The strong Nagata conjecture

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It is proved that there exist wild coordinates in the polynomial algebra in three variables over a field of characteristic zero. This result implies the famous Nagata conjecture.

Let $P_n = K[x_1, x_2, \dots, x_n]$ be the polynomial algebra in n variables $x_1, x_2, \dots, x_n$ over a field K of characteristic 0. A K -automorphism $\phi \in \text{Aut}_K P_n$ is called elementary if ϕ preserves all but one x_i for $i = 1, \dots, n$. In other words, a K -automorphism $\phi \in \text{Aut}_K P_n$ is elementary if it can be expressed as

$$\phi : (x_1, \dots, x_n) \rightarrow (x_1, \dots, x_{j-1}, ax_j + q, x_{j+1}, \dots, x_n),$$

where $0 \neq a \in K, q \in K[x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n]$. A K -automorphism $\phi \in \text{Aut}_K P_n$ is called tame if ϕ can be decomposed as a product of elementary automorphisms. Otherwise ϕ is called wild. A polynomial $p \in P_n$ is called a coordinate polynomial (or just a coordinate for short) if there exists $\phi \in \text{Aut}_K P_n$ such that $\phi(p) = x_1$. Moreover, a coordinate $p \in P_n$ is called a tame coordinate if there exists a tame automorphism $\phi \in \text{Aut}_K P_n$ such that $\phi(p) = x_1$. Otherwise p is called a wild coordinate. The notion of the tame and wild coordinates plays an important role in the study of automorphisms of polynomial algebras in refs. 1–5. Jung (6) in 1942 [$\text{Char}(K) = 0$] and van der Kulk (7) in 1953 [$\text{Char}(K) > 0$] proved that all automorphisms of P_2 are tame. As an immediate consequence, all coordinates of P_2 are tame as well. Nagata (8) conjectured in 1972 that there exist wild automorphisms in $\text{Aut}_K P_3$. The famous Nagata conjecture was recently proved by Shestakov and Umirbaev (9–11). However, the following strong version of the Nagata conjecture remains open.

Strong Nagata Conjecture

There exist wild coordinates of P_3 . The history of the strong Nagata conjecture naturally goes back to 1972 when the Nagata conjecture was formulated. However, it was formulated formally in June 2002 by J.-T.Y. in an algebra seminar held at the University of Hong Kong. Obviously the strong Nagata conjecture implies the Nagata conjecture but not vice versa, since it is possible that an image f of x under a wild automorphism $\phi: (x, y, z) \rightarrow (f, g, h)$ of $P_3 = K[x, y, z]$ can be taken to x by a tame automorphism.

In this article, we settle the strong Nagata conjecture affirmatively. Our result is the following.

Main Theorem

There exist wild coordinates of P_3 . In particular, all wild coordinates of $K[z][x, y]$ (see refs. 1 and 2) are also wild coordinates of $P_3 = K[x, y, z]$. Moreover, the image of x, y, z under a wild automorphism in $\text{Aut}_K(P_3)$ must contain at least two wild coordinates.

Before proving the main theorem in the next section, let us describe the main idea of this article introduced by J.-T.Y. when he formulated the strong Nagata conjecture in the same algebra seminar [this idea was motivated by a similar idea used by Shpilrain and Yu (12) to effectively classify parametrized curves; see also ref. 13]: a given coordinate $p = p(x, y, z) \in P_3 = K[x, y, z]$ can be faithfully parametrized as a triple $\{x(u, v), y(u, v), z(u, v)\} \subset P_2 = K[u, v]$, where $p(x(u, v), y(u, v), z(u, v)) = 0$ and $K[x(u, v), y(u, v), z(u, v)] = K[u, v]$. It can be seen that $p =$

$p(x, y, z)$ is a tame coordinate of P_3 if and only if there exists a sequence of elementary automorphisms of P_3 that takes the triple $(x(u, v), y(u, v), z(u, v))$ to $(0, u, v)$. This condition can be effectively determined by an algorithm motivated by ideas of Shestakov and Umirbaev (9–11). By the algorithm, we are able to prove that all wild coordinates of $K[z][x, y]$ (1, 2) are also wild coordinates of $P_3 = K[x, y, z]$, hence we obtain many wild coordinates of P_3 . Note that a tame automorphism takes a wild coordinate to a wild coordinate; this way we also obtain many wild coordinates.

In the sequel we use the following conventions: (i) K is a field of characteristic zero; (ii) all automorphisms are K -automorphisms unless specified otherwise, and $\text{Aut} P_n$ always means $\text{Aut}_K P_n$; (iii) algebraic (linear) independency (dependency) always means K -algebraic (K -linear) independency (dependency); (iv) an automorphism ϕ of P_n with $\phi(x_i) = p_i$ is denoted $(p_1, \dots, p_n)$ sometimes, and each p_i is called a coordinate of ϕ ; and (v) $\bar{p}$ denotes the highest homogeneous part of a polynomial $p \in P_n$.

Proof of the Main Theorem

We start with some preliminaries. In ref. 14, the Poisson bracket of elements of the algebra P_n was defined. If $f, g \in P_n$, then

$$[f, g] = \sum_{1 \leq i < j \leq n} [x_i, x_j] \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} \right).$$

The Poisson bracket satisfies the Leibniz identity, i.e.

$$[x \cdot y, z] = [x, z] \cdot y + x \cdot [y, z]. \quad [1]$$

For every element $f \in P_n$, the highest homogeneous part $\bar{f}$ and the degree $\deg f$ can be defined in an ordinary way, if we put $\deg(x_i) = 1$, where $1 \leq i \leq n$.

