A pressure-gradient mechanism for vortex shedding in constricted channels

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Numerical simulations of the unsteady, two-dimensional, incompressible Navier–Stokes equations are performed for a Newtonian fluid in a channel having a symmetric constriction modeled by a two-parameter Gaussian distribution on both channel walls. The Reynolds number based on inlet half-channel height and mean inlet velocity ranges from 1 to 3000. Constriction ratios based on the half-channel height of 0.25, 0.5, and 0.75 are considered. The results show that both the Reynolds number and constriction geometry have a significant effect on the behavior of the post-constriction flow field. The Navier–Stokes solutions are observed to experience a number of bifurcations: steady attached flow, steady separated flow (symmetric and asymmetric), and unsteady vortex shedding downstream of the constriction depending on the Reynolds number and constriction ratio. A sequence of events is described showing how a sustained spatially growing flow instability, reminiscent of a convective instability, leads to the vortex shedding phenomenon via a proposed streamwise pressure-gradient mechanism. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4841576>]

I. INTRODUCTION

It is well known that the internal flow in a partially constricted channel, the external flow in a separating-reattaching boundary layer, or the canonical problems of flow past a backward-facing step or cylinder can experience unsteady shedding of vortices at sufficiently high Reynolds numbers.^{1–5} Recently, this vortex shedding phenomenon has been observed in numerical simulations and experiments on the hemodynamics of stenosed blood vessels.^{6–8} The vortex shedding phenomenon is important because it significantly impacts surface forces, can affect local mass transfer rates, generates vibrations, and it is expected that breakdown of shed vortices plays a critical role in transition to turbulence.⁹

Real flows in channel geometries are typically three-dimensional, particularly at high Reynolds numbers. However, it is accepted that in the early stages of shear layer roll-up into discrete vortices, the flow is two-dimensional.³ Farther downstream, particularly near the reattachment point, the shed vortices are subject to three-dimensional instabilities. In addition, Alizard, Cherubini, and Robinet¹⁰ summarize research that suggests that vortex shedding frequencies are not altered by flow three-dimensionality. Thus, it is reasonable in this study on the origins of vortex shedding to consider a two-dimensional flow.

We define shedding to be a process by which a fluid structure with coherent rotational motion, a vortex or recirculation region, splits into two distinct flow structures with the latter advecting with the bulk flow. A brief review of vortex shedding in channel and pipe flows is discussed next with a focus on the causes of shedding.

A. Vortex shedding phenomena

Sobey and Drazin¹¹ numerically consider the flow in a two-dimensional channel having periodic boundary conditions in the streamwise direction and a symmetric cosine-shaped constriction. Various

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constriction sizes are considered. The constriction ratio, ζ , is defined as the maximum constriction height normalized by the half-channel height. The constriction aspect ratio, γ (ratio of streamwise width of constriction, σ^* , to full channel height, $2h$) ranges from 1.33 to 3.2. Reynolds numbers vary from 1 to 150 based on half-channel height and mean inlet velocity. Unless otherwise noted, all Reynolds numbers subsequently referred to throughout the text are defined based on half-channel height or radius and mean inlet velocity. For a Reynolds number of 150, they observe unsteady, periodic flow characterized by shedding of vortices. However, the cause of the shedding is not addressed. The relatively low value of Reynolds number for shedding compared to later studies may be related to the background noise level (i.e., numerical accuracy) or the use of periodic boundary conditions.

A recent study by Griffith *et al.*⁴ investigates the wake behavior from a one-sided semi-circular blockage described by a single parameter (ζ) in a two-dimensional channel. Vortex shedding occurs above a Reynolds number of 1000 for a blockage ratio of $\zeta = 0.5$. A mechanism explaining the vortex shedding is not provided, possibly because they find that these flow features occur at Reynolds numbers above the critical value based on linear stability analysis for transition to three-dimensional flow.

Experiments and direct numerical simulations of transitional pulsatile flow through a constricted three-dimensional channel are made by Beratlis, Balaras, and Kiger.¹² They observe vorticity roll-up at the downstream end of the shear layer that eventually sheds. A potential explanation for the initial vortex shedding is given based on common features with inviscid round jets accelerating in a quiescent, unbounded environment. In the inviscid jet problem studied by Gharib, Rambod, and Shariff,¹³ the vortex ring “pinches” off after the circulation reaches a comment value based on a variational principle. Beratlis *et al.*¹² did not find the same value of the universal formation number as in the study of Gharib *et al.*¹³ and suggests that this could be related to the jet being bounded by walls that play an important role in the flow dynamics. Other vortices are observed to shed after the initial vortex, and this is attributed to a Kelvin–Helmholtz-like instability of the shear layer.

Direct numerical simulations of stenotic flow in a tubular geometry with a steady inlet condition are made by Varghese *et al.*⁶ Steady flow downstream of the stenosis is predicted for Reynolds numbers of 250 and 500 for an axisymmetric constriction. Introduction of a geometric perturbation given by a 0.05 eccentricity in the stenosis results in shedding of vortices. Vortices are shed with a Strouhal number of 0.5 based on throat velocity and diameter. It is suggested that vortices are shed due to a wave-like roll-up instability that propagates along the shear layer similar to the Kelvin–Helmholtz instability. In addition, turbulence statistics indicate the presence of velocity fluctuations in the region immediately downstream of the throat. These fluctuations are believed by the authors to be due to the oscillatory wave-like motion of the stenotic jet (i.e., flapping).

Recently, Vetel *et al.*⁸ experimentally investigates transition to turbulence in a pipe geometry having a smooth axisymmetric constriction. When the Reynolds number is increased above the critical value of 200, an unsteady flow occurs with low-frequency oscillations of the reattachment point and the shedding of vortices. The authors link the shedding phenomenon to a periodic discharge of the unstable recirculation region.

Numerical and experimental investigations of stenotic flow in an axisymmetric circular tube having a steady inlet flow are performed by Griffith *et al.*¹⁴ The effect of blockage size on the type of instability (convective or absolute) is studied. At sufficiently large Reynolds numbers, experiments indicate a convective instability occurring downstream of the blockage. This instability manifests as small waves in the shear layer that are amplified as they propagate downstream. Linear stability analysis identifies an absolute instability in the form of azimuthal modes. For a blockage ratio of 0.75, the critical Reynolds number for absolute instability is 335, and from Figure 18 in their paper, the critical Reynolds number is estimated to be 1185 for a blockage ratio of 0.5. Note that the numerically predicted absolute instability modes are not observed in the experiments.

Our constricted channel problem involves shear-layer confinement by two bounding walls. For comparison, therefore, we identify two other known mechanisms of vortex shedding in free shear layers (no walls) and boundary layers (one wall).

Marquillie and Ehrenstein² study the onset of nonlinear oscillations in a separating boundary layer created by flow over a one-sided wall-mounted bump. Although not an internal flow, this

flow bears a close resemblance to a constricted channel with small constriction ratios. The flow is shown to be convectively unstable. The authors find that if the Reynolds number exceeds a critical value, the flow experiences self-sustained two-dimensional low-frequency fluctuations near the start of the separation region (“flapping”) and higher frequency oscillations (aperiodic vortex shedding) farther downstream. Using an extrapolation method to obtain steady solutions above the critical Reynolds number for unsteady flow, they observe that the onset of unsteady behavior coincides with topological flow changes such as rupture of the elongated recirculation region near the reattachment point. The topological changes are found to trigger a rapid transition to an absolute instability. Topological changes in the base flow are also connected with the onset of unsteadiness in separation bubbles investigated by Cherubini *et al.*¹⁵

Vortices may shed via the classic inviscid Kelvin–Helmholtz instability in a free shear layer, where the shear layer rolls up into a sequence of vortices that subsequently detach from the layer. In addition, in axisymmetric jet flow into a quiescent medium, vortex rings are shed via a process termed “pinch-off.” This occurs when the growing vortex ring is fed a critical amount of vorticity from the shear layer, thereby reaching a maximum circulation. This pinch-off process is based on the Kelvin–Benjamin variational principle¹⁶ and found to have a universal time scale referred to as the “formation number.”¹⁷ The maximum vorticity (or circulation) principle has also been used by Kiya and Sasaki¹⁸ in their investigation of the structure of a turbulent separation bubble occurring in flow past a blunt plate. They explain that the shedding of vortices occurs when a sufficient amount of vorticity is accumulated.

