Reversible diffusion-influenced reactions of an isolated pair on some two dimensional surfaces

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We investigate reversible diffusion-influenced reactions of an isolated pair in two dimensions. To this end, we employ convolution relations that permit deriving the survival probability of the reversible reaction directly in terms of the survival probability of the irreversible reaction. Furthermore, we make use of the mean reaction time approximation to write the irreversible survival probability in restricted spaces as a single exponential. In this way, we obtain exact and approximative expressions in the time domain for the reversible survival probability for three different two dimensional spatial domains: The infinite plane, the annular domain, and the surface of a sphere. Our obtained results should prove useful in the context of membrane-bound reversible diffusion-influenced reactions in cell biology. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4830218>]

I. INTRODUCTION

Diffusion-influenced reactions¹ can be modeled by solutions of the Smoluchowski equation in the presence of certain types of boundary conditions (BC). Among all solutions, Green's functions (GF) are of particular importance, because they permit constructing the solution for any given initial distribution and they can be used to calculate other relevant quantities, for instance, the survival probability.²⁻⁴ However, in most cases, the GF is not directly observable, but its mathematical derivation can be somewhat involved. Therefore, it is worthwhile looking for alternative approaches that dispense with the GF of the Smoluchowski equation altogether and still permit the calculation of observable quantities in a less labor intensive way.

One example of such an approach is provided by convolution relations^{5,6} that relate the survival probability of the reversible reaction to the survival probability of the irreversible reaction. The advantage of this method lies in the fact that the survival probability of the irreversible reaction is already known for a number of cases. Hence, the reversible survival probability can be calculated directly in terms of the irreversible survival probability, without the need to solve the corresponding complicated initial-boundary problem of the Smoluchowski equation. Even if the irreversible survival probability is not known, it can be derived as the solution of its equation of motion, the Sano-Tachiya (S-T) equation,⁷ which is also more easily solved than the underlying Smoluchowski equation.

The convolution relations provide another advantage. In the case of restricted spatial domains, the survival probability of the irreversible reaction is typically given by a sum of exponentials with various decay times. By means of the so-

called mean reaction time approximation,⁸ and except at very short times, the survival probability can be written as a single exponential with a characteristic decay time that can be identified with the mean reaction time. Combining these approximative expressions for the irreversible survival probability with the convolution relations immediately generates approximative expressions for the reversible survival probability as well.

In this paper, we will use these methods to study the reversible diffusion-influenced reaction of an isolated pair in two dimensions (2D). Diffusion in 2D is of particular interest for several reasons. Conceptually, it is distinct because 2D is the critical dimension regarding recurrence and transience of random walks⁹ and because the steady-state solution of the diffusion equation is inconsistent with the boundary condition at infinity.¹⁰ In view of biological applications, the theoretical treatment of the 2D case facilitates a better understanding of reversible membrane-bound reactions in cell biology, shedding new light on phenomena like signal-induced heterogeneities and receptor clustering.¹¹ Finally, from a more technical point of view, the mathematical treatment of the 2D case appears more cumbersome in general than its 1D and 3D counterparts,^{4,12} emphasizing the need for less labor intensive methods.

The paper is organized as follows. After introducing the general theoretical context, we will discuss the central convolution relations. We will use these relations to obtain exact and approximative solutions for three different 2D spatial domains: The infinite plane, the annular domain, and the surface of a sphere. Although the case of the infinitely extended plane was already solved by using a GF of the 2D Smoluchowski equation,⁴ the cases of the annular domain and the spherical surface have not yet been solved. Here we solve all the three cases by using the convolution relations. Finally, we will compare and discuss the results. In the Appendix, we will present an alternative derivation of the convolution relations.

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II. THEORY

A. Reversible geminate recombination

We consider an isolated pair of two disklike molecules A and B with diffusion constants D_A and D_B , respectively. The molecules may bind when their separation equals the encounter distance a to form a bound molecule AB . In the case of a reversible reaction, the bound molecule may dissociate to form an unbound pair $A + B$ again. This system can equivalently be described as the diffusion of a point-like particle with diffusion constant $D = D_A + D_B$ around a static disk with radius a . Reactions are introduced by imposing boundary conditions at the disk's surface. The probability density function (PDF) $p(r, t|r_0)$ yields the probability to find the particles at a separation r at time t , provided that the separation was initially r_0 at time $t = 0$. We assume that the system is centrosymmetric and that the interaction potential vanishes. Then, the time evolution of $p(r, t|r_0)$ is governed by the Smoluchowski equation.¹³ More precisely, $p(r, t|r_0)$ is the GF for the following initial-value problem:

$$\frac{\partial}{\partial t} p(r, t|r_0) = \mathcal{L}_r p(r, t|r_0), \quad b > r > a, \quad (1)$$

$$p(r, 0|r_0) = \frac{\delta(r - r_0)}{2\pi r}, \quad (2)$$

where the differential operator $\mathcal{L}_r$ is defined by

$$\mathcal{L}_r = \frac{D}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r}. \quad (3)$$

Note that $b = \infty$ in the case of the infinitely extended plane.

Furthermore, one has to specify a BC at the “outer” boundary. In the case of the infinitely extended plane, one requires the BC to be as follows:

$$\lim_{r \rightarrow \infty} p(r, t|r_0) = 0, \quad (4)$$

while for the annular domain one imposes the BC as

$$\left. \frac{\partial}{\partial r} p(r, t|r_0) \right|_{r=b} = 0. \quad (5)$$

Regardless of the underlying spatial domain, the GF describing the diffusion-influenced reaction is only defined for $r \geq a > 0$ and one has to specify a BC for $r = a$ that takes into account the behavior at the encounter distance. The irreversible association reaction is taken into account by the radiation BC¹⁴ that is characterized by an intrinsic association constant κ_a ,

$$-J(a, t|r_0) = \kappa_a p(a, t|r_0). \quad (6)$$

Here, $J(r, t|r_0)$ refers to the total diffusional flux:

$$-J(r, t|r_0) = 2\pi r D \frac{\partial}{\partial r} p(r, t|r_0). \quad (7)$$

To describe reversible reactions, the radiation BC has to be generalized to the back-reaction BC that involves additionally an intrinsic dissociation constant κ_d ,²⁻⁴

$$-J(a, t|r_0) = \kappa_a p(a, t|r_0) - \kappa_d [1 - S(t|r_0)]. \quad (8)$$

Note that for $\kappa_d = 0$, Eq. (8) reduces to the radiation BC Eq. (6), as it should.

