt{\lambda} e^{i\theta}) d\theta = \text{Index}(T - z) \pi R^2. \end{aligned}$$

Dividing by πR^2 and letting R approach zero gives us at once $G^c(z) = \text{Index}(T - z)$.

We have, therefore, proved the following statement:

THEOREM 4. *With z not in the essential spectrum of T , $\text{Index}(T - z) = \delta_z(\lambda) = G^c(z - (\delta + i\gamma))$ when λ is positive and both λ and $\gamma^2 + \delta^2$ are sufficiently small.*

This theorem establishes the index conjecture.

Remark: Helton and Howe proved a weaker result (11). They showed that their planar measure dP referred to in the first remark earlier was constant on the components of the complement of the essential spectrum of T (from the Poisson bracket formula), and they were able to identify the constant as the index. Theorem 1 added to their result gives another proof of the index conjecture.

A simple corollary of the preceding Theorem 4 is the following statement:

THEOREM 5. *A sufficient condition for z to be in the essential spectrum of T is that $\delta_z(\lambda)$ not be equal to an integer in a positive neighborhood of $\lambda = 0$.*

Since G is the complete unitary invariant of T only when $TT^* - T^*T$ has one dimensional range, it is impossible to recover the essential spectrum entirely in terms of G in general. However, as a test of the utility of the principal function in this case, we show how the essential spectrum may be characterized when D has one dimensional range.

First we need a lemma. Consider the perturbation problem $T_z^*T_z \rightarrow T_z^*T_z + D = T_zT_z^*$. Let the Kato subspace H_K be the smallest invariant subspace of $T_z^*T_z$ that contains the range of D . Clearly $T_z^*T_z = T_zT_z^*$ on the orthogonal complement of H_K .

The following result is an easy observation:

LEMMA. *Spectrum $(T_z^*T_z)$ is included in the union of the spectra of $T_z^*T_z$ and $T_zT_z^*$ restricted to the Kato subspace.*

A result of C. Putnam asserts (7) that z is in the essential spectrum of T when D is compact and T is seminormal, if and only if both $T_zT_z^*$ and $T_z^*T_z$ are singular. Assume for definiteness that D has one dimensional range and is positive. Then the phase shift $\delta_z(\lambda)$ corresponding to the one-dimensional perturbation problem takes values between zero and one. However, if both $T_z^*T_z$ and $T_zT_z^*$ are singular, then $\delta_z(\lambda)$ can not be constantly (a.e.) equal to zero or to one in a positive neighborhood of $\lambda = 0$, for $\det(1 + D(T_z^*T_z - l)^{-1}) = \exp \int \delta_z(\lambda) (d\lambda/\lambda - l)$ while $\exp - \int \delta_z(\lambda) (d\lambda/\lambda - l) = \det(1 - D(T_zT_z^* - l))^{-1}$ and, thus (as the reader may easily see), the first determinant is analytic at zero iff $\delta_z(\lambda) = 0$ in a positive neighborhood of zero while the second determinant is analytic near zero iff $\delta_z(\lambda) = 1$ in a positive neighborhood of zero. But these determinants can be rewritten in terms of the spectral measures of $T_z^*T_z$ and $T_zT_z^*$ because D has one dimensional range. If $D = (\cdot, k)k$ then $\det(1_h + D(T_z^*T_z -$

$l)^{-1}) = 1 + \int (d\|E(\lambda)k\|^2/\lambda - l)$ where $E(\lambda)$ is the spectral resolution corresponding to the operator $T_z^*T_z$, while $\det(1_h - D(T_zT_z^* - l)^{-1}) = 1 - \int (d\|F(\lambda)k\|^2/\lambda - l)$ where $F(\lambda)$ is the spectral resolution corresponding to $T_zT_z^*$. Thus, if both $T_z^*T_z$ and $T_zT_z^*$ are singular, the phase shift can not be zero or one in a positive neighborhood of zero.