Recent research by Obabko and Cassel^{19,20} suggests a purely viscous mechanism as a cause of high-Reynolds number shedding in wall-bounded shear flows subject to an adverse stream-wise pressure gradient. For sufficiently large Reynolds numbers, the shedding process is initiated by intermittent ejections of near-wall secondary vorticity having a “spike-like” character. Each boundary-layer event ejects fluid away from the wall and splits the primary recirculation region into discrete co-rotating vortices that are then shed with the flow. In this mechanism, a sufficiently large adverse pressure gradient is a necessary condition in order for an ejection event to occur within the boundary layer.

B. Current hypotheses

From the above literature review, we find that researchers identify a number of possible explanations for the origins of the vortex splitting and subsequent shedding in both internal and external flows. The phenomenon is postulated to be caused by: (a) a Kelvin–Helmholtz-like instability in the shear layer, (b) topological flow changes inside the recirculation region, (c) an eruption of secondary vorticity bisecting the main recirculation region (also termed bubble bisection), (d) a local absolute instability near the centre of the recirculation region, (e) a global instability, and (f) vortex pinch-off (i.e., splitting) due to exceeding a maximum value of circulation based on the Kelvin–Benjamin variational principle.

The above hypotheses are tested to see if they apply to the partially constricted channel problem. Our previous research finds that hypothesized mechanisms (c) through (f) do not explain the vortex shedding in the present partially constricted channel problem.²¹ In addition, the mathematical foundation for hypothesis (a), i.e., how to extend the Kelvin–Helmholtz instability to flows with viscosity, and both finite Reynolds numbers and finite shear-layer thicknesses has not been developed. Thus, explanation (a) cannot be tested and is not a truly viable hypothesis. Therefore, of the existing hypotheses, only (b) regarding topological changes remains as a potential explanation for vortex shedding in this geometry. It is also possible that another hypothesis can be found to explain vortex shedding in this problem. Note that this listing of postulated vortex shedding mechanisms is primarily focused on those relevant to two-dimensional contexts.

C. Present study

The goal of the present investigation is to identify the origin of vortex splitting and shedding in partially constricted channels. We seek to accomplish this by looking more closely at the

FIG. 1. Portion of a two-dimensional partially constricted channel.

consequences of a sustained perturbation immediately downstream from the constriction on the resulting flow behavior. Further, our results may explain the topological flow changes that researchers such as Theofilis *et al.*²² suggest are present at the onset of unsteady flow behavior. Our findings may lead to novel methods for suppressing or controlling the vortex shedding phenomenon.

The constricted channel problem is formally defined in Sec. II. Numerical solutions to the Navier–Stokes equations and representative power spectra are shown in Sec. III. We identify a spatially growing flow instability in Sec. IV. Based on the instability and a streamwise pressure-gradient mechanism, we propose a sequence of events leading to vortex splitting and shedding in Sec. V. Finally, a discussion, including the generality of the proposed vortex splitting mechanism to other flows and the influence of bounding walls on vortex shedding follows in Sec. VI.

II. PROBLEM FORMULATION

The flow of a homogeneous, isothermal, incompressible, Newtonian fluid in a rigid, infinite, two-dimensional constricted channel is investigated numerically. This section includes a description of the channel geometry, discusses the constriction model employed, and provides the governing equations and boundary conditions. A portion of the channel geometry is illustrated in Figure 1 where flow is from left to right. The half-channel inlet height is denoted by h , maximum constriction height by b , width by σ^* , and aspect ratio by γ . The aspect ratio, γ , is defined as the ratio of streamwise width of the constriction to full channel height, i.e., $\gamma = \sigma^*/2h$. The channel length approaches infinity in the upstream and downstream directions. In order to account for a symmetric constriction, we define $\pm T(x^*)$ to be the distance from the channel centerline to the channel walls in the dimensional (x^*, y^*) Cartesian coordinate system.

A. Constriction model

A review of the literature shows that a number of different models of a constriction geometry have been proposed, particularly in the application to stenotic flows. These models include trapezoidal, semicircular, sinusoidal, or Gaussian distributions of the obstruction in the streamwise direction.^{23–25} The semicircular and trapezoidal options have a potential limitation from a numerical perspective in that a discontinuity in the wall slope exists in the two locations where the channel intersects the constriction shape. Therefore, the constriction on the upper and lower walls are modeled in the present investigation as having a symmetric Gaussian distribution in the streamwise direction. In dimensional form, this is

$$T(x^*) = h - be^{-\frac{1}{2}\left[\frac{x^*}{\sigma^*}\right]^2}. \quad (1)$$

Note that the constriction ratio parameter ζ is defined as the ratio of the maximum Gaussian height, b , to the half-channel height, h ($\zeta = b/h$). Constriction ratios of 0.25, 0.50, and 0.75 are considered.

The streamwise width of the Gaussian distribution is set through the parameter σ^* . Thus, we have a two-parameter (ζ, σ^*) model for the geometric constriction. It is known from previous stenotic flow studies including Solzbach *et al.*²⁴ and Stroud *et al.*²⁵ that the constriction height and width significantly affect the post-constriction flow field. We set the approximate constriction width at the wall to be equal to the full channel height, $2h$, in order to consider a low aspect ratio constrictions (i.e., $\gamma = 1$). Accordingly, σ^* is taken to be 0.3. Low aspect ratio constrictions appear to be common in stenotic flows. For example, a review by Hammes *et al.*²⁶ found that stenoses often have widths close to the vessel diameter.

B. Governing equations and boundary conditions

The unsteady, two-dimensional Navier–Stokes equations are non-dimensionalized using characteristic velocity and length scales taken to be the mean inlet velocity $\bar{U}$ and the half-channel height at the inlet h . Therefore, non-dimensional variables are defined as

$$x = \frac{x^*}{h}, \quad y = \frac{y^*}{h}, \quad t = \frac{t^* \bar{U}}{h}, \quad \sigma = \frac{\sigma^*}{h}, \quad u = \frac{u^*}{\bar{U}}, \quad v = \frac{v^*}{\bar{U}}, \quad \psi = \frac{\psi^*}{h \bar{U}}, \quad \omega = \frac{\omega^* h}{\bar{U}}, \quad (2)$$

where * quantities are in dimensional form, (x, y) are the non-dimensional Cartesian coordinates, (u, v) are the non-dimensional Cartesian components of the velocity vector, and ψ and ω are the non-dimensional streamfunction and vorticity, respectively. The constriction model in non-dimensional form becomes

$$T(x) = 1 - \zeta e^{-\frac{1}{2}[\frac{x}{\sigma}]^2}. \quad (3)$$

The governing equations are cast in the vorticity-streamfunction formulation, which in non-dimensional Cartesian coordinates is

$$\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = \frac{1}{Re} \left[\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right], \quad (4)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega, \quad (5)$$

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad (6)$$

where the Reynolds number based on inlet half-channel height, h , mean inlet velocity $\bar{U}$, and kinematic viscosity is defined as

$$Re = \frac{h \bar{U}}{\nu}. \quad (7)$$

The solution for steady, fully developed Poiseuille flow in a channel is imposed at the inlet and outlet, i.e., as $|x| \rightarrow \infty$. Thus, we have

$$\psi(y) = \frac{3}{2} \left[y - \frac{y^3}{3} \right], \quad \omega(y) = 3y, \quad u(y) = \frac{3}{2} [1 - y^2], \quad v = 0. \quad (8)$$

The boundary conditions imposed on the channel walls at the bottom and top of the physical domain are no-slip, impermeable, rigid walls. This is expressed as

$$u = 0, \quad v = 0, \quad \psi = \mp 1 \quad \text{at} \quad y(x) = \mp T(x). \quad (9)$$

The fully developed plane Poiseuille solution is also applied throughout the channel as the initial condition. Thus, due to the imposed constriction at $t = 0^+$, the flow is accelerated instantaneously from the initial plane Poiseuille flow and then left to evolve to a steady or unsteady solution.