Knowledge of the GF permits to derive further important quantities, in particular the survival probability $S(t|r_0)$, i.e., the probability that a pair of molecules with initial distance r_0 is separated at time t :

$$S(t|r_0) = 2\pi \int_a^b p(r, t|r_0) r dr, \quad (9)$$

where $b = \infty$ in the case of the infinitely extended plane.

So far, the theory was formulated for the initially unbound state, but quite analogously one can treat the case of an initially bound state. The corresponding PDF is denoted by $p(r, t|*)$ and it also satisfies the Smoluchowski equation (Eq. (1)). However, the initial condition is now

$$p(r, t = 0|*) = 0, \quad (10)$$

and the back-reaction BC becomes

$$-J(a, t|*) = \kappa_a p(a, t|*) - \kappa_d [1 - S(t|*)]. \quad (11)$$

The “survival probability” $S(t|*)$ is the probability that an initially bound pair of molecules is found unbound at time t :

$$S(t|*) := 2\pi \int_a^b p(r, t|*) r dr. \quad (12)$$

The notion of a “survival probability” is somewhat misleading in the reversible case.^{4,6} For the irreversible reaction, a non-vanishing probability to observe an unbound isolated pair at a time t necessarily means that no association reaction has occurred before. For the reversible reaction, by contrast, association and subsequent dissociation events may occur possibly many times before t and hence contribute to the probability of finding an isolated pair separated at a time t . Therefore, the quantities $S(t|r_0)$ and $S(t|*)$ could be more precisely referred to as a separation probability of the initially unbound and bound pairs, respectively.⁶ Nevertheless, the term “survival probability” has been commonly used in the literature³ also for the reversible case. In the following, we will use survival probability interchangeably with separation probability.

We would like to point out that we will make use of the notation $p_{\text{irr}}(r, t|r_0)$, $S_{\text{irr}}(t|r_0)$ and $p_{\text{rev}}(r, t|r_0)$, $p_{\text{rev}}(r, t|*)$, $S_{\text{rev}}(t|r_0)$, $S_{\text{rev}}(t|*)$ to refer to the GF and survival probability in the case of the irreversible reaction and to those in the case of the reversible reaction, respectively. The GF and survival probability in the case of the irreversible reaction are assumed to satisfy a radiation BC (Eq. (6)).

Turning to the case of a spherical surface, we consider an isolated pair of sphere-like particles with encounter radius a diffusing on the surface of a sphere of radius u , cf. Ref. 8 and Fig. 1. Without loss of generality, one particle can be thought of as fixed at the south pole $\theta = \pi$, while the other particle diffuses around characterized by the diffusion constant $D = D_A + D_B$. The position of the diffusing particle is determined by the angle θ , or equivalently by $z = \cos \theta$. Now, the initial-value problem takes the form⁸

$$\frac{\partial}{\partial t} p_{\text{rev}}(z, t|z_0) = \mathcal{L}_z p_{\text{rev}}(z, t|z_0), \quad 1 > z > -\cos \alpha, \quad (13)$$

FIG. 1. Geometry of an isolated pair of molecules on a spherical surface.

$$p_{\text{rev}}(z, 0|z_0) = \frac{\delta(z - z_0)}{2\pi R^2}, \quad (14)$$

where the differential operator $\mathcal{L}_z$ is defined by

$$\mathcal{L}_z = \frac{D}{R^2} \frac{\partial}{\partial z} \left[(1 - z^2) \frac{\partial}{\partial z} \right]. \quad (15)$$

The particles are at contact if $z = z_a \equiv -\cos \alpha$, see Fig. 1, and the back-reaction BC reads

$$\begin{aligned} 2\pi a \frac{D}{R} \sin \alpha \frac{\partial}{\partial z} p_{\text{rev}}(z, t|z_0)|_{z=z_a} \\ = \kappa_a p_{\text{rev}}(z = z_a, t|z_0) - \kappa_d [1 - S_{\text{rev}}(t|z_0)], \end{aligned} \quad (16)$$

where the survival probability takes the form

$$S_{\text{rev}}(t|z_0) = 2\pi R^2 \int_{-\cos \alpha}^1 p_{\text{rev}}(z, t|z_0) dz. \quad (17)$$

Again, for $\kappa_d \rightarrow 0$, one recovers from Eq. (16) the radiation BC.⁸ The “outer” BC at $z = 1$ is

$$\lim_{z \rightarrow 1} p_{\text{rev}}(z, t|z_0) < \infty. \quad (18)$$

Again, the case of the initially bound state can be treated analogously to the case of the initially bound state in terms of the GF $p_{\text{rev}}(z, t|*)$ that satisfies Eq. (13), the initial condition

$$p_{\text{rev}}(z, t|*) = 0, \quad (19)$$

and the corresponding BC. The survival probability is now given by

$$S_{\text{rev}}(t|*) = 2\pi R^2 \int_{-\cos \alpha}^1 p_{\text{rev}}(z, t|*) dz. \quad (20)$$

B. Survival probabilities and convolution relations

Instead of calculating the GF corresponding to the quite complicated initial- and boundary-value problems for the initially unbound and bound states described in Sec. II A, we will employ convolution equations relating $S_{\text{rev}}(t|*)$ to $S_{\text{irr}}(t|r_0)$:

$$S_{\text{rev}}(t|*) = \kappa_d \int_0^t P_{\text{rev}}(\tau|*) S_{\text{irr}}(t - \tau|a) d\tau, \quad (21)$$

$$\frac{\partial P_{\text{rev}}(t|*)}{\partial t} = -\kappa_d P_{\text{rev}}(t|*) - \kappa_d \int_0^t P_{\text{rev}}(\tau|*) \frac{\partial S_{\text{irr}}(t - \tau|a)}{\partial t} d\tau, \quad (22)$$

where $P_{\text{rev}}(t|*)$ denotes the fraction of the bound state at time t :

$$P_{\text{rev}}(t|*) = 1 - S_{\text{rev}}(t|*). \quad (23)$$

Equations (21) and (22) were first derived in Ref. 5, see also Ref. 6. Note that in Ref. 5 the convolution equations are actually formulated in a more general form that also takes into account the possibility of a decay of the bound state according to the reaction scheme $A + B \rightleftharpoons C \rightarrow D$.

