Existence of simply connected algebraic surfaces of general type with positive and zero indices

(Galois covering/fundamental group)

BORIS MOISHEZON* AND MINA TEICHER†

*Columbia University, New York, NY 10027; and †Bar-Ilan University, Ramat-Gan, Israel

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ABSTRACT In the classification problem of algebraic surfaces of general type, an important conjecture states that for simply connected such surfaces Chern numbers satisfy the inequality $c_1^2 \leq 2c_2$ (or equivalently, the index $\tau \leq 0$). We disprove this conjecture by computing fundamental groups of Galois coverings corresponding to generic CP^2 projections of projective embeddings of $CP^1 \times CP^1$ related to linear systems $|a l_1 + b l_2|$, $a \geq 3$, $b \geq 2$. Also, we proved the existence of simply connected minimal surfaces of general type with zero index (e.g., $c_1^2 = 2c_2$). Previously, it was conjectured that these are exactly the surfaces uniformizable in the polydisk. So this conjecture is also disproved.

Algebraic surfaces have as discrete topological invariants the Chern numbers c_1^2 and c_2 . For simply connected surfaces, this pair of invariants is equivalent to the pair of invariants b_2 and τ , where b_2 is the second Betti number and τ is the index of the symmetric bilinear form on $H^2(X, \mathbb{Z})$, given by the cup product.

In 1976 an important inequality between c_1^2 and c_2 was discovered. This is the Bogomolov–Miyaoka–Yau inequality $c_1^2 \leq 3c_2$.

Aside from certain types of surfaces, most algebraic surfaces are the so-called surfaces of general type. Many surfaces of general type are simply connected. After the above-mentioned Bogomolov–Miyaoka–Yau inequality was discovered, strong empirical evidence suggested that for simply connected surfaces of general type the stronger inequality $c_1^2 \leq 2c_2$ must hold. The inequality $c_1^2 \leq 2c_2$ is equivalent to the fact that the index $\tau = \frac{1}{3}(c_1^2 - 2c_2)$ is not positive. So the following conjecture was formulated a few years ago (1).

Bogomolov Watershed Conjecture. An algebraic surface of general type with positive index has a nontrivial fundamental group or, equivalently, an infinite fundamental group.

Recently we succeeded in disproving this conjecture by constructing a series of simply connected surfaces of general type with a positive index. This result completely changed the total picture of surfaces of general type.

It has also been conjectured that surfaces of general type with $c_1^2 = 2c_2$ (or $\tau = 0$) are uniformizable in the polydisk. Our examples show that this conjecture is also false.

We took $X = CP^1 \times CP^1$, $l_1 = CP^1 \times pt$, $l_2 = pt \times CP^1$, $E = al_1 + bl_2$, where a and b are positive integers. Let X_{ab} be the projective embedding of X in some P^N with respect to $|E|$. Let $f: X_{ab} \rightarrow CP^2$ be the generic projection. Let $\tilde{f}: \tilde{X}_{ab} \rightarrow CP^2$ be the Galois cover of X_{ab} with respect to f . $Y = \tilde{X}_{ab}$ is the example we presented. Miyaoka (2) was the first to notice that Galois covers corresponding to generic projections very often have positive indices. In our case $\tau(\tilde{X}_{ab}) = (n!)(3)(ab - 3a - 3b + 5)$, where $n = 2ab$. So $\tau(\tilde{X}_{ab}) > 0$ when $a \geq 6$, $b \geq 5$ or $b = 4$, $a > 7$ and $\tau(\tilde{X}_{ab}) = 0$ when $a = b$

$= 5$ or $a = 7$, $b = 4$. For $a \geq 2$, $b \geq 2$ each $\tilde{X}_{ab}$ is a minimal surface of general type ($K_{\tilde{X}_{ab}}$ is ample). We have proved the following theorem.

THEOREM. For $a \geq 3$, $b \geq 2$, $\Pi_1(\tilde{X}_{ab})$ is a finite commutative group. For a and b relatively prime, $\Pi_1(\tilde{X}_{ab}) = 0$.

Outline of Proof

Reducing the Problem to the Affine Case. It is convenient to replace CP^2 by its “generic” affine part $\mathbb{C}^2$. Let S be the branch curve of f in $\mathbb{C}^2$. Let X_{Gal}^{aff} be the part of X_{Gal} lying over $\mathbb{C}^2 \subseteq CP^2$. It is evident that $\Pi_1(X_{Gal}^{aff}) \rightarrow \Pi_1(X_{Gal})$ is surjective. Therefore, if we prove that $\Pi_1(X_{Gal}^{aff})$ is finite or trivial, we have proven the same for $\Pi_1(X_{Gal})$. So from now on we are interested in $\mathcal{A} = \Pi_1(X_{Gal}^{aff})$.

