Monte Carlo estimation of the electric field in stellarators

(neoclassical transport/quasineutrality/BETA computer codes/magnetohydrodynamic equilibrium)

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ABSTRACT The BETA computer codes have been developed to study ideal magnetohydrodynamic equilibrium and stability of stellarators and to calculate neoclassical transport for electrons as well as ions by the Monte Carlo method. In this paper a numerical procedure is presented to select resonant terms in the electric potential so that the distribution functions and confinement times of the ions and electrons become indistinguishable.

In the BETA codes the variational principle of ideal magnetohydrodynamics is implemented as a finite difference scheme to compute the equilibrium of a toroidal plasma in three dimensions without symmetry (1). The magnetic field B is represented by flux functions and Clebsch potentials in the two forms

$$B = \nabla s \times \nabla \psi = \nabla \phi + \zeta \nabla s.$$

The toroidal flux s is assumed to be a single-valued function with nested toroidal level surfaces $s = \text{constant}$, and the pressure is supposed to be a function of s alone.

A difficulty with the existence of smooth equilibria satisfying these hypotheses is circumvented by interpreting the numerical solution of the problem as an asymptotic approximation that is valid for small but positive mesh size. Numerical experiments with a line tracing code suggest that in favorable cases the required flux surfaces behave well on the relatively fine length scale of the electron gyroradius ρ_e . Toroidal and poloidal angles $\tilde{\phi}$ and $\tilde{\psi}$ with normalized periods 0 or 2π are obtained from ϕ and ψ by an affine substitution involving the rotational transform ι . This enables us to introduce s , $\tilde{\psi}$, and $\tilde{\phi}$ as a system of flux coordinates in which to perform transport calculations based on a Fourier decomposition of the function

$$\frac{1}{B^2} = \sum b_{mn}(s) \cos(m\tilde{\psi} - n\tilde{\phi})$$

appearing in the work of Boozer and Kuo-Petravic (2). For that purpose one assumes that the magnetic field is in a steady state and does not interact with the motion of collections of test particles. The BETA equilibrium code generates reliable values of the Fourier coefficients b_{mn} that are needed.

The neoclassical transport of either ions (i) or electrons (e) is governed by a linear drift kinetic equation of the form

$$f_i + \rho[B + \nabla \times (\rho B)] \cdot \nabla f = \nabla \cdot \{ \nu \exp(-Mv^2/2T) \nabla [\exp(Mv^2/2T) f] \}$$

for the relevant distribution function f_i or f_e , where the differential operators on the left are applied in physical space and those on the right are applied in velocity space. In dimensionless units, ρ stands for the parallel gyroradius,

which depends on the strength of the magnetic field, on the magnetic moment, and on the electric potential Φ , while ν stands for the collision frequency, M for the mass, v for the speed, and T for the temperature. A Monte Carlo method of split time is used to integrate the drift kinetic equation, with calls to an Adams ordinary differential equation solver for the guiding center orbits alternating with iterations of a random walk implementation of the Fokker–Planck collision operator (3).

The scaling with gyroradius is such that for larger collision frequencies the electrons tend to escape faster than the ions by a factor equal to the square root of the mass ratio M_i/M_e . Nevertheless, the BETA codes are capable of evaluating electron transport because what turns out to matter most in the computation is the number of mean free paths traversed by the test particles.

The particle confinement time τ is estimated by examining the exponential decay of expected values of integrals of the distribution function f that have been selected judiciously to work for electrons as well as ions (3). Let the leading term in an expansion of f be represented as the product

$$f = e^{-t/\tau} F$$

of a function of t and a factor F independent of t . Then f satisfies the homogeneous drift kinetic equation, so by separation of variables F must be the steady solution of a related inhomogeneous equation associated with the source term $S = F/\tau$. Thus the decay time τ can be expressed as a quotient

$$\tau = \|F\|/\|S\|$$

of norms more comparable to the conventional definition of the energy confinement time τ_E as temperature divided by power. In this context it should be observed that the choice we have specified for the source S corresponds to optimal confinement and might not yield accurate results for an undesirable method of heating the plasma.