A Dirichlet boundary condition for wall vorticity exists through the use of Jensen's method (see, for example, Spatz²⁷). In this method, the wall vorticity is calculated from known values of the streamfunction by applying the Poisson equation for streamfunction (5) at the wall.

C. Coordinate transformation

The infinite non-dimensional physical domain ($-\infty < x < \infty$) and ($-T(x) \leq y \leq T(x)$) is transformed into a finite computational domain ($-1 < \xi < 1$) and ($-1 \leq \eta \leq 1$). The physical domain (x, y) is mapped to the computational domain (ξ, η) using the following coordinate transformations:

$$\xi = \frac{2}{\pi} \tan^{-1} \left[\frac{x - x_a}{a} \right], \quad \eta = \frac{y}{T(x)}, \quad (10)$$

where grid points are concentrated in the streamwise direction near $x = x_a$ in the physical domain through parameter a , improving spatial resolution in the constricted region with increasing a . With this transformation we have a non-uniform grid in the physical domain and a uniform grid in the computation domain.

The outlet boundary in the physical domain is located at approximately $x = 1908$ with typical values for the transformation being $a = 3$ and $\xi = 0.999$. Resolution of the grid as the outlet boundary is approached as a function of both parameter a and the number of grid points, N_x . In a typical unsteady shedding case, $N_x = 2049$. This results in 48 grid points located from $x = 40$ (typical x location where vortices have dissipated due to viscous diffusion) to the outlet boundary. In addition, moving the outlet location farther downstream did not modify the flow solution for any case considered. Thus, the Poiseuille boundary conditions are found to be valid at the outlet and do not influence the shedding behavior.

Using the transformation (10), the spatial derivatives are transformed into the computational domain. Substitution of the transformed derivatives into the streamfunction equation (5) yields the equation in computational coordinates. In general form, this is

$$A(\xi) \frac{\partial^2 \psi}{\partial \xi^2} + B(\xi) \frac{\partial \psi}{\partial \xi} + C(\xi, \eta) \frac{\partial^2 \psi}{\partial \eta^2} + D(\xi, \eta) \frac{\partial \psi}{\partial \eta} = F(\xi, \eta) \frac{\partial^2 \psi}{\partial \xi \partial \eta} - \omega, \quad (11)$$

where

$$A(\xi) = \Gamma_x^2(\xi), \quad (12a)$$

$$B(\xi) = \Gamma_x(\xi) \Gamma_x'(\xi), \quad (12b)$$

$$C(\xi, \eta) = \Gamma_x^2(\xi) G_1^2(\xi, \eta) + \Gamma_y^2(\xi), \quad (12c)$$

$$D(\xi, \eta) = T'(\xi) \Gamma_y(\xi) G_1(\xi, \eta) \Gamma_x^2(\xi) - B(\xi) G_1(\xi, \eta) - \Gamma_x^2(\xi) [G_2(\xi, \eta) - T'(\xi) \Gamma_y(\xi) G_1(\xi, \eta)], \quad (12d)$$

$$F(\xi, \eta) = 2\Gamma_x^2(\xi) G_1(\xi, \eta). \quad (12e)$$

Functions Γ_x , Γ_y , G_1 , and G_2 are defined by

$$\Gamma_x(\xi) = \frac{[1 + \cos(\pi \xi)]}{\pi a}, \quad (13a)$$

$$\Gamma_y(\xi) = \frac{1}{T(\xi)}, \quad (13b)$$

$$G_1(\xi, \eta) = \frac{\eta T'(\xi)}{T(\xi)}, \quad (13c)$$

$$G_2(\xi, \eta) = \frac{\eta T''(\xi)}{T(\xi)}. \quad (13d)$$

Note that a prime, ', denotes differentiation with respect to ξ . Similarly, substitution of the transformed spatial derivatives into (4) yields the vorticity-transport equation in computational coordinates

$$\frac{\partial \omega}{\partial t} = R(\xi) \frac{\partial^2 \omega}{\partial \xi^2} + S(\xi, \eta) \frac{\partial^2 \omega}{\partial \eta^2} + Q(\xi, \eta) \frac{\partial^2 \omega}{\partial \xi \partial \eta} + G(\xi, \eta) \frac{\partial \omega}{\partial \xi} + H(\xi, \eta) \frac{\partial \omega}{\partial \eta}, \quad (14)$$

where

$$R(\xi) = \frac{1}{Re} A(\xi), \quad (15a)$$

$$S(\xi, \eta) = \frac{1}{Re} C(\xi, \eta), \quad (15b)$$

$$Q(\xi, \eta) = -\frac{2}{Re} \Gamma_x^2(\xi) G_1(\xi, \eta), \quad (15c)$$

$$G(\xi, \eta) = \frac{1}{Re} B(\xi) - \Gamma_x(\xi) u(\xi, \eta), \quad (15d)$$

$$H(\xi, \eta) = G_1(\xi, \eta) \Gamma_x(\xi) u(\xi, \eta) - \Gamma_y(\xi) v(\xi, \eta) + \frac{1}{Re} D(\xi, \eta). \quad (15e)$$

The streamwise and wall-normal velocities (6) are transformed accordingly to

$$u(\xi, \eta) = \Gamma_y(\xi) \frac{\partial \psi}{\partial \eta}, \quad (16a)$$

$$v(\xi, \eta) = -\Gamma_x(\xi) \left[\frac{\partial \psi}{\partial \xi} - G_1(\xi, \eta) \frac{\partial \psi}{\partial \eta} \right]. \quad (16b)$$

The boundary and initial conditions, (8) and (9), are similarly transformed into computational coordinates through the use of (10). Thus, the boundary conditions at the bottom and top of the computational domain are

$$u = 0, \quad v = 0, \quad \psi = \mp 1 \quad \text{at} \quad \eta = \mp 1. \quad (17)$$

The inlet and outlet conditions, i.e., $|\xi| \rightarrow 1$, become

$$u(\eta) = \frac{3}{2} [1 - \eta^2], \quad v = 0, \quad \psi(\eta) = \frac{3}{2} \left[\eta - \frac{\eta^3}{3} \right], \quad \omega(\eta) = 3\eta. \quad (18)$$

D. Numerical methods

The vorticity-streamfunction formulation of the incompressible Navier–Stokes equations (11)–(16b) is solved numerically using a finite-difference scheme suited to two-dimensional, unsteady, incompressible flows. The scheme is second-order accurate in both space and time in the computational domain. This approach is described in Obabko²⁸ and Boghosian.²¹

Simulations are made for various grid resolutions ($N_\xi \times N_\eta$). Grid independence is defined when the maximum of the absolute values of vorticity and streamfunction change by less than approximately 1% with a doubling of the grid. Adequate grid resolution is a function of both the Reynolds number and constriction ratio. For steady, symmetric solutions, grid independent results are obtained with a resolution of 513×129 . For steady, asymmetric solutions, a resolution of 1025×129 is used. Finally, for the unsteady solutions, i.e., vortex shedding, simulations are made on grids of 2049×129 , 2049×257 and 4097×513 . The cases involving shedding are inherently unsteady and are initiated by a sustained flow instability (to be shown in Sec. IV). The spatiotemporal

evolution of this instability is dependent on the grid resolution (discussed next). We note that the unsteady cases are typically run to a final non-dimensional time of 300 (unless the case is initiated from a previous solution). This allows for numerous shedding cycles (even considering any startup transients) since the time for a shedding event ranges from about 1 to 3 over the range of Reynolds numbers and constriction ratios considered.