Introducing $\Pi_1(X_{Gal}^{aff})$ as $\text{Ker } \Pi_1(\mathbb{C}^2 - S, u_0)/\langle \cdot \rangle \rightarrow S_n$. For f the generic projection

$$X_{Gal} = \overline{X \times \dots \times X - \Delta}, \\ \begin{matrix} CP^2 & CP^2 \\ (n \text{ times}) \end{matrix}$$

where Δ is the set of n -tuples $(a_1, \dots, a_n)$ such that $a_i = a_j$ for some $i \neq j$. Let $\tilde{f}: X_{Gal}^{aff} \rightarrow \mathbb{C}^2$ be the natural projection corresponding to f .

There is a natural homomorphism $\Pi_1(\mathbb{C}^2 - S, u_0) \xrightarrow{\Psi} S_n$ for S the branch curve of f and u_0 any point not in S . Ψ is in fact surjective. So we have $1 \rightarrow \text{ker } \Psi \rightarrow \Pi_1(\mathbb{C}^2 - S, u_0) \rightarrow S_n \rightarrow 1$.

We choose a good system of generators for $\Pi_1(\mathbb{C}^2 - S, u_0)$. Let $\pi: \mathbb{C}^2 \rightarrow \mathbb{C}$ be the projection on the first coordinate. Let $M' \subseteq S \subseteq \mathbb{C}^2$ be the points of S at which $\pi|_S$ is not étale. Let $M = \pi(M')$ (M is finite). Let $u \in \mathbb{C} - M$. $(\pi|_S)^{-1}(u)$ is a “good” fiber. Let us take u real, “far enough” from the “bad” points. Let u_0 be a point in $\tilde{\mathbb{C}}^1 = \pi^{-1}(u)$, $u_0 \notin S$.

Let N be a finite set in $\tilde{\mathbb{C}}^1$. $N = \{q_j, \dots, q_p\}$. Let γ_j be a path from u_0 to q_j , such that two γ_j do not meet each other at any point but u_0 . Let η_j be a small circle around q_j . Let γ_j' be the part of γ_j outside η_j . Take $\Gamma_j = \gamma_j \eta_j (\gamma_j')^{-1}$, $\{\Gamma_j\}$ freely generate $\Pi_1(\tilde{\mathbb{C}}^1 - N, u_0)$. $\{\Gamma_j\}_{j=1}^p$ is called a good system of generators for $\Pi_1(\tilde{\mathbb{C}}^1 - N, u_0)$.

We are interested in the case in which $N = \tilde{\mathbb{C}}^1 \cap S$. Then we have a surjection $\Pi_1(\tilde{\mathbb{C}}^1 - S, u_0) \xrightarrow{\nu} \Pi_1(\mathbb{C}^2 - S, u_0) \rightarrow 0$. $\{\nu(\Gamma_j)\}$ then generates $\Pi_1(\mathbb{C}^2 - S, u_0)$. By abuse of notation, we shall denote $\nu(\Gamma_j)$ by Γ_j .

Since f is stable, Γ_j induces a transposition in S_n . So $\Gamma_j^2 \in \text{ker } \Psi$. Let $\langle \Gamma_j^2 \rangle$ be the normal subgroup generated by $\{\Gamma_j^2\}$. Then $\langle \Gamma_j^2 \rangle \subseteq \text{ker } \Psi$. By the standard isomorphism theorems, we have

$$1 \rightarrow \text{ker } \Psi / \langle \Gamma_j^2 \rangle \rightarrow \Pi_1(\mathbb{C}^2 - S, u_0) / \langle \Gamma_j^2 \rangle \rightarrow S_n \rightarrow 1.$$

One can show that $\text{ker } \Psi = \Pi_1(X_{Gal}^{aff} - \tilde{f}^{-1}(S))$ and $\text{ker } \Psi / \langle \Gamma_j^2 \rangle = \Pi_1(X_{Gal}^{aff})$. Denote $\text{ker } \Psi / \langle \Gamma_j^2 \rangle$ by $\mathcal{A}$ and $\Pi_1(\mathbb{C}^2 - S$,

FIG. 1. X_0 is a union of planes.

$u_0)/\langle \Gamma_j^2 \rangle$ by $\bar{\Pi}_1$. We thus have $0 \rightarrow \mathcal{A} \rightarrow \bar{\Pi}_1 \xrightarrow{\Psi} S_n \rightarrow 1$. By abuse of notation, we shall also denote $\bar{\Pi}_1 \rightarrow S_n$ by Ψ . We need to compute $\mathcal{A}$, the kernel of $\Psi: \bar{\Pi}_1 \rightarrow S_n$.