Experience has shown that relatively short runs of the BETA transport code suffice to determine both the ion confinement time τ_i and the electron confinement time τ_e . Consequently it becomes difficult to avoid facing up to the anomaly that τ_e is much bigger than τ_i in many cases. Our analysis of magnetic surfaces indicates that the existence of islands or ergodic regions does not explain this phenomenon for good stellarator configurations. It is, therefore, natural to turn to the behavior of the electric potential Φ for an answer (4–6).

In practice the speed of sound c makes the term on the left in Coulomb's law

$$\frac{1}{c^2} \Delta \Phi = f_i - f_e$$

so small that the first three or four significant digits in the charge separation $f_i - f_e$ have to vanish. Thus the law is not

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Table 1. Dependence of the Fourier coefficients $C_{n/m}$ of the charge separation on the Fourier coefficients $P_{n/m}$ of the electric potential for Heliotron E

ν	$P_{0/0}$	$P_{0/1}$	$P_{1/2}$	$P_{1/20}$	τ_i	τ_e	$C_{0/1}$	$C_{1/2}$	$C_{1/20}$
1	-0.5	0.05	-0.01	0.00	85	65	0.01	-0.03	0.00
1	-0.5	0.05	0.00	0.00	70	70	0.01	0.12	0.01
1	0.0	0.00	0.00	0.00	70	100	-0.11	-0.03	-0.01
1	0.0	0.05	0.00	0.00	85	25	0.09	-0.08	-0.02
1	1.0	-0.05	0.00	0.00	380	210	0.01	0.01	0.00
1	1.0	0.00	0.00	0.00	1200	110	0.12	-0.08	-0.02
4	-0.7	0.05	0.00	0.00	65	75	0.01	0.05	0.00
4	-0.5	-0.10	0.00	0.00	11	480	-0.22	-0.06	0.00
4	-0.5	0.00	-0.10	0.00	35	80	-0.06	-0.16	0.01
4	-0.5	0.00	0.00	0.00	25	150	-0.07	0.01	0.00
4	-0.5	0.04	0.00	0.00	40	110	-0.01	0.00	-0.01
4	0.0	0.00	0.00	0.00	40	300	0.20	0.12	-0.02
4	1.0	-0.05	0.00	0.00	125	1200	-0.03	0.06	-0.01
4	1.0	0.05	0.00	0.00	320	100	0.16	0.01	-0.01
16	-1.5	0.05	0.00	0.00	400	600	0.15	-0.02	-0.01
16	-1.0	0.00	0.00	0.00	30	50	-0.07	-0.01	-0.01
16	-1.0	0.03	0.00	0.00	85	90	0.04	0.00	0.01
16	-0.5	0.00	0.00	0.00	14	190	0.05	0.04	0.02
16	-0.5	0.05	0.00	0.00	25	55	0.08	0.04	-0.01
16	0.0	0.00	0.00	-0.02	11	25	0.00	0.04	0.00
16	0.0	0.00	0.00	0.00	25	450	0.23	0.10	0.02
16	2.0	0.00	0.00	0.00	215	3000	0.06	0.01	0.00
64	-3.0	0.00	0.00	0.00	200	1000	-0.03	-0.01	0.00
64	-1.5	0.00	0.00	-0.02	25	110	0.00	0.00	-0.03
64	-1.5	0.00	0.00	0.00	60	700	-0.02	0.00	-0.02
64	-1.5	0.02	0.02	0.00	70	800	0.02	0.06	-0.01
64	-1.5	0.05	0.00	0.00	65	300	0.04	0.00	0.01
64	-0.5	0.05	0.00	0.00	35	500	0.16	0.09	0.00
64	0.0	0.00	0.00	0.00	30	1800	0.06	0.06	0.01
64	1.0	0.00	0.00	0.00	40	1200	0.06	0.02	-0.04

Average $\beta = 0.02$; gyroradii $\rho_i = 0.01$ and $\rho_e = 0.0002$ in units of the plasma radius. The confinement times are given in milliseconds when the magnetic field strength is 1 tesla, and $\nu = 1$ for the reactor regime specified by Boozer and Kuo-Petravic (2).

so much a Poisson equation for Φ as it is a quasineutrality requirement that may be restated in the form

$$F_i = F_e.$$

We choose to interpret this together with the condition $\tau_i = \tau_e$ as a set of ambipolarity equations for the determination of Φ . The equations are to be solved numerically by a spectral method.