But $\delta_z(\lambda)$ is an integer near zero iff $G_z^p(\lambda, e^{i\theta})$ is an integer near zero, by Theorem 3 above. Hence, since the assumption that D is positive and has one dimensional range implies that $G_z^p(\lambda, e^{i\theta})$ takes values between zero and one, we have proved the following result:

THEOREM 6. *If T^* is completely hyponormal and $TT^* - T^*T$ has one dimensional range, then the essential spectrum of T consists of those points z for which $G(\delta, \gamma)$ is not (exclusively) equal to zero or to one almost everywhere as $\delta + i\gamma$ ranges over a neighborhood of z .*

We can give this result a more geometric flavor if we restrict ourselves to the case that either U or V has finite spectral multiplicity, for in this case it is known that $G(\delta, \gamma)$ assumes only the values zero and one. [See Pincus (1), where it is shown how the spectral multiplicity of both U and V are determined from $G(\delta, \gamma)$.] Hence under this additional hypothesis, the essential spectrum of T consists of those points z for which every disk with center z intersects both the support of $G(\delta, \gamma)$ and its complement in a set of positive planar measure.

This last result was conjectured several years ago (see ref. 10) but in too much generality.

In order to illustrate this theorem, we give an example. It was proved in ref. 1 that T has a singular integral representation:

$$Lx(\lambda) = [\lambda + iA(\lambda)]x(\lambda) + \frac{1}{\pi} \int \frac{b(\lambda)^*b(\mu)}{\mu - \lambda} x(\mu) d\mu$$

where $A(\lambda)$ and $b(\lambda)$ are certain matrices. $A(\lambda)$ is Hermitian with essentially bounded components and $b(\lambda)$ has only one nonzero column consisting of essentially bounded functions of λ . $x(\lambda)$ has only a finite number of components under the assumption that U has finite spectral multiplicity. Then it is shown (1) that

$$\exp \int \frac{G(t, \lambda)}{t - l} dt = \det \left\{ \frac{1 + b(\lambda)^*(A(\lambda) - l)^{-1}b(\lambda)}{1 - b(\lambda)^*(A(\lambda) - l)^{-1}b(\lambda)} \right\}$$

from which the principal function is easily determined (10).

Remark: Results about the essential spectrum of singular integral operators with scalar coefficients of the above type have been obtained by K. Clancey (9) through the use of "localization" techniques. Theorem 6 gives a somewhat more general result. We also remark that the bilinear functional defined by Eq. 2 of Theorem 1 above can clearly be extended to $C_0^2(R^2)$, and if G is constant on an open set 0 and if R and S in $C_0^2(R^2)$ are supported in 0, then $L(R, S) = 0$ by a simple generalization of the Dubois Raymond lemma. Hence the support of the functional L must be a subset of the essential spectrum of T . It is not true, however, that the support of L must always coincide with the essential spectrum of T even if T is completely hyponormal. For, as we have already noted, it is shown (19) that G can be chosen to be any positive summable function with compact support. Thus, by Theorems

4 and 5 we could find an operator T , satisfying our conditions, with $G = 1/2$ inside the unit disc and $G = 0$ outside the unit disc. In this case the unit disc will be both the spectrum and the essential spectrum of T . The support of L in this case is just the unit circle.

Discrete spectrum and invariant subspaces—when D has one dimensional range

Throughout this section we assume that T^* is completely hyponormal and that D has one dimensional range.

THEOREM 7. *Under these hypotheses, z is an eigenvalue for T iff there exists a ball B_z about z such that*

$$\iint_{B_z} \frac{1 - G^c(\zeta)}{|\zeta - z|^2} dA < \infty.$$

We will prove this statement. Let us begin by noting that T_z has 0 for an eigenvalue iff 0 is in the point spectrum of the restriction of $T_z^*T_z$ to the Kato space, since $T_z^*T_z = T_zT_z^*$ on H_K and the point spectrum of T_z^* is empty (because of the assumption of complete hyponormality).