There are cases where a grid independent solution is not possible owing to the presence of an instability. For example, the canonical Kelvin–Helmholtz instability problem does not allow for a grid independent solution. The growth rate of the instability is proportional to the wavenumber.³¹ Thus, as the wavelength approaches zero, the growth rate approaches infinity. As a result, smaller mesh sizes used in grid independence checks will allow for the introduction of smaller disturbances with faster growth rates. Thus, the instability always occurs at an earlier point in time, and grid independence is not possible. The effects of the wave instability on the Navier–Stokes solutions in the present investigation are also dependent on a given grid resolution. Finer grids produce smaller truncation-error perturbations leading to a change in the growth rates of the wave instability. The different growth rates will lead to a small offset in the timing and/or location of the initial shedding between two grid resolutions. Thus, some differences are expected in the solution between two different grids owing to the instability. However, these differences are small and do not affect the pressure-gradient mechanism and have a negligible effect on the calculated shedding frequencies.

III. NAVIER–STOKES SOLUTIONS

A. Flow bifurcations

Solutions to the Navier–Stokes equations are obtained for a range of Reynolds numbers $1 \leq Re \leq 3000$ and constriction ratios, $\zeta = 0.25, 0.5$, and 0.75 . A plot of the $Re - \zeta$ parameter space is shown in Figure 2 indicating the conjectured thresholds between the identified regimes denoted by bold solid lines. The closed circles represent cases simulated.

FIG. 2. Flow regime map in two-dimensional constricted channel flow as a function of Reynolds number (Re) and constriction ratio (ζ) for $\sigma = 0.3$. Solid lines indicate estimated boundaries between the various flow regimes. Symbol (●) represents cases simulated, and (x) signifies separated flow boundary for similar geometry from Sobey and Drazin.¹¹

The simulations reveal several types of flow regimes depending on Re and ζ . For very low Reynolds numbers and/or constriction ratios, we observe steady flow without separation. Based on our parameter resolution, the critical Reynolds number for steady separated flow is $50 < Re < 100$ for $\zeta = 0.25$, $8 < Re < 10$ for $\zeta = 0.5$, and $2 < Re < 5$ for $\zeta = 0.75$. In Figure 2, we plot the critical Reynolds number for steady separation from the channel study of Sobey and Drazin¹¹ with the symbol $\times$ for comparison. There is reasonably good agreement for the separated flow prediction despite the different constriction geometries (Gaussian vs. sinusoidal and different aspect ratios).

At Reynolds numbers larger than these critical values, the adverse streamwise pressure gradient created by the flow expansion is sufficiently large and results in flow separation with two symmetric recirculation regions behind the constriction and a high velocity jet issuing from the constriction that expands downstream. The rapid acceleration through the constriction results in plug-like, or blunt, velocity profiles at the throat.

A dominant flow feature is the shear layers that divide the jet and recirculation regions. The separation point is found to move upstream with increasing Re . The symmetric recirculation regions on the upper and lower walls become asymmetric with respect to the channel centerline as the Reynolds number is increased via a pitchfork symmetry-breaking bifurcation,¹¹ similar to the results in the sudden expansion problem, see, for example, Refs. 40 and 44. This asymmetry leads to the jet deflecting towards the upper or lower wall. However, one of the asymmetric solutions is the dominant solution owing to the asymmetry introduced by the sweep direction in the Alternating Direction Implicit (ADI) numerical method.²⁹ Tsui and Wang²⁹ show that if the starting point is the lower wall, one asymmetric solution is obtained, and if the horizontal sweeping starts at the upper wall, the other solution is obtained. The critical Reynolds number of the bifurcation point for asymmetric flow decreases with increasing constriction ratio. With further increase of the Reynolds number, the length of the larger recirculation region increases. Due to the flow expansion across the main recirculation region, an adverse pressure gradient is created, and additional recirculation regions may occur downstream in an alternating pattern on each wall (see, for example, Figure 5 in Durst *et al.*⁴⁴).

For sufficiently large Reynolds numbers and/or constriction ratios, the flow becomes unsteady, characterized by vortex shedding and shear-layer fluctuations. Vortices are shed by a splitting of the primary recirculation region, and the shedding location moves upstream with larger Reynolds numbers. Several cases demonstrating vortex shedding are shown next.

B. Vortex shedding behaviors

Figure 3 shows the instantaneous streamlines for a Reynolds number of 1000 and constriction ratio of 0.5 for times between 195 and 218. The corresponding vorticity plots are provided in Figure 4. Other vorticity contour plots are found in the thesis of Boghosian.²¹ At this stage, the flow is well beyond the time needed for initial transients to convect out of the domain due to the impulsively applied constriction, and this time sequence illustrates several flow features. The shedding occurs within a range of streamwise locations on the lower wall from $x = 7$ to $x = 10$. Smaller secondary recirculation regions can form between the wall and the main recirculation region and appear to split the main recirculation region (bubble bisection hypothesis) as can be the case in wall-bounded shear layers.^{19,20} However, it is determined that the secondary recirculation regions do not split the larger recirculation region and are a non-contributing, i.e., passive, participant in the splitting process for confined shear layers.²¹ The process of lower wall vortex splitting and subsequent shedding is different than on the upper wall owing to the flow asymmetry. The recirculation region on the upper wall immediately downstream of the constriction appears to grow and shrink over a small range centered around $x = 5$, but it does not split. However, vortices are observed to form and shed from near $x = 9$ as shown in parts (c), (e), and (k), not as a result of a splitting event, but due to the adverse pressure gradient created by the abrupt flow expansion across the primary recirculation region on the lower wall. Shed vortices may merge farther downstream as visible in parts (k) and (l). Thus, vortices are shed due to aperiodic vortex splitting on the lower wall and from the flow expansion-induced adverse pressure gradient on the upper wall in this case. The shear layers that form at the constriction are observed to have an unsteady fluctuation, or “flapping,” that is not evident in the time interval

FIG. 3. (a)–(l) Instantaneous streamlines for $Re = 1000$ and $\zeta = 0.5$ illustrating vortex shedding. Note dots in this figure correspond to (x, y) locations used in power spectra analysis, cf. Figures 6 and 7.

FIG. 4. (a)–(l) Instantaneous vorticity contours for $Re = 1000$ and $\zeta = 0.5$ corresponding to Figure 3. Over the time range of 195–218, the minimum and maximum vorticity is approximately -178 and 145 , respectively, and we use 30 equally spaced contour levels.