The convolution relations, Eqs. (21) and (22), can be derived as follows. First, we consider Eq. (21). The amount of the bound state which is converted to the unbound state between time τ and $\tau + d\tau$ is given by $\kappa_d P_{\text{rev}}(\tau|*) d\tau$. When the bound state is converted to the unbound state, the latter is formed with the two constituent reactants separated by the reaction distance a . The probability that the unbound state which is formed at time τ with the two constituent reactants separated by a is still alive at time t is given by $S_{\text{irr}}(t - \tau|a)$. Therefore, the fraction of the unbound state is given by Eq. (21).

Let us now turn to Eq. (22). The first term on the rhs represents the decay of the bound state between t and $t + dt$. The second term represents the formation of the bound state between t and $t + dt$. It is calculated in the following way. The amount of the bound state which is converted to the unbound state between time τ and $\tau + d\tau$ is given by $\kappa_d P_{\text{rev}}(\tau|*) d\tau$. When the bound state is converted to the unbound state, the latter is formed with the two constituent reactants separated by the reaction distance a . The probability that the unbound state which is formed at time τ with the two constituent reactants separated by a is still alive at time t is given by $S_{\text{irr}}(t - \tau|a)$. When the unbound state decays, the bound state is formed. The rate at which the unbound state formed between time τ and $\tau + d\tau$ with the two constituent reactants separated by a will decay to form the bound state at time t is given by $-\partial S_{\text{irr}}(t - \tau|a)/\partial t$. Therefore, the formation rate of the bound state is given by the second term on the rhs of Eq. (22).

Employing the method used in Ref. 5 to derive Eqs. (21) and (22), one can derive analogous convolution equations for the initially unbound state:⁶

$$S_{\text{rev}}(t|r_0) = S_{\text{irr}}(t|r_0) + \kappa_d \int_0^t P_{\text{rev}}(\tau|r_0) S_{\text{irr}}(t - \tau|a) d\tau, \quad (24)$$

$$\begin{aligned} \frac{\partial P_{\text{rev}}(t|r_0)}{\partial t} = & -\kappa_d P_{\text{rev}}(t|r_0) \\ & - \kappa_d \int_0^t P_{\text{rev}}(\tau|r_0) \frac{\partial S_{\text{irr}}(t - \tau|a)}{\partial t} d\tau - \frac{\partial S_{\text{irr}}(t|r_0)}{\partial t}, \end{aligned} \quad (25)$$

where

$$P_{\text{rev}}(t|r_0) = 1 - S_{\text{rev}}(t|r_0). \quad (26)$$

In addition to the derivation discussed above, we will give an alternative derivation of the convolution relations, Eqs. (21), (22), (24), and (25), in the Appendix.

One of the virtues of the convolution relations (Eqs. (21) and (24)) is that they can be easily solved in the Laplace domain.

$$\tilde{S}_{\text{rev}}(s|*) = \frac{\kappa_d}{s} \frac{\tilde{S}_{\text{irr}}(s|a)}{1 + \kappa_d \tilde{S}_{\text{irr}}(s|a)}, \quad (27)$$

$$\tilde{S}_{\text{rev}}(s|r_0) = \frac{1}{s} + \frac{\tilde{S}_{\text{irr}}(s|r_0) - s^{-1}}{1 + \kappa_d \tilde{S}_{\text{irr}}(s|a)}. \quad (28)$$

The survival probability $S_{\text{irr}}(t|r_0)$ describing the irreversible case is already known for a number of different cases. Otherwise, $S_{\text{irr}}(t|r_0)$ can be found as the solution of the S-T equation. In the following, we will exploit this fact to calculate $S_{\text{rev}}(t|*)$, $S_{\text{rev}}(t|r_0)$ in a less labor intensive way than via the use of GF.

III. CALCULATION OF THE SURVIVAL PROBABILITY

A. Infinitely extended plane

First, we consider an isolated pair reversibly reacting in the infinitely extended plane. The survival probability in the irreversible case is known, see Chap. 13.5, Eq. (13) in Ref. 15.

$$s \tilde{S}_{\text{irr}}(s|r_0) = 1 - \frac{h K_0(qr_0)}{q K_1(qa) + h K_0(qa)}, \quad (29)$$

where K_0 , K_1 refer to the modified Bessel functions of second kind, Sec. 9.6 in Ref. 16, and we defined

$$q = \sqrt{\frac{s}{D}}, \quad h = \frac{\kappa_a}{2\pi a D}. \quad (30)$$

Inserting the expression for $\tilde{S}_{\text{irr}}(s|r_0)$ in Eq. (28) yields

$$s \tilde{S}_{\text{rev}}(s|r_0) = 1 - \frac{hq K_0(qr_0)}{(q^2 + \kappa_D) K_1(qa) + hq K_0(qa)}, \quad (31)$$

where

$$\kappa_D = \frac{\kappa_d}{D}. \quad (32)$$

The inversion theorem for the Laplace transformation can be applied to find the corresponding expression for $\tilde{S}_{\text{rev}}(s|r_0)$ in the time domain

$$S_{\text{rev}}(t|r_0) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \tilde{S}_{\text{rev}}(s|r_0) ds. \quad (33)$$

To calculate the Bromwich contour integral we first note that $\tilde{S}_{\text{rev}}(s|r_0)$ is multi-valued and has a branch point at $s = 0$. Therefore, we use the contour shown in Fig. 2 with a branch cut along the negative real axis, cf. Chap. 12.3, Fig. 40 in Ref. 15. Furthermore, we note that the $1/s$ term yields 1 and

FIG. 2. Integration contour used in Eq. (34).