In units of the temperature T let us express Φ as a Fourier series

$$\Phi = P_{0/0}(1 - s) + \sum P_{n/m} \cos(m\tilde{\psi} - n\tilde{\phi})$$

with nontrivial dependence on the angles $\tilde{\psi}$ and $\tilde{\phi}$. The radial factor $P_{0/0}$ may serve in a conventional way to satisfy the ambipolarity condition $\tau_i = \tau_e$. On the other hand, the law of conservation of energy for any test particle shows that the mirror effect of wells in the electric potential Φ changes with the sign of the charge for ions and electrons. Therefore, it is reasonable to ask that the remaining Fourier coefficients $P_{n/m}$ of Φ fulfill the requirement that the corresponding Fourier coefficients $C_{n/m}$ of the steady state charge separation

$$F_i - F_e = \sum C_{n/m} \cos(m\tilde{\psi} - n\tilde{\phi})$$

vanish. These coefficients can be estimated reliably from expected values of related trigonometric functions in the Monte Carlo solution of the drift kinetic equation.

It is more customary to find the electric potential Φ from Ohm's law

$$\nabla\Phi + u \times B = \eta J.$$

If the resistivity η vanishes then the parallel component of the electric field vanishes, too, and we are led to the partial differential equation

$$\frac{\partial\Phi}{\partial\phi} = 0.$$

This means that $\Phi = \Phi(s, \psi)$ depends only on the flux functions s and ψ . If the rotational transform ι is irrational, it also follows that Φ is a function of s alone and that is the hypothesis usually made in the theory of neoclassical transport.

Conceptually it is not proper to ask that Φ satisfy two different equations. Therefore, we impose quasineutrality as an absolute requirement and only hope to recover Ohm's law in some approximate sense. Thus we replace dependence on ψ by the assumption that small resonant terms of the form $P_{n/m} \cos(m\tilde{\psi} - n\tilde{\phi})$ may appear in the Fourier representation of Φ with the ratio n/m nearly equal to the rotational transform ι . Corresponding contributions to the parallel component of the electric field are supposed to be orders of magnitude smaller than the radial component.

The BETA transport code provides a successful implementation of the numerical method we have proposed to calculate Φ . It is quite feasible to calculate the first significant figure in the low-order Fourier coefficients $C_{n/m}$ or in the difference of confinement times $\tau_i - \tau_e$. In general these quantities do not vanish, and truncated ambipolarity equa-

tions of the kind we have described furnish diagonally dominant relations for the corresponding coefficients $P_{n/m}$. A determination of the radial electric field is more complicated, but at least in some cases a more or less unique result for the particle confinement time $\tau = \tau_i = \tau_e$ may emerge that seems to represent an average of the values of τ_i and τ_e when $\Phi = 0$. Individual runs of the code using 256 orbits take only an hour or two on the CRAY computer, but a convincing solution of the ambipolarity equations is very time consuming because it requires many runs.

In Table 1 we present numerical data for the Heliotron E experiment providing evidence of the existence of roots of the equations for quasineutrality that is at least convincing for low collision frequency. A survey of the table does give an impression of the way in which the electric field affects transport. A radial term of magnitude comparable to the temperature tends to improve the ion confinement appreciably, whereas resonant contributions that are orders of magnitude smaller may correspondingly reduce the electron confinement time. This phenomenon may help to explain the

anomalous electron transport that has been observed in stellarator experiments. It seems consistent with the known sensitivity of transport to resonance for configurations with little or no shear. These conclusions are borne out by further computations we have made for the TJ-II Helicac.

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