FIG. 5. (a)–(j) Instantaneous streamlines for $Re = 3000$ and $\zeta = 0.5$ illustrating periodic vortex shedding.

shown; however, the fluctuation is a small deviation from any of the subfigures, and the jet is always impinging on the upper wall. Similar behavior is observed for $Re = 400$, $\zeta = 0.75$.²¹

The instantaneous streamfunction contours are shown for a Reynolds number of 3000 and $\zeta = 0.5$ in Figure 5 over a time range of 121.4–125.0. The starting time of the plots is lower than the previous case, because this simulation has been initialized from a previous solution. Again, there are interesting differences compared to the lower Reynolds number of 1000. Taking $t = 121.4$ as a starting point, we show an instantaneous plot every 0.4 time units. A general sequence for vortex shedding is as follows. The vortex is split immediately downstream of the constriction on both walls between $t = 121.8$ and 122.2 shown in parts (b) and (c). The split vortex on the lower wall immediately downstream of the constriction then reconnects back with the original vortex on

the downstream side of the constriction at $t = 123$ as observed in part (e). This reconnected vortex then splits again at $t = 123.8$ as shown in parts (f) and (g) and sheds downstream. After this, the recirculation region on the constriction grows in size from $t = 124.2$ to 125 (parts (h) through (j)) and then splits as in part (c) such that parts (a) and (j) are nearly identical. This process repeats in a periodic manner, with a clearly observable shedding frequency. The shedding location moves upstream compared to the lower Reynolds number cases, now occurring near $x = 1$ on both walls.

The trend from our results is that as the Reynolds number increases, the flow first exhibits vortex shedding with multiple frequencies, and then at sufficiently large Reynolds number, a single dominant frequency occurs, i.e., periodic shedding (cf. Figures 3 and 5). Flows without a single dominant shedding frequency, i.e., that do not exhibit periodic shedding, are also observed in the flow over a bump by Marquillie and Ehrenstein,² for flow in a sudden expansion by Latornell and Pollard,⁴¹ and to a much smaller degree in the sudden expansion study by Fearn, Mullin, and Cliffe.⁴⁰ Thus, multiple frequencies appear to be a common feature in flows with vortex shedding.

C. Vortex shedding and shear-layer connection

We obtain the power spectra for four cases involving unsteady vortex shedding to examine frequency information contained in both the shear layer and in the flow downstream of the constriction. Discrete Fourier transforms of the streamwise velocity at points in the shear layer are computed and compared to results in the vortex shedding region. Details of the power spectra computations and further cases are provided in the thesis of Boghosian.²¹ A representative case is discussed below.

Figure 6 shows the streamwise velocity and power spectrum for the case with Reynolds number of 1000 and $\zeta = 0.5$ at a location of $(x, y) = (13, -0.9)$ in the region where vortex shedding occurs above the lower wall (see the dot at this location in Figure 3). Note that the vertical scale in the power spectrum is arbitrarily normalized. The sign changes in part (a) indicate that vortices are convecting past the specified point. Although there is no single clearly dominant shedding frequency, one peak is identified at a Strouhal number, $St = fh/\bar{U} = 0.16$. The peak observed at $St = 0.3$ may be a harmonic of this shedding frequency. The power spectra is again computed for the same cases as for the vortex shedding but at locations in the shear layer near the constriction. Figure 7 shows the streamwise velocity and power spectrum for the same case ($Re = 1000$ and $\zeta = 0.5$) at a location in the shear layer immediately downstream of the constriction, $(x, y) = (1, -0.5)$. See the dot at this location in Figure 3. The vortex shedding frequency of $St = 0.16$ (cf. Figure 6) is clearly observed in the shear-layer power spectrum in part (b) of this later figure.

The fact that the dominant vortex shedding frequencies are evident, even pronounced, in the upstream shear layers immediately downstream of the constriction (but well upstream from where the shedding occurs) suggests that there is a strong connection between the local dynamics within the confined shear layers and subsequent vortex shedding farther downstream. In the power spectra of all cases considered, it is noted that the primary vortex shedding frequency downstream of the constriction is the same as the frequency of the streamwise pressure-gradient within the shear layers located near the constriction. The shedding frequency, however, differs from the lower shear-layer flapping frequency (see details in Boghosian²¹). Therefore, our results suggest that the vortex-shedding mechanism originates in the upstream shear layer but is not connected with the local shear-layer flapping.

By only considering the dominant shedding frequency, we can obtain a Strouhal-Reynolds number correlation. For a constriction ratio of $\zeta = 0.5$ at Reynolds numbers of 1000, 1500, and 3000, we have primary Strouhal numbers of 0.16, 0.18, and 0.32. From this, we find a Strouhal-Reynolds number correlation of $St = 0.0017Re^{0.6524}$. We can compare this to the relation of Roshko (provided in Ahlborn *et al.*⁴⁵) for the laminar flow past a circular cylinder of $St = 0.212(1 - 21.2/Re_D)$, where the Reynolds number is based on the cylinder diameter. Using our Reynolds numbers based on full-channel height, we find Strouhal numbers of 0.2098, 0.2105, and 0.2113 using Roshko's relation. The Strouhal numbers calculated using Roshko's relation are nearly constant over the Reynolds number range and are not significantly different from our values considering the differences of geometry (cylinder vs. Gaussian) and flow type (internal vs. external). This indicates that the mechanism for vortex shedding may have a common origin.

FIG. 6. (a) Streamwise velocity and (b) power spectrum at $(x, y) = (13, -0.9)$ for $Re = 1000$ and $\zeta = 0.5$.

Our research indicates two distinct shear-layer phenomena. In this research, we focus on the shear layer and its connection with vortex shedding. We also observe a low-frequency shear layer flapping near the constriction at Strouhal numbers less than approximately $St = 0.1$. However, we do not investigate this second shear-layer phenomenon. We do believe the flapping to be another consequence of the time-dependent behavior as a result of the convective-like wave instability. At this location near the constriction, the instability has a relatively small amplitude (that varies in time) and this may slightly alter the local flow through the pressure-gradient mechanism (described in

FIG. 7. (a) Velocity oscillations in shear-layer and (b) power spectrum at $(x, y) = (1, -0.5)$ for $Re = 1000$ and $\zeta = 0.5$.

Sec. V) resulting in the flapping. Note that the findings of Yang and Voke⁴³ and Deck and Thorigny⁴² suggest that the cause of shear-layer flapping is due to convective instabilities in the shear layer.

The appearance of the vortex shedding frequencies in the shear-layer spectrum is typical of all cases for which shedding is observed and suggests that the vortex shedding process occurring downstream is intimately connected with dynamics upstream in the shear layers. This connection is related to a spatially growing wave instability and is discussed next.

IV. FLOW INSTABILITY

The power spectrum in Sec. III (and other examples shown in Boghosian²¹) indicates that shedding frequencies occurring downstream are present in the upstream shear layer near the constriction. Evidence from this research indicates that this connection is based on a spatially growing wave instability.

A case with vortex shedding having a Reynolds number of 400 and constriction ratio, $\zeta = 0.75$, is shown in Figure 8 to illustrate the spatially growing wave instability. This figure shows the streamwise pressure gradient displaying a growing oscillatory perturbation at a moving location in space at two wall-normal locations (wall is at $\eta = -1$). The perturbations decay farther downstream as they exit the unstable region (approximately from $x = 2$ to $x = 8$). The initial transient due to the suddenly imposed constriction is observed at a time of $t = 1$, moves out of the unstable region by a time of approximately 8 and is out of the domain by the time of 20. As the initial transient convects out of the unstable region, the flow from upstream of the constriction arrives at the unstable region and is similarly perturbed. Note that for times beyond $t = 5$, the amplified pressure gradient owing to the wave instability exceeds that of the geometry-induced pressure-gradient peaks (visible near $x = 0$). This case is representative of all vortex shedding cases and shows that a sustained, spatially growing oscillatory wave instability exists in the flow downstream of the constriction for sufficient Reynolds number and constriction ratio. Cases that do not exhibit vortex shedding do not display such a growing time-dependent disturbance.