hence, we obtain

$$2\pi i [S_{\text{rev}}(t|r_0) - 1] = - \int_{C_2} e^{st} \tilde{S}_{\text{rev}}(s|r_0) ds - \int_{C_4} e^{st} \tilde{S}_{\text{rev}}(s|r_0) ds. \quad (34)$$

Thus, it remains to calculate the integrals $\int_{C_2}$, $\int_{C_4}$. To this end, we choose $s = Dx^2 e^{i\pi}$ and use Appendix 3, Eq. (25) in Ref. 15:

$$K_n(xe^{\pm\pi i/2}) = \pm \frac{1}{2} \pi i e^{\mp n\pi i/2} [-J_n(x) \pm i Y_n(x)], \quad (35)$$

where $J_n(x)$, $Y_n(x)$ denote the Bessel functions of first and second kind, respectively (Sec. 9.1 in Ref. 16). It follows that

$$\int_{C_2} e^{st} \tilde{S}_{\text{rev}}(s|r_0) ds = -2h \int_0^\infty e^{-Dx^2 t} [J_0(xr_0) - i Y_0(xr_0)] \times \frac{\alpha(x) + i\beta(x)}{\alpha(x)^2 + \beta(x)^2} dx. \quad (36)$$

Here, we have defined

$$\alpha(x) = (x^2 - \kappa_D) J_1(xa) + hx J_0(xa), \quad (37)$$

$$\beta(x) = (x^2 - \kappa_D) Y_1(xa) + hx Y_0(xa). \quad (38)$$

To evaluate the integral along the contour C_4 we choose $s = Dx^2 e^{-i\pi}$ and after an analogous calculation one finds that

$$\int_{C_4} e^{st} \tilde{S}_{\text{rev}}(s|r_0) ds = - \left(\int_{C_2} e^{st} \tilde{S}_{\text{rev}}(s|r_0) ds \right)^*, \quad (39)$$

where * means complex conjugation. Thus, one finally arrives at the exact expression for the survival probability in the time domain

$$S_{\text{rev}}(t|r_0) = 1 - a \int_0^\infty e^{-Dx^2 t} P(x, a) T(x, r_0) dx, \quad (40)$$

where⁴

$$T(x, r_0) = \frac{J_0(xr_0)\beta - Y_0(xr_0)\alpha}{[\alpha^2 + \beta^2]^{1/2}}, \quad (41)$$

$$P(x, r_0) = -\frac{1}{x} \frac{\partial}{\partial r_0} T(x, r_0) = \frac{J_1(xr_0)\beta - Y_1(xr_0)\alpha}{[\alpha^2 + \beta^2]^{1/2}}. \quad (42)$$

Next, we turn to the case of the initially bound state and use Eqs. (27) and (29) to obtain

$$\tilde{S}_{\text{rev}}(s|*) = \frac{\kappa_D}{s} \frac{K_1(qa)}{(q^2 + \kappa_D)K_1(qa) + hqK_0(qa)}. \quad (43)$$

We can use again the inversion theorem (Eq. (33)) to calculate the expression for $S_{\text{rev}}(t|*)$ in the time domain. The actual calculation is very similar to the one presented for $S_{\text{rev}}(t|r_0)$, therefore we only give the result

$$S_{\text{rev}}(t|*) = 1 - 2\pi \frac{\kappa_d}{\kappa_a} a^2 \int_0^\infty e^{-Dx^2 t} P^2(x, a) \frac{dx}{x}. \quad (44)$$

A comparison with Ref. 4, where Eqs. (40) and (44) were obtained by first calculating the GF, shows that the exact expressions for $S_{\text{rev}}(t|*)$, $S_{\text{rev}}(t|r_0)$ can be more easily derived via the route of the convolution relations.

B. Annular domain

Next, we consider the case of the annular domain. The expression for $S_{\text{irr}}(s|r_0)$ is known (Chap. 13.4, Eqs. (3) and (4) in Ref. 15):

$$\begin{aligned} \tilde{S}_{\text{irr}}(s|r_0) &= \frac{1}{s} + \frac{h}{s} \frac{[K_1(qb)I_0(qr_0) + I_1(qb)K_0(qr_0)]}{K_1(qb)[qI_1(qa) - hI_0(qa)] - I_1(qb)[qK_1(qa) + hK_0(qa)]}. \end{aligned} \quad (45)$$

Again we employ Eq. (28) to arrive at the survival probability in the Laplace domain:

$$\tilde{S}_{\text{rev}}(s|r_0) = \frac{1}{s} + \frac{hq[K_1(qb)I_0(qr_0) + I_1(qb)K_0(qr_0)]}{s\Delta(s)}, \quad (46)$$

where

$$\Delta(s) = \psi(qa)K_1(qb) - \phi(qa)I_1(qb) \quad (47)$$

and

$$\psi(qa) = (q^2 + \kappa_D)I_1(qa) - hqI_0(qa), \quad (48)$$

$$\phi(qa) = (q^2 + \kappa_D)K_1(qa) + hqK_0(qa). \quad (49)$$

Next, we again apply the inversion theorem (Eq. (33)) to calculate the survival probability in the time domain. In contrast to the case of the infinite plane, the integrand Eq. (46) is now single-valued and we choose the integration contour depicted in Fig. 3. Then, the Bromwich contour integral is

FIG. 3. Integration contour used in Eq. (50).

determined by

$$\begin{aligned} S_{\text{rev}}(t|r_0) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \tilde{S}_{\text{rev}}(s|r_0) ds \\ &= \sum_n \text{Res}_{s_n} [e^{st} \tilde{S}_{\text{rev}}(s|r_0)], \end{aligned} \quad (50)$$

where the sum goes over all poles of $e^{st} \tilde{S}_{\text{rev}}(s|r_0)$. The second term in Eq. (46) has a simple pole at $s_0 = 0$ and non-vanishing simple poles at $s_n = -D\xi_n^2$, i.e., one has

$$\Delta(s) \Big|_{s=-D\xi_n^2} = 0, \quad (51)$$

where $\pm\xi_n \neq 0$ are the roots of

$$\alpha(\xi_n)Y_1(\xi_n b) - \beta(\xi_n)J_1(\xi_n b) = 0, \quad (52)$$

with $\alpha(x)$, $\beta(x)$ defined by Eqs. (37) and (38). To arrive at Eq. (52), we used Eq. (51) as well as Eq. (35) and

$$I_n(xe^{\pm\pi i/2}) = e^{\pm n\pi i/2} J_n(x), \quad (53)$$

cf. Appendix 3, Eq. (26) in Ref. 15.