Thus, z will be an eigenvalue for T iff $d\mu_z(\cdot)$ has a point mass at zero where $d\mu_z$ is a positive measure defined by the equation

$$\exp \int_0^\infty \frac{\delta_z(\lambda)}{\lambda - l} d\lambda = 1 + \int_0^\infty \frac{d\mu_z(\lambda)}{\lambda - l}$$

But a known result (15) asserts that $d\mu_z(\cdot)$ has a point mass at zero if and only if

$$\int_0^1 \frac{1 - \delta_z(\lambda)}{\lambda} d\lambda < \infty.$$

This is the same as

$$\int_0^1 \frac{1}{s^2} \int_0^{2\pi} (1 - G_z^p(s, e^{i\theta})) d\theta ds < \infty.$$

If we set $s^2 e^{i\theta} = \zeta$, we can rewrite this condition as

$$\int_{|\zeta| \leq 1} \frac{1 - G^c(\zeta + z)}{|\zeta|^2} dA < \infty.$$

The theorem follows after a translation of this result by z .

Remark: It follows from the theorem that G must be approximately continuous at an eigenvalue and take the value one. In particular, it is easily seen that if G^c dominates the characteristic function of some open set O , then every point in O must be an eigenvalue for T . However, there are compact sets F with empty interior so that if we set $G^c =$ characteristic function of F , then the integral in the theorem will be finite for almost all z (with respect to area) in F . On the other hand, there are compact sets F having no interior so that the integral with $G^c =$ characteristic function of F is identically infinite for all z . Thus the point spectrum of T may be empty (see ref. 3).

Of course, if T has an eigenvalue, then T^* has a nontrivial closed invariant subspace. What is quite interesting is that T^* can have invariant subspaces—in fact, polar invariant subspaces, that is, subspaces invariant both for the partial isometry and the positive factors entering into the polar decomposition of T —without T having eigenvalues. Conditions for the existence of such subspaces can be given in terms of the principal function, and the two theorems below give an illustration of what can happen.

THEOREM 8. *If the measure $d\mu_z(\cdot)$ has a singular component, then T^* has a polar invariant subspace.*

We note that conditions on the phase shift that imply the presence of a singular component are known (see refs. 15 and 17).

We now state a result that illustrates that invariant subspaces can occur even when the principal function never takes on integer values.

THEOREM 9. *T^* has a polar invariant subspace if there exists a complex number z and a set Σ of positive λ measure such that*

$$\int_{\Sigma} d\lambda \int_0^{2\pi} \log |\sin 2\pi G_z^p(\lambda, e^{i\theta})| d\theta > -\infty.$$

These theorems are proved through the analysis of a certain canonical model associated with T by means of the principal function. The proofs will appear (see ref. 20).

We note that it is shown in ref. 13 how the deficiency indices of certain unbounded singular integral operators may be computed by a “change of variables” involving the principal function. *Similar computations may be performed for Wiener-Hopf and for Toeplitz operators.*

Characteristic operator functions

Let W be a partial isometry, let V be a bounded symmetric operator, and let D be a symmetric operator with one dimensional range. Let $WV = (V + D)W$, and assume, for simplicity, that the least subspace that contains the range of D and that is invariant for V reduces W .

When the deficiency indices of W are finite, it is possible to express the Nagy-Foias characteristic operator function explicitly in terms of a certain principal function. We will illustrate this in the special case where V and $V + D$ have eigenvalues $\alpha_1, \dots, \alpha_n$, and $\beta_1, \dots, \beta_m$.

It is shown in ref. 4 that $\tilde{W}$, the minimal unitary dilation of W , satisfies $\tilde{W}\tilde{V} = (V + D)\tilde{W}$ for certain self adjoint extensions $\tilde{V}$ and D of V and D to the dilation space of W . With $D = \tilde{K}^*\tilde{K}$, a factorization of D through an auxiliary one-dimensional Hilbert space h , we can define $\Phi(l, z) = 1 + 1/\pi \tilde{K}\tilde{W}(W - z)^{-1}(V - l)^{-1}\tilde{K}^*$ as an operator on h (i.e., a complex number) and establish the representation $\Phi(l, z) = \exp(1/2\pi i) \int_{-\infty}^{\infty} \int_{|\tau|=1} g(\lambda, \tau) (d\tau/\tau - z) (d\lambda/\lambda - l)$ which expresses $\Phi(l, z)$ in terms of a function $g(\lambda, \tau)$ which is, for fixed λ in $\sigma_{ac}(V)$, the characteristic function of the positive arc on the unit circle extending from $f(\lambda)s(\lambda)$ to $f(\lambda)$ evaluated at τ . [Here $s(\lambda) = \exp -2\pi i\delta(\lambda)$ and $\det [1 + D(V - l)^{-1}] = \exp \int [\delta(\lambda)/\lambda - l] d\lambda$ while $f(\cdot)$ is a certain unimodular function defined on $\sigma_{ac}(V)$ with the property that $W = W - f(V)$ where W_- is the wave operator $s - \lim_{t \rightarrow -\infty} e^{it(V+D)} e^{-itV} P_{ac}(V)$.]