The flow in the partially constricted channel is strongly nonparallel. Thus, a formal local stability analysis (temporal or spatial) is not appropriate. However, the spatially growing wave instability bears some resemblance to a convective instability and may be referred to as “convective-like.” This is discussed next.

A. Relation to a convective instability

In any experiment or numerical simulation, there will inevitably be sustained disturbances either from imperfections in the geometry, inlet flow, or background noise, or numerical truncation errors. Some flow instabilities, e.g., a convective instability, can amplify such sustained disturbances above a critical Reynolds number leading to order-one changes in the flow field. Convective instabilities, in particular, often occur in spatially developing shear flows and are known to be extremely sensitive to noise. If the perturbations are sustained, the instability can lead to time-dependent behavior.³⁰ In the present problem, we find a sustained spatially growing wave instability in all vortex shedding cases that is reminiscent of a convective instability.

A convective instability is defined per Drazin.³¹ “The flow is convectively unstable if a sufficiently small perturbation grows above a given threshold at no fixed point of the flow but does grow at a moving point.” The results (cf. Figure 8) from the present simulations are consistent with this definition. However, because the flow is nonparallel, we cannot show formally that the wave instability is a convective instability based on a local spatial stability analysis. Nonetheless, there is precedence for using the term convective instability in the context of nonparallel flow. Convective instabilities may be dominant in the related problem of flow in a partially constricted tube (stenotic flows) with a steady inlet velocity.^{6,14,32,34} In addition, convective instabilities are identified in the two-dimensional flow over a bump,² the backward-facing step,^{30–33} and separated boundary-layer.¹⁰ In each of these problems, the base flow is strongly nonparallel.

Kaiktsis *et al.*³⁰ found the flow in the backward-facing step to be convectively unstable in a large portion of the domain and observed that large spatial growth of small-amplitude sustained disturbances by the convective instability can lead to unsteady behavior. Griffith *et al.*¹⁴ shows experimentally for stenotic flows that the convective instability plays an important role in the transition of the flow to an unsteady state. For separated boundary-layer problems, unsteadiness in the flow is attributed to convective instabilities from the continuous effects of noise.¹⁰ Furthermore, transient growth studies suggest that the optimal perturbation may be in the form of a convective-type instability. For example, Alizard *et al.*¹⁰ and Cherubini *et al.*¹⁵ identify the optimal perturbation for the occurrence of unsteadiness in a separated boundary-layer flow over a flat plate as “convective

FIG. 8. (a)–(h) A spatially growing oscillatory perturbation in the streamwise pressure gradient for $Re = 400$ with $\zeta = 0.75$ for two wall-normal locations, $\eta = -0.75$ (greater peak values) and $\eta = -1$ (lower peak values). Note that the initial transient location is indicated when present.

waves” or a disturbance that is localized near the separation point of the recirculation region and amplified along the shear layer as it convects downstream through a pseudo-resonance of global modes. Blackburn, Sherwin, and Barkley³³ also find that optimal growth dynamics mimic a convective instability in a stenotic flow. They further suggest that transient growth studies are the appropriate method to quantify convective instabilities. Thus, a precedence is established for using the term convective instability in nonparallel flows despite the fact that a formal local (parallel) analysis is not appropriate in such cases.³⁴ Following the recent trend established by the above researchers and others, we suggest in this problem that the response of the flow downstream of the constriction is consistent with the definition of a convective instability. To avoid confusion with the formal conditions for a convective instability, we choose to use the term “convective-like” instability. Also, a transient growth analysis is to be performed for the present problem and will be the subject of a subsequent paper.

V. PRESSURE-GRADIENT MECHANISM

A sequence of events is described starting with a sustained, oscillatory growing perturbation that can lead to vortex splitting and ultimately to vortex shedding in partially constricted channels.

At a sufficiently large Reynolds number and/or constriction ratio, it is well established that the constriction geometry-induced adverse pressure gradient results in the formation of the initial, or primary, recirculation regions immediately downstream of the constriction. This occurs owing to the pressure forces overcoming the low streamwise momentum fluid near the channel walls as a result of deceleration of the post-constriction flow. The result is local flow separation and flow reversal. Note that the streamwise pressure-gradient amplification owing to the wave instability can eventually exceed the geometry-induced streamwise pressure gradient (cf. Figure 8). Therefore, it is plausible to suggest that the instability-induced pressure gradient can also cause reversal of low momentum fluid. One place where low momentum fluid is always present is along the centerline of the primary recirculation regions (cf. Figures 13 and 14). Now we aim to show how the amplified pressure gradient field due to the instability affects vortices or recirculation regions already present in the flow, ultimately leading to vortex shedding.

The hypothesized sequence of events leading to vortex shedding is shown in Figure 9. Stages (a) and (b) in the figure are found in existing research and are present in the current investigation. Although it has been suggested by others (e.g., Varghese *et al.*⁶ and Griffith *et al.*¹⁴) that an instability may play a role in vortex shedding, the physical mechanism identified by the sequence (c) through (f), through which it leads to vortex splitting and shedding, to our knowledge has not been described previously. In this mechanism, the growing streamwise pressure gradient first leads to an internal vortex splitting characterized by local extrema in the streamfunction and then to external vortex splitting through its action on the low-momentum fluid within the recirculation region. Thus, stages (c) through (f) are considered in more detail with respect to how a wave instability in the shear layer can ultimately lead to vortex shedding.

From plotting contours of instantaneous streamfunction with streamwise pressure gradient for points in time close to vortex splitting, it is observed that vortex splitting is closely linked with the adverse and favorable pressure gradients on the upstream and downstream side, respectively, of the zero pressure-gradient contour. The streamwise pressure gradient is observed to have a (+, 0, −) pattern near the splitting location. For example, consider the combined contour plot of instantaneous streamlines and zero pressure-gradient contours prior to vortex splitting and after vortex splitting as shown in Figure 10 (cf. corresponding streamwise pressure gradient in Figure 8). Note that for the $\partial p/\partial x = 0$ contour starting at the lower wall near $x = 5.25$, the adverse pressure gradient is located upstream of the zero line and the favorable condition is located downstream of the zero line and that this zero pressure-gradient line approximately divides the main recirculation regions along both the upper and lower walls. The magnitude of the streamwise pressure gradient is greater in part (b) of Figure 10 owing to the growth of the wave instability in time and is sufficiently large locally near $x = 5.5$ to result in vortex splitting. Similar behavior is observed in Figure 11 for a different initial condition with $Re = 1000$ with $\zeta = 0.5$. Note that a secondary recirculation region is observed below the splitting point on the lower wall in the above figures and appears to play a role in splitting the

FIG. 9. (a)–(f) Hypothesized sequence of events leading from sustained wave instability to vortex splitting and shedding via proposed streamwise pressure-gradient mechanism.

larger vortex, i.e., eruption (or also termed bubble bisection) hypothesis.³⁵ However, this secondary recirculation region plays a passive role when it occurs in the present case. Its existence may be explained by the variation of the streamwise pressure gradient in the wall-normal direction at a given streamwise location. The pressure gradient at the wall is lower than in the shear layer (cf. Figure 8) but still sufficiently large to cause a secondary or local recirculation region at the wall. However, there is no eruption of vorticity as in Cassel and Obabko.³⁵

We now describe the detailed steps of how the spatially growing streamwise pressure gradient can modify a vortex or recirculation region *already* present (in this case due to the constriction geometry). The idealized vortex or recirculation region is shown schematically in Figure 12(a). By its nature, the recirculation region is bisected by a line of zero streamwise momentum. Also, based on the Navier–Stokes results, the local streamwise velocity profile $u = u(y)$ is nearly linear in the wall-normal direction across the centerline of the recirculation region. The amplification of the streamwise pressure gradient owing to the wave instability is present across the full height of the channel with a maximum in the shear layer. Part (b) of Figure 12, however, shows this spatially growing pressure gradient located above the recirculation region for illustration purposes.