The residue at the pole $s = 0$ can be determined by employing the small argument expansion of the modified Bessel functions, see Sec. 9.6, Eqs. (9.6.7)–(9.6.9) in Ref. 16. We obtain

$$\begin{aligned} \text{Res}_{s=0} \left[e^{st} \frac{hq[K_1(qb)I_0(qr_0) + I_1(qb)K_0(qr_0)]}{s\Delta(s)} \right] \\ = \frac{ha}{\frac{\kappa_D}{2}(a^2 - b^2) - ha}. \end{aligned} \quad (54)$$

Taking into account the contribution from the $1/s$ term in Eq. (46), we arrive at

$$\begin{aligned} S(t|r_0) &= \frac{\pi(b^2 - a^2)}{\pi(b^2 - a^2) + K_a} \\ &= \sum_{n \neq 0} \text{Res}_{s_n} \left[e^{st} \frac{hq[K_1(qb)I_0(qr_0) + I_1(qb)K_0(qr_0)]}{s\Delta(s)} \right], \end{aligned} \quad (55)$$

where we introduced the following equilibrium constant:

$$K_a = \frac{\kappa_a}{\kappa_d}. \quad (56)$$

To calculate the remaining residues at the non-vanishing poles $s_n = -D\xi_n^2$, we use

$$\begin{aligned} \text{Res}_{s_n = -D\xi_n^2} \left[e^{st} \frac{hq[K_1(qb)I_0(qr_0) + I_1(qb)K_0(qr_0)]}{s\Delta(s)} \right] \\ = e^{st} \frac{hq[K_1(qb)I_0(qr_0) + I_1(qb)K_0(qr_0)]}{s \frac{d}{ds} \Delta(s)} \Big|_{s=-D\xi_n^2}. \end{aligned} \quad (57)$$

The calculation of the denominator

$$s \frac{d}{ds} \Delta(s) \Big|_{s=-D\xi_n^2} = \frac{1}{2} q \frac{d}{dq} \Delta(s) \Big|_{q=i\xi_n} \quad (58)$$

is greatly facilitated by the identities

$$\begin{aligned} \frac{\psi(qa)}{I_1(qb)} \Big|_{q=i\xi_n} &= \frac{\phi(qa)}{K_1(qb)} \Big|_{q=i\xi_n} = -\frac{\alpha(x)}{J_1(xb)} \Big|_{x=\xi_n} \\ &= -\frac{\beta(x)}{Y_1(xb)} \Big|_{x=\xi_n} \equiv \rho, \end{aligned} \quad (59)$$

which follow from Eqs. (51)–(53). Moreover, we make use of the following identities:

$$\begin{aligned} xI'_v(x) + vI_v(x) &= xI_{v-1}(x), \\ xK'_v(x) + vK_v(x) &= -xK_{v-1}(x), \\ I_v(x)K'_v(x) - K_v(x)I'_v(x) &= -\frac{1}{x}, \\ I_v(x)K_{v+1}(x) + K_v(x)I_{v+1}(x) &= \frac{1}{x}, \end{aligned} \quad (60)$$

see Appendix 3, Eqs. (13), (15), (21), and (22) in Ref. 15, and find in this way

$$s \frac{d}{ds} \Delta(s) \Big|_{s=-D\xi_n^2} = \frac{\alpha^2(\xi_n) - J_1^2(\xi_n b) \varkappa}{2\alpha(\xi_n)J_1(\xi_n b)}, \quad (61)$$

where

$$\varkappa = (\xi_n^2 - \kappa_D)^2 - 2a^{-1}h\kappa_D + h^2\xi_n^2. \quad (62)$$

The numerator on the rhs of Eq. (57) can be evaluated by use of Eqs. (35) and (53). Everything taken together, we obtain for the survival probability in an annular domain

$$\begin{aligned} S_{\text{rev}}(t|r_0) &= \frac{\pi(b^2 - a^2)}{\pi(b^2 - a^2) + K_a} \\ &\quad - h\pi \sum_{n \neq 0} e^{-D\xi_n^2 t} \xi_n \frac{C(\xi_n r_0, \xi_n b) \alpha(\xi_n) J_1(\xi_n b)}{\alpha^2(\xi_n) - J_1^2(\xi_n b) \varkappa}, \end{aligned} \quad (63)$$

where

$$C(xr_0, xb) = J_0(xr_0)Y_1(xb) - Y_0(xr_0)J_1(xb). \quad (64)$$

Finally, we consider the case of the initially bound state. Equations (27) and (45) yield

$$\tilde{S}_{\text{rev}}(s|*) = \frac{\kappa_D[K_1(qb)I_1(qa) - I_1(qb)K_1(qa)]}{s\Delta(s)}. \quad (65)$$

The corresponding expression in the time domain can be found along similar lines as presented for $S_{\text{rev}}(t|r_0)$; therefore, we only provide the following result:

$$\begin{aligned} S_{\text{rev}}(t|*) &= \frac{\pi(b^2 - a^2)}{\pi(b^2 - a^2) + K_a} \\ &\quad - \kappa_D \pi \sum_{n \neq 0} e^{-D\xi_n^2 t} \frac{G(\xi_n a, \xi_n b) \alpha(\xi_n) J_1(\xi_n b)}{\alpha^2(\xi_n) - J_1^2(\xi_n b) \varkappa}, \end{aligned} \quad (66)$$

where

$$\begin{aligned} G(xr_0, xb) &= J_1(xr_0)Y_1(xb) - Y_1(xr_0)J_1(xb) \\ &= -\frac{1}{x} \frac{\partial}{\partial r_0} C(xr_0, xb). \end{aligned} \quad (67)$$