Finite Presentation of $\Pi_1(C_2\text{-alg Curve})$ by Using Braids. The braid group of $\mathbb{C}^1$ and some finite subset N is the group of homotopy equivalence classes of diffeomorphisms of $\mathbb{C}^1$ that fix N and are identical at infinity.

There is a classical theorem of Zariski and Van Kampen from the 1930s that all the relations in $\Pi_1 = \Gamma_1(\mathbb{C}^2 - S, u_0)$ come from the braid group of $\bar{\mathbb{C}}_1$ and $\bar{\mathbb{C}}^1 \cap S, B_q = B_q[\bar{\mathbb{C}}^1, \bar{\mathbb{C}}^1 \cap S]$ (see refs. 3 and 4). The theorem says that Π_1 is generated by a good system of generators for $\Pi_1(\mathbb{C}^1 - S, u_0)$. The relations are induced from the singularities of M via the braid monodromy $\Theta: \Pi_1(C - M, u) \rightarrow B_q$ (see ref. 4). We use explicit formulas for braid monodromy for branch curves corresponding to generic projections of surfaces X_{ab} . These formulas were obtained in ref. 5.

To compute the braid monodromy, a degeneration $f_0: X_0 \rightarrow \mathbb{C}^2$ of $f: X_{ab} \rightarrow \mathbb{C}^2$ was introduced. X_0 is a union of planes as shown in Fig. 1, in which each triangle corresponds to a projective plane and each 1-simplex corresponds to a projective line. Two planes in X_0 intersect by a line or by a point or do not intersect. The "branch curve" of f_0 is the union of the intersection lines. Denote it by S_0 . When redegnerating lines are doubled, we have twice as many points in $S \cap \mathbb{C}^1$ as we had in $S_0 \cap \mathbb{C}^1$. Let $p = \#S_0 \cap \mathbb{C}^1$. Thus, a good system

of generators for $\Pi_1(\mathbb{C}^1 - S, u_0)$ is $\{\Gamma_i, \Gamma_j\}_{i,j=1}^p$, in which each i corresponds to one of the lines in S_0 .

Each of the intersection lines is the intersection line of exactly two planes. We have a global numeration for the planes (Fig. 1). Take Γ_i . It corresponds to a line L_i . Assume L_i to be the intersection line of the planes numerated α and β . It turns out that $\Psi(\Gamma_j) = \Psi(\Gamma_j) =$ the transposition $(\alpha\beta)$.

$\bar{\Pi}_1 \xrightarrow{\Psi} S_n \rightarrow 1$ Splits. We have used a modified version of Artin's presentation of S_n . S_n is generated by the set of transpositions with the following relations: $[\alpha, \beta] = 1$ iff α and β have no common index, $\langle \alpha, \beta \rangle = (\alpha\beta\alpha\beta) = 1$ iff α and β have one common index, and $\langle \alpha_6 \rangle = \alpha_5\alpha_4\alpha_3\alpha_2\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ whenever $\alpha_6, \dots, \alpha_1$ is a cycle of transpositions in which each one has a common index with the following one and α_1 has a common index with α_6 .

Using the above presentation of $\bar{\Pi}_1$ we were able to choose a set of elements in $\bar{\Pi}_1$ that are preimages of the transpositions and satisfy the above relations and thus Ψ splits by ρ .

Using the Reidemeister-Schreier Method. The Reidemeister-Schreier method is applicable (p. 85 of ref. 6) since we have a finite presentation of $\bar{\Pi}_1$ and Ψ splits. By that we obtain a system of generators for $\mathcal{A}$ as follows: $A_{\sigma:j} = \rho(\sigma)\Gamma_j\Gamma_j^{-1}\rho(\sigma)^{-1}$, $j = 1, \dots, \sigma \in S_n$. Using various relations in $\bar{\Pi}_1$ we find that $\mathcal{A}$ is actually generated by $(A_{\sigma:2})$, $\sigma \in S_n$.

We obtained the following key relations in $\mathcal{A}$: (i) $A_{\sigma:2} = A_{\sigma^{-1}:2}^{-1}$, (ii) $A_{\sigma:2} = A_{\sigma':2}$ whenever σ and σ' agree on $\Psi(\Gamma_2)$, and (iii) $A_{\Gamma_1:2} = A_{\Gamma_2\Gamma_1:2}A_{1:2} = A_{1:2}A_{\Gamma_2\Gamma_1:2}$ and proved the commutativity of $\mathcal{A}$ based on these relations.

Each Generator Is of Finite Order. For X_{ab} we proved that each generator satisfies the condition $(A_{\sigma:2})^a = (A_{\sigma:2})^b = 1$. Thus $\mathcal{A}$ is finite. Furthermore, for a and b relatively prime $A_{\sigma:2} = 1 \forall \sigma$ and then $\mathcal{A} = 0$.

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