The defect operators and defect spaces of W are defined to be $D_W = (I - W^*W)^{1/2}$, $D_{W^*} = (I - WW^*)^{1/2}$ and $\mathfrak{D}_W = \overline{D_W \mathfrak{H}}$, $\mathfrak{D}_{W^*} = \overline{D_{W^*} \mathfrak{H}}$. The characteristic operator function defined on the set of those z for which the operator $I - zW^*$ is boundedly invertible, is defined to be (21)

$$\theta_W(z) = [-W + zD_{W^*}(I - zW^*)^{-1}D_W] \mathfrak{D}_W.$$

The values of $\theta_W(z)$ are bounded operators from the eigenspace of V to that of $V + D$. We can choose an orthonormal basis consisting of the eigenvectors $f_1, f_2, \dots, f_n$ of V which spans $\mathfrak{D}_W$, and an orthonormal basis $g_1, g_2, \dots, g_m$ consisting of eigenvectors of $V + D$, which spans $\mathfrak{D}_{W^*}$.

THEOREM 10.

$$(\theta_W(z)f_i, g_j) = A_{i,j} \cdot z \cdot \exp \frac{1}{2\pi} \int_{-\pi}^{\pi} k_{i,j}(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} d\theta$$

where

$$k_{i,j}(\theta) = \ln |(f_i, g_j)| + \frac{1}{2} P \int_{-\infty}^{\infty} \frac{g(\lambda, e^{i\theta}) - \delta(\lambda)}{\lambda - \beta_j} d\lambda - \frac{1}{2} P \int_{-\infty}^{\infty} \frac{g(\lambda, e^{i\theta}) - \delta(\lambda)}{\lambda - \alpha_i} d\theta.$$

is in $L_1(-\pi, \pi)$, (f_i, g_j) is real, $A_{i,j} = \text{Sgn}(f_i, g_j)$, and $\delta(\lambda) = (1/2\pi) \int_{-\pi}^{\pi} g(\lambda, e^{i\theta}) d\theta$.

This theorem makes it possible to compute the spectrum of W . It also enables us to find the spectral multiplicity of the unitary part (if any) of W as well as the spectral multiplicity of the minimal unitary dilation of W .

Furthermore, it enables us in some cases to find all the invariant subspaces of W in terms of g . For example, this can be accomplished when both V and $V + D$ have only one eigenvalue.

Note added in proof. The theory of the determining function extends to elements T belonging to a C^* algebra equipped with a trace, for example a II_{∞} factor. The determining function is defined as an element of the C^* algebra generated by T and results analogous to the foregoing can be obtained. There is a naturally defined principal function that is invariant under trace class perturbations of T and that extends the index of $T-z$ to the plane, and there is a Mikhlin functional calculus and the analogues of theorems 1, 2, 3, and 4 in this setting.

In particular if $\mathcal{A}$ is the C^* algebra generated by translation operators on the Hilbert space $L_2(0, \infty)$, then there is a faithful representation ρ of $\mathcal{A}$ into a type II_{∞} factor. Each element of $\rho(\mathcal{A})$ is determined modulo the II_{∞} -compact ideal by an almost periodic function on the real line. For a Toeplitz operator with symbol f which is an exponential polynomial, the associated principal function is almost everywhere dA equal to the mean motion associated with $f - z$, z not in $\sigma_{\text{ess}}(T)$.

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