C. Spherical surface

By making use of the S-T equation, the survival probability in the irreversible case has been calculated in Ref. 8,

$$S_{\text{irr}}(t|z) = \sum_{n=1}^{\infty} c_n e^{-t/\tau_n} P_{v_n}(z), \quad (68)$$

where the functions P_{v_i} denote those fundamental solutions of the Legendre differential equation that are finite at $z = 1$. The numbers v_i are the roots of the radiation boundary condition of the S-T equation. Furthermore, we have

$$\tau_i = \frac{\tau}{v_i(v_i + 1)}, \quad \tau = \frac{R^2}{D}, \quad (69)$$

and

$$c_i = \frac{\int_{-\cos \alpha}^1 P_{v_i}(z) dz}{\int_{-\cos \alpha}^1 P_{v_i}^2(z) dz}. \quad (70)$$

Using the Laplace transform of Eq. (68), we have

$$\tilde{S}_{\text{irr}}(s|z) = \sum_{n=1}^{\infty} \frac{c_n P_{v_n}(z)}{s + \tau_n^{-1}}, \quad (71)$$

and using Eqs. (27) and (28), we can immediately obtain exact expressions for $\tilde{S}_{\text{rev}}(s|z)$, $\tilde{S}_{\text{rev}}(s|*)$ in the Laplace domain.

However, to derive expressions in the time domain we will take another route. As described in more detail in Ref. 8, the survival probability in a restricted space is typically given by a sum of exponentials that can be approximated by a single exponential, with the exception of very short times:

$$S_{\text{irr,app}}(t|z) = \exp \left[-\frac{t}{\tau(z)} \right], \quad (72)$$

where the decay time $\tau(z)$ may be identified with the mean reaction time, which is given by⁸

$$\tau(z) = \int_0^{\infty} t \frac{\partial}{\partial t} [1 - S_{\text{irr}}(t|z)] dt = \int_0^{\infty} S_{\text{irr}}(t|z) dt. \quad (73)$$

Now, the Laplace transform of the approximative expression (Eq. (72)) is simple:

$$\tilde{S}_{\text{irr, app}}(s|z) = \frac{1}{s + \tau(z)^{-1}}, \quad (74)$$

and via Eqs. (27) and (28) we immediately obtain

$$\tilde{S}_{\text{rev, app}}(s|*) = \frac{\kappa_d}{s} \frac{1}{s + \tau^{-1}(z_a) + \kappa_d}, \quad (75)$$

$$\begin{aligned} \tilde{S}_{\text{rev, app}}(s|z) &= \frac{1}{s + \tau^{-1}(z)} \left[1 - \frac{\kappa_d}{s + \tau^{-1}(z_a) + \kappa_d} \right] \\ &+ \tilde{S}_{\text{rev, app}}(s|*), \end{aligned} \quad (76)$$

where $z_a = -\cos \alpha$. Then, the corresponding expressions in the time domain become

$$S_{\text{rev, app}}(t|*) = \frac{\kappa_d}{\kappa_d + \tau^{-1}(z_a)} \left[1 - e^{-t[\tau^{-1}(z_a) + \kappa_d]} \right], \quad (77)$$

$$\begin{aligned} S_{\text{rev, app}}(t|z) &= \frac{\kappa_d}{\kappa_d + \tau^{-1}(z_a)} + e^{-t/\tau(z)} \frac{\tau^{-1}(z) - \tau^{-1}(z_a)}{\tau^{-1}(z) - \tau^{-1}(z_a) - \kappa_d} \\ &- e^{-t[\kappa_d + \tau(z_a)^{-1}]} \left[\frac{\kappa_d}{\kappa_d + \tau^{-1}(z_a)} + \frac{\kappa_d}{\tau^{-1}(z) - \tau^{-1}(z_a) - \kappa_d} \right]. \end{aligned} \quad (78)$$

Note that

$$S_{\text{rev, app}}(t = 0|*) = 0, \quad (79)$$

$$S_{\text{rev, app}}(t = 0|z) = 1, \quad (80)$$

as it should be. Furthermore, for the steady-state we obtain

$$\lim_{t \rightarrow \infty} S_{\text{rev, app}}(t|*) = \lim_{t \rightarrow \infty} S_{\text{rev, app}}(t|z) = \frac{\kappa_d}{\kappa_d + \tau^{-1}(z_a)}, \quad (81)$$

and find that the steady-state is independent of the initial state, as expected.

It is now instructive to calculate $S_{\text{rev, app}}(t|r_0)$, $S_{\text{rev, app}}(t|*)$ for the annular domain and to compare the results with the case of a sphere. To this end, we first note that the approximate expressions, Eqs. (77) and (78), for the survival probabilities are still valid also in the case of the annular domain, due to the general nature of the relations, Eqs. (21) and (24), and the approximation (Eq. (72)). It remains to calculate the mean passage time $\tau(r_0)$ for the case of the annular domain. By means of Eq. (73) it follows that

$$\tau(r_0) = \lim_{s \rightarrow 0} \tilde{S}_{\text{irr}}(s|r_0). \quad (82)$$

Using Eq. (45) and the series expansion of I_ν and K_ν , Sec. 9.6, Eqs. (9.6.10)–(9.6.11) in Ref. 16, we obtain

$$\tau^P(r_0) = \frac{b^2}{2D} \ln(r_0/a) - \frac{1}{4D} (r_0^2 - a^2) + \frac{1}{2haD} (b^2 - a^2). \quad (83)$$

There is an alternative way of deriving Eq. (83). The mean reaction time $\tau(r_0)$ is related to $S_{\text{irr}}(t|r_0)$ through Eq. (73). On the other hand, $S_{\text{irr}}(t|r_0)$ satisfies the S-T equation. Therefore,

the mean reaction time can be calculated from the differential equation and the associated boundary conditions obtained by integrating the S-T equation over t from 0 to infinity.^{8,17}

Henceforth, we will use $\tau^P(r_0)$, $\tau^S(z)$ to denote the mean reaction time for the case of the annular domain and of the sphere, respectively. The mean reaction time of an isolated pair on a spherical surface has been calculated in Ref. 8. In the notation of the present article, one has at contact, $z = -\cos \alpha = -1 + \frac{a^2}{2R^2}$,

$$\tau^S(z = -\cos \alpha) = 4\pi \frac{R^2}{\kappa_a} \left[1 - \frac{a^2}{4R^2} \right]^{1/2}. \quad (84)$$