At some finite time, the leading part of the amplified streamwise pressure gradient perturbation wave passes across the recirculation region. The wave contains a train of regions with adverse, zero, and favorable pressure gradients (+, 0, −), corresponding to the oscillatory nature of the disturbance. At some streamwise location at which the wave has amplified a sufficient amount, the low fluid momentum at and near the centerline of the vortex cannot overcome the local pressure gradient (adverse upstream and favorable downstream of the $\partial p/\partial x = 0$ contour), and thus the fluid is “turned back.” Part (b) shows a schematic of the vortex topology at this stage. The picture

FIG. 10. (a) and (b) Pressure-gradient mechanism and vortex splitting for $Re = 400$ and $\zeta = 0.75$. $\partial p/\partial x = 0$ contours with instantaneous streamlines. Note case is restarted from $Re = 100$.

presented here in part (b) is the idealized model of Figure 10(a). The result is what we term an “internal splitting” with two local extrema in the streamfunction. The eventual splitting is due to the dominance of the pressure forces over the inertial forces. This type of structure can also exist in a steady solution near the transition to unsteady flow, in which case the pressure gradient pattern of

FIG. 11. (a) and (b) Pressure-gradient mechanism and vortex splitting for $Re = 1000$ and $\zeta = 0.5$. $\partial p/\partial x = 0$ contours with instantaneous streamlines. Note case has Poiseuille initial condition.

$(+, 0, -)$ is stationary. See Figure 3.1(i) in the thesis of Boghosian²¹ for an example. This steady streamfunction pattern is also observed in the recirculation zone as “dimpling” by Griffith *et al.*⁴

As time advances, the pressure gradient wave continues to grow spatially owing to the wave instability. Inside the primary recirculation region, the magnitude of the streamwise velocities near

FIG. 12. (a)–(e) Effect of instability-induced growing streamwise pressure gradient on existing recirculation region.

the zero x -momentum line become smaller and smaller due to the amplified pressure gradient. Thus, the flow along those streamlines are more susceptible to being overcome, and flow reversed, by the spatially evolving pressure gradient. Over time, it is observed that the developing pressure-gradient field reverses the direction of more and more low momentum fluid. This reversal can be seen to occur in the vicinity of the nearly vertical $\partial p/\partial x = 0$ contour line. Thus, the internal splitting propagates outward in the wall-normal direction from the zero x -momentum (or $u = 0$) line. This is shown in part (c) of Figure 12.

Finally, at some point in time the growth in the streamwise pressure gradient is sufficiently large such that the recirculation region is at the onset of splitting along a vertical line. Essentially, the recirculation region is being pulled apart near the $\partial p/\partial x = 0$ contour line by the growing adverse and favorable pressure gradients on the upstream and downstream sides, respectively. This idealized model is shown in part (d). The splitting location (point) occurs where the flow is such that $u = v = \partial p/\partial x = 0$, thus, having zero streamwise and wall-normal momentum. In addition, it is noted that the splitting point is a location of zero kinetic energy. At this point, the flow is at the onset of splitting vertically all the way through the main or primary recirculation region. Any infinitesimal increase in the streamwise pressure gradient will cause what we term “external splitting” of the recirculation region. This occurs as the instability continues to evolve and grow in time and the recirculation region splits into two distinct vortices. This is shown in part (e). The latter of the two vortices is then advected downstream with the bulk flow, i.e., shedding occurs.

Recall that the wave instability amplifies the streamwise pressure gradient beyond the geometry-induced pressure-gradient value near the constriction. It is noted that without the presence of the wave instability, the streamwise pressure gradient would be limited to that produced by the geometry (constriction and aspect ratios) and Reynolds number, which is insufficient to cause splitting and shedding. Indeed, no cases have been observed in which vortex shedding occurs without being preceded by the growing instability wave that originates far upstream of the shedding location.

Results from the simulations confirm this idealized model of vortex splitting and shedding. Figure 10(b) shows an example of vortex splitting. This can be seen in the lower recirculation region near $(x, y) = (5.25, -0.6)$. The zero streamwise pressure gradient (dark bold line) separates the two developing vortices, and the splitting point on the zero x -momentum line can be inferred. Similar results are found in Figures 13 and 14 for different Reynolds numbers and constriction ratios. These latter figures also demonstrate that the zero x -momentum and zero streamwise pressure gradient line intersections define where a splitting (and subsequent shedding) is initiated regardless of the Reynolds number or constriction ratio.

In summary, a spatially growing wave instability is clearly present and leads to amplification of the streamwise pressure gradient. Adverse and favorable portions of the growing streamwise pressure gradient can impact existing recirculation regions (setup by the adverse pressure gradient due to the constriction), where low momentum fluid exists. Such low momentum fluid is present along the centerline of the primary recirculation region by definition. Flow reversal then follows naturally when the pressure forces exceed the inertia forces. Hence, we have vortex splitting and shedding owing to the wave instability-induced streamwise pressure gradient. Thus, because the wave instability is sustained in time, the pressure gradient mechanism provides a route leading to unsteady vortex shedding behavior.

It is important to note that vortex shedding is only observed in cases for which sustained growing oscillatory disturbances are present in the streamwise pressure-gradient field. The spatial growth of the streamwise pressure gradient is tracked in time and observed to occur in all cases much earlier in time and well upstream of the vortex splitting and shedding event. As a result, the amplified pressure gradient is truly a cause, not a consequence, of vortex shedding through its action on the low-momentum fluid in the recirculation region. In fact, it is the presence, not the source, of the oscillatory streamwise pressure gradient that is the key to the vortex splitting and shedding mechanism. That is, in the present scenario it is a spatially growing wave instability that produces the alternating adverse and favorable pressure gradient behavior. However, such a behavior is not unique to the convective-like instability and we believe could also be reproduced via other means; for example, through an imposed external forcing or inherent nonlinearities in the Navier–Stokes equations. We note that the above mechanism remains essentially unchanged with increased

FIG. 13. (a) and (b) Instantaneous streamlines with zero x -momentum (long dashed line through recirculation region center) and zero pressure-gradient contours (short dashed line) for $Re = 400$ and $\zeta = 0.5$.

FIG. 14. (a) and (b) Instantaneous streamlines with zero x -momentum (long dashed line through recirculation region center) and zero pressure-gradient contours (short dashed line) for $Re = 400$ and $\zeta = 0.75$.

Reynolds number and constriction ratios. In these cases, the streamwise location of the mechanism is observed to move farther upstream with a reduced wavelength in the pressure gradient instability.

VI. DISCUSSION

We identify various flow regimes including steady (symmetric and asymmetric) separated flow, unsteady vortex shedding, and shear-layer fluctuations in a two-dimensional partially constricted channel depending on the constriction ratio and Reynolds numbers (cf. Figure 2). Based on a sustained spatially growing flow instability (reminiscent of a convective instability) and the resulting spatio-temporal streamwise pressure gradient development, we provide a phenomenological model that describes the process of vortex splitting and ultimately vortex shedding in two-dimensional partially constricted channels.