Now, to facilitate a comparison between the two cases, we follow Ref. 8 and consider a pair of molecules in a circle of radius $b = 2R$ such that the area available for diffusion is equal to that on the surface of a sphere with radius R . Then, by virtue of Eq. (83) the mean reaction time becomes at contact $r_0 = a$,

$$\tau^P(r_0 = a) = 4\pi \frac{R^2}{\kappa_a} \left[1 - \frac{a^2}{4R^2} \right]. \quad (85)$$

Clearly, one has

$$\tau^S(z = -\cos \alpha) > \tau^P(r_0 = a). \quad (86)$$

Using this relation and Eqs. (77) and (78), we can compare the survival probabilities in the case of the annular domain and the sphere. In particular, we find for the ultimate fate of the molecule pair

$$\lim_{t \rightarrow \infty} S_{\text{rev, app}}^S(t|*) > \lim_{t \rightarrow \infty} S_{\text{rev, app}}^P(t|*). \quad (87)$$

IV. CONCLUDING REMARKS

In this article, we investigated the reversible diffusion-influenced reaction of an isolated pair in two dimensions for the case of an infinite and annular domain and the surface of a sphere. As the central theoretical tool we employed the convolution relations, Eqs. (21) and (24), that permit to express the survival probability of the reversible reaction directly in terms of the survival probability of the irreversible reaction, which is already known for many cases. Compared to approaches that involve the explicit derivation of the GF of the underlying Smoluchowski equation, the discussed method considerably reduces the complexity of the necessary calculations as demonstrated by the derivation of the exact expressions, Eqs. (40), (44), (63), and (66), for the case of the infinite plane and the annular domain, respectively. Moreover, we combined the approach based on the convolution relations with the mean reaction time approximation method that gives the irreversible survival probability in restricted spaces as a single exponential (Eq. (72)). Thus, we derived approximate expressions in the time domain for the reversible survival probability for the case of the surface of a sphere, Eqs. (77) and (78). We derived analogous approximate expressions for the survival probability in the case of the annular domain. In this way, we could readily compare the influence of the underlying spatial domains on the behavior of the reversible diffusion-influenced system. In particular, we found that the

ultimate separation probability of an isolated pair is larger in the case of a sphere than in the case of an annular domain (Eq. (87)).

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APPENDIX: ALTERNATIVE DERIVATION OF THE CONVOLUTION RELATIONS

In this appendix, we present an alternative derivation of the convolution equations, Eqs. (21), (22), (24), and (25). Without loss of generality, we consider the case of the infinitely extended plane.

We start by considering the reversible GF $p_{\text{rev}}(r, t|r_0)$ and the irreversible GF $p_{\text{irr}}(r, t|r_0)$ that satisfy both the initial-value problem (Eqs. (1) and (2)). The GF $p_{\text{irr}}(r, t|r_0)$ is subject to the radiation BC (Eq. (6)), which describes an irreversible association, instead of the back-reaction BC (Eq. (8)).

For the irreversible GF, we also consider the adjoint form of the Smoluchowski equation and of the radiation BC, i.e.,

$$-\frac{\partial p_{\text{irr}}(r', t'|r, t)}{\partial t} = \mathcal{L}_r p_{\text{irr}}(r', t'|r, t) \quad (\text{A1})$$

and

$$2\pi a D \frac{\partial p_{\text{irr}}(r', t'|r, t)}{\partial r} \Big|_{r=a} = \kappa_a p_{\text{irr}}(r', t'|a, t). \quad (\text{A2})$$

Now, we multiply Eq. (1) with $p_{\text{irr}}(r', t'|r, t)$ and Eq. (A1) with $p_{\text{rev}}(r, t|r_0)$ and subtract the resulting equations from each other. The result is

$$\begin{aligned} & \frac{\partial}{\partial t} [p_{\text{irr}}(r', t'|r, t) p_{\text{rev}}(r, t|r_0)] \\ &= D \frac{1}{r} \frac{\partial}{\partial r} \left[p_{\text{irr}}(r', t'|r, t) r \frac{\partial}{\partial r} p_{\text{rev}}(r, t|r_0) \right. \\ & \quad \left. - p_{\text{rev}}(r, t|r_0) r \frac{\partial}{\partial r} p_{\text{irr}}(r', t'|r, t) \right]. \end{aligned} \quad (\text{A3})$$

Next, we integrate both sides of Eq. (A3) over time $\int_0^{t'} dt$ and over space $2\pi \int_a^\infty dr r$.

First, we focus on the lhs of Eq. (A3). After the integration over time it becomes

$$\frac{1}{2\pi r} [\delta(r' - r) p_{\text{rev}}(r, t'|r_0) - \delta(r - r_0) p_{\text{irr}}(r', t'|r)], \quad (\text{A4})$$

where we have used the initial condition (Eq. (2)). The integration over space is now trivial and we finally arrive at

$$\begin{aligned} & 2\pi \int_a^\infty dr r \int_0^{t'} dt \frac{\partial}{\partial t} [p_{\text{irr}}(r', t'|r) p_{\text{rev}}(r, t|r_0)] \\ &= p_{\text{rev}}(r', t'|r_0) - p_{\text{irr}}(r', t'|r_0). \end{aligned} \quad (\text{A5})$$

Now, we turn to the rhs of Eq. (A3). We integrate first over space and obtain the expression

$$\begin{aligned} & -2\pi a D \left[p_{\text{irr}}(r', t'|a, t) \frac{\partial}{\partial r} p_{\text{rev}}(r, t|r_0) \Big|_{r=a} \right. \\ & \quad \left. - p_{\text{rev}}(a, t|r_0) \frac{\partial}{\partial r} p_{\text{irr}}(r', t'|r, t) \Big|_{r=a} \right], \end{aligned} \quad (\text{A6})$$

because the GF vanishes for $r \rightarrow \infty$. Using the boundary conditions, Eqs. (8) and (A2), we arrive at

$$\begin{aligned} & 2\pi \int_a^\infty dr r \int_0^{t'} dt [\text{rhs of Eq. (A3)}] \\ &= \kappa_d \int_0^{t'} dt [1 - S_{\text{rev}}(t|r_0)] p_{\text{irr}}(r', t'|a, t). \end{aligned} \quad (\text{A7})$$

Everything taken together, we finally obtain

$$\begin{aligned} & p_{\text{rev}}(r', t'|r_0) - p_{\text{irr}}(r', t'|r_0) \\ &= \kappa_d \int_0^{t'} dt [1 - S_{\text{rev}}(t|r_0)] p_{\text{irr}}(r', t'|a, t). \end{aligned} \quad (\text{A8})$$

We note that

$$p_{\text{irr}}(r', t'|a, t) = p_{\text{irr}}(r', t' - t|a). \quad (\text{A9})$$

Using this identity and switching the notation $t \leftrightarrow t'$, $r' \rightarrow r$, Eq. (A8) takes the form

$$\begin{aligned} & p_{\text{rev}}(r, t|r_0) = p_{\text{irr}}(r, t|r_0) \\ & \quad + \kappa_d \int_0^t dt' [1 - S_{\text{rev}}(t'|r_0)] p_{\text{irr}}(r, t - t'|a) \end{aligned} \quad (\text{A10})$$

of a Dyson-like equation connecting the reversible GF $p_{\text{rev}}(r, t|r_0)$ subject to a back-reaction BC (Eq. (8)) with the irreversible GF $p_{\text{irr}}(r, t|r_0)$ subject to a radiation BC (Eq. (6)). Dyson-like equations relating the non-reactive GF subject to a reflective BC with $p_{\text{irr}}(r, t|r_0)$ and $p_{\text{rev}}(r, t|r_0)$ have been considered in Ref. 18, and Refs. 4 and 19, respectively. A useful feature of Eq. (A10) is that it gives rise to the convolution relations, as we shall see now.

In fact, by integrating over space $2\pi \int_a^\infty dr r$ and using the definition Eq. (9), the convolution relation (Eq. (24)) immediately follows from Eq. (A10).

$$S_{\text{rev}}(t|r_0) = S_{\text{irr}}(t|r_0) + \kappa_d \int_0^t [1 - S_{\text{rev}}(t'|r_0)] S_{\text{irr}}(t - t'|a) dt'. \quad (\text{A11})$$

Next, we differentiate both sides of Eq. (A11) with respect to time and obtain

$$\begin{aligned} & \frac{\partial S_{\text{rev}}(t|r_0)}{\partial t} \\ &= \frac{\partial S_{\text{irr}}(t|r_0)}{\partial t} + \kappa_d [1 - S_{\text{rev}}(t|r_0)] S_{\text{irr}}(0|a) \\ & \quad + \kappa_d \int_0^t [1 - S_{\text{rev}}(t'|r_0)] \frac{\partial S_{\text{irr}}(t - t'|a)}{\partial t} dt' \\ &= \kappa_d P_{\text{rev}}(t|r_0) + \kappa_d \int_0^t P_{\text{rev}}(t'|r_0) \frac{\partial S_{\text{irr}}(t - t'|a)}{\partial t} dt' \\ & \quad + \frac{\partial S_{\text{irr}}(t|r_0)}{\partial t}, \end{aligned}$$

where we used $S_{\text{irr}}(0|a) = 1$. Obviously,

$$\frac{\partial S_{\text{rev}}(t|r_0)}{\partial t} = -\frac{\partial P_{\text{rev}}(t|r_0)}{\partial t}, \quad (\text{A12})$$

and hence Eq. (25) follows.

Finally, we would like to point out that Eqs. (21) and (22) can be obtained along the same lines. The major difference is that the reversible GF $p_{\text{rev}}(r, t|*)$ satisfies the initial condition (Eq. (10)) instead of Eq. (2). This difference leads to

$$p_{\text{rev}}(r, t|*) = \kappa_d \int_0^t dt' [1 - S_{\text{rev}}(t'|*)] p_{\text{irr}}(r, t - t'|a), \quad (\text{A13})$$

cf. Refs. 4, 18, and 19. Hence, Eqs. (21) and (22) follow.

¹S. A. Rice, *Diffusion Limited Reactions* (Elsevier, New York, 1985).

²N. Agmon, *J. Chem. Phys.* **81**, 2811 (1984).

³H. Kim and K. Shin, *Phys. Rev. Lett.* **82**, 1578 (1999).

⁴T. Prüstel and M. Meier-Schellersheim, *J. Chem. Phys.* **137**, 054104 (2012).

⁵M. Tachiya, "Theory of diffusion-controlled dissociation and its applications to charge separation," in *Extended Abstract of Annual Meeting on Photochemistry, Tsu, Japan, 1980* (Japan. Photochem. Assoc., 1980), pp. 256–257.

⁶N. Agmon and A. Szabo, *J. Chem. Phys.* **92**, 5270 (1990).

⁷H. Sano and M. Tachiya, *J. Chem. Phys.* **71**, 1276 (1979).

⁸H. Sano and M. Tachiya, *J. Chem. Phys.* **75**, 2870 (1981).

⁹D. Toussaint and F. Wilczek, *J. Chem. Phys.* **78**, 2642 (1983).

¹⁰C. Emeis and P. Fehder, *J. Am. Chem. Soc.* **92**, 2246 (1970).

¹¹I. Bethani, S. Skanland, I. Dikic, and A. Acker-Palmer, *EMBO J.* **29**, 2677 (2010).

¹²T. Prüstel and M. Meier-Schellersheim, *J. Chem. Phys.* **138**, 104112 (2013).

¹³M. von Smoluchowski, *Z. Phys. Chem.* **92**, 129 (1917).

¹⁴F. Collins and G. Kimball, *J. Colloid Sci.* **4**, 425 (1949).

¹⁵H. Carslaw and J. Jaeger, *Conduction of Heat in Solids* (Clarendon Press, New York, 1986).

¹⁶M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover, New York, 1965).

¹⁷A. Szabo, K. Schulten, and Z. Schulten, *J. Chem. Phys.* **72**, 4350 (1980).

¹⁸A. Szabo, G. Lamm, and G. Weiss, *J. Stat. Phys.* **34**, 225 (1984).

¹⁹I. Gopich, K. Solntsev, and N. Agmon, *J. Chem. Phys.* **110**, 2164 (1999).