We find that for sufficient constriction ratio and Reynolds number, a sustained wave instability is present and leads to oscillatory, spatial growth in the streamwise pressure gradient downstream of the constriction. At some location downstream, adverse and favorable portions of the amplified pressure-gradient becomes sufficiently large and act on low momentum fluid that is present by definition along the centerline of the initial, or primary, recirculation regions. The low momentum fluid cannot overcome the local pressure gradients and is “turned back,” i.e., flow reversal occurs, as pressure forces exceed inertial forces. This is similar to how the adverse pressure gradient on the downstream side of the constriction leads to the initial recirculation regions by acting on the low momentum fluid near the walls. In both cases, there are adverse pressure gradients acting on low momentum fluid leading to flow reversal. The initial adverse pressure gradient is due to the geometry and the second is a result of the flow instability. The wave instability induced pressure-gradient mechanism is observed to be present in every case exhibiting vortex splitting and shedding and is not present in any case that does not have shedding. In this problem, the shedding location occurs at a point where $\partial p / \partial x = u = v = 0$. Thus, the pressure-gradient mechanism can be viewed as a direct physical link between the local wave instability and the resulting global unsteady vortex shedding farther downstream.

It is believed that the pressure-gradient mechanism through its structural change of the primary recirculation region (by affecting low momentum fluid) is the cause of the topological flow changes suggested by Theofilis *et al.*²² and others that are present at the onset of unsteady behavior. The wave instability induced pressure-gradient mechanism is consistent with the hypothesis put forth by a number of researchers such as Chedron *et al.*,³⁶ Varghese *et al.*,⁶ and Griffith *et al.*¹⁴ in that vortices are shed due to a Kelvin–Helmholtz-like instability that propagates and grows along the shear layer. We believe that together the wave instability and pressure-gradient mechanism offers a more descriptive and physical explanation of this Kelvin–Helmholtz-like instability and its implications. Recall that formally the Kelvin–Helmholtz instability is an inviscid instability and does not have a known counterpart in viscous shear layers.

With increasing Reynolds number and/or constriction ratio, the length of the initial recirculation regions increase. As a result, the instability-induced pressure gradient perturbation has a longer recirculation region upon which to act. Thus, longer recirculation regions are more susceptible to vortex shedding, i.e., the pressure gradient has a greater spatial extent over which it can become amplified.

Thus, efforts to control the streamwise pressure-gradient in the shear layer immediately downstream of the constriction may lead to suppression of the vortex-shedding phenomenon with minimal energy input.

Our transition to unsteady flow occurs at Reynolds numbers above that predicted by linear stability theory for the transition to three-dimensionality in related geometries.^{9,14} It is possible that a nonlinear stability analysis could show the transition to unsteady flow at larger Reynolds numbers than predicted by linear theory and thus more in agreement with our findings. However, we do not expect significant changes in the mechanism when considering a three-dimensional flow, as it is accepted that in the early stages of shear layer roll-up into discrete vortices, the flow is essentially two-dimensional.³ Possibly there would be an azimuthal component to the spatially growing wave instability, and thus the splitting location might vary in this direction.

We would like to continue this research in several areas. We plan on performing three-dimensional simulations to look for the effect of flow three-dimensionality on the pressure-gradient mechanism. In addition, we are looking at simplifying the mechanism down to its basic essence, i.e., spatially growing pressure-gradient wave acting on low momentum fluid (such as found in opposing shear layers) through an asymptotic analysis. Also, simulations whereby we perturb a steady solution containing low momentum fluid (e.g., opposing shear layer) with a pressure wave will be considered to determine its consequences.

We end with a discussion on how our investigation relates to previous shedding research and try to provide a description of the response of a shear layer considering both a pressure gradient and the nature of the confinement.

A. Relation to previous research

The Strouhal numbers for vortex shedding agree reasonably well with the findings for other internal geometries, e.g., Wee *et al.*³⁷ for backward-facing steps, Ahmed and Giddens³⁷ and Varghese *et al.*⁶ for stenotic flows. The general finding is a Strouhal number for vortex shedding of $O(0.1)$. Some of the differences in Strouhal number predictions are likely related to the specific noise level in a given study as the transition to unsteadiness may be based on a sustained convective instability. The seeming universality of the Strouhal number raises several questions. Is this pressure-gradient mechanism a general feature of wall-bounded and confined flows? For these flows, are the eruption or pressure-gradient mechanisms, or both, present in vortex shedding cases? If so, what determines which one dominates? Note that in research involving high Reynolds numbers, both mechanisms are present, although only the eruption mechanism is elucidated.^{19,28} For the case of $\zeta = 0.75$ in the present investigation, the pressure-gradient mechanism is observed for Reynolds numbers as low as 400, and the viscous eruption mechanism has not been observed for Reynolds numbers up to 1×10^4 . At this Reynolds number, there are indications of a Rayleigh instability in this problem similar to that observed in the unsteady boundary-layer problem.³⁸ It is also possible that the magnitude of the amplified pressure gradient is insufficient to cause splitting owing to the eruption mechanism for any Reynolds number in this geometry.

Vortex splitting and shedding by the streamwise pressure-gradient mechanism appears to be present in a wide range of fluid dynamics problems, and we cite several illustrations. First, in the investigation by Obabko²⁸ mentioned above on the unsteady separation induced by a thick-core vortex convecting above a plane wall, which is considered a model problem for dynamic stall, the streamwise pressure-gradient mechanism can be inferred to be present through wall pressure-gradient and streamfunction plots. Second, in the study of flow around an impulsively started cylinder by Koumoutsakos and Leonard,¹ Figures 7, 10, 25, and 30 show vortex structures during shedding that are strikingly similar to Figures 3 and 5 in this paper where the pressure gradient mechanism is present. Contours of streamwise pressure gradient at various times would confirm or deny the presence of the proposed mechanism in a given problem; however, these plots have not been provided. Third, the structure of two-dimensional separation is described in the study by Pauley, Moin, and Reynolds.³⁹ In Figures 4e and 4f of their paper, we see the familiar structures before and after splitting and suspect the pressure-gradient mechanism to be present. What is interesting in this case is that vortex splitting and shedding are occurring without a constriction or area change, rather it is caused by an externally imposed adverse streamwise pressure gradient created by varying the amount of suction on the upper wall of a channel. Finally, in the sudden-expansion or backward-facing step problem, the mechanism may also be present. For example, research by Kaiktsis *et al.*³⁰ demonstrating a convective instability in the flow over a backward-facing step, shows (their Figure 9) two local streamfunction maxima and a slight “kink” in the first recirculation region for the steady solution at $Re = 1000$ with an expansion ratio of 2. This is very similar to steady solutions showing the local extrema in streamfunction (internal splitting) we have found at the onset of unsteadiness (presented in Boghosian²¹). The above examples may be an indication that the proposed streamwise pressure-gradient mechanism has a general nature. If this is the case, then tracking zero streamwise pressure gradient contours along the low-momentum centerlines of vortices and recirculation regions

will prove to be a harbinger of subsequent vortex splitting and shedding. This requires further study to confirm.

B. Influence of confinement on shedding

The effect of bounding walls (i.e., viscosity) in the constricted channel problem is to provide a means to generate the initial recirculation regions. Once present, the pressure-gradient mechanism does not rely on the bounding walls to initiate vortex shedding. In this problem, the presence of the pressure-gradient mechanism and the existence of low momentum regions are both necessary (and possibly sufficient) to allow for vortex shedding. This is in contrast to the external flow over a wall-bounded shear layer, i.e., boundary layer case, where for sufficiently large Reynolds numbers the wall plays an important role in the primarily viscous eruption mechanism.^{19,35}

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