The next lemma was proved in ref. 14.

Lemma 1. Two polynomials $f, g \in P_n$ are algebraically dependent if and only if $[f, g] = 0$.

Lemma 2. Elements $f, g \in P_n$ are algebraically dependent if and only if there exists $t \in P_n$ such that $f, g \in K[t]$.

Proof: It was proved in ref. 14 that the centralizer $C(f) = \{h \in P_n \mid [f, h] = 0\}$ of any element $f \in P_n$ is a polynomial algebra in one indeterminate, i.e., there exists $t \in P_n$ such that $C(f) = K[t]$. By Lemma 1, $g \in C(f)$ if and only if f and g are algebraically dependent.

Two generated subalgebras of the algebra P_n were studied in ref. 14. The lower degree bound of an element in such a subalgebra obtained there plays a major role in the study of the automorphisms in $\text{Aut} P_3$ in refs. 9–11 as well as in the study of coordinates of P_3 in this article. Recall that a pair of elements (f, g) of the algebra P_n is called reduced (15), if $\bar{f} \notin K[\bar{g}]$ and $\bar{g} \notin K[\bar{f}]$. According to ref. 14, an algebraically independent reduced pair of elements (f, g) of the algebra P_n is called $*$ -reduced if $\bar{f}$ and $\bar{g}$ are algebraically dependent.

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Let f, g be a $*$ -reduced pair of elements of P_n and $k = \deg f < m = \deg g$. Put

$$l = \frac{k}{\gcd(k, m)}, \quad s = \frac{m}{\gcd(k, m)},$$

$$N = N(f, g) = \frac{km}{\gcd(k, m)} - k - m - \deg[f, g],$$

where $\gcd(k, m)$ is the greatest common divisor of k and m . We sometimes call the above $*$ -reduced pair (f, g) an l -reduced pair in the sequel. Let $G(x, y) \in K[x, y]$. It was proved in ref. 14, if $\deg_y G(x, y) = lq + r$, $0 \leq r < l$, then

$$\deg G(f, g) \geq qN + mr, \quad [2]$$

and if $\deg_x G(x, y) = sq_1 + r_1$, $0 \leq r_1 < s$, then

$$\deg G(f, g) \geq q_1N + kr_1. \quad [3]$$

We consider the set of triples $\theta = (f_1, f_2, f_3)$, where $f_i \in P_n$. $\deg \theta := \deg f_1 + \deg f_2 + \deg f_3$ is called the degree of the triple θ . Recall that an elementary transformation of the triple (f_1, f_2, f_3) changes only one coordinate f_i to the element of the form $\alpha f_i + g$, where $0 \neq \alpha \in K$, $g \in K[\{f_1, f_2, f_3\} - f_i]$. The notation $\theta \rightarrow \tau$ means that the triple τ can be obtained from θ by a single elementary transformation. Moreover, $\theta \Rightarrow \tau$ means that there exists a sequence of triples $\theta = \theta_0, \theta_1, \dots, \theta_s = \tau$ such that

$$\theta_0 \rightarrow \theta_1 \rightarrow \dots \rightarrow \theta_s.$$

As usual, an automorphism θ of the algebra P_3 such that $\theta(x_i) = f_i$, $1 \leq i \leq 3$, is also denoted by the triple $\theta = (f_1, f_2, f_3)$. Note that the automorphism $\theta \in \text{Aut}P_3$ is tame if and only if

$$(f_1, f_2, f_3) \Rightarrow (x_1, x_2, x_3).$$

A triple $\theta = (f_1, f_2, f_3)$ of elements of the algebra $P_2: = K[u, v]$ is called a tame triple, if

$$(f_1, f_2, f_3) \Rightarrow (0, u, v).$$

Otherwise it is wild.

A polynomial f_1 in the triple (f_1, f_2, f_3) of P_2 is reducible, if there exists $g \in K[f_2, f_3]$ such that $f_1 = \bar{g}$, otherwise it is irreducible. Put $f'_1 = c(f_1 - g)$, where $0 \neq c \in K$, then $\deg(f'_1) < \deg(f)$ and $\deg(f'_1, f_2, f_3) < \deg \theta$. In this situation we say f_1 is reduced in θ by the triple (f'_1, f_2, f_3) . We may define the reducibility of f_2 and f_3 similarly. θ is elementarily reducible (or θ admits an elementary reduction) if one of f_i is reducible.

In the sequel the next lemma is useful.

Lemma 3. Let $f, g, h \in P_2$ and let $K[f, g, h] = P_2$. If f, g are algebraically dependent, then there exists $t \in P_2$ such that $K[f, g] = K[t]$ and $P_2 = K[t, h]$.

Proof: By Lemma 2 there exists a $t \in P_2$ such that $f, g \in K[t]$. So $P_2 = K[t, h]$. If $K[f, g]$ is a proper subring of $K[t]$, then $K[f, g, h]$ is a proper subring of $K[t, h] = P_2$. Therefore $K[f, g] = K[t]$.

Lemma 3 implies the following.

Lemma 4. Let $f, g, h \in P_2$ and let $K[f, g, h] = P_2$. If f, g are algebraically dependent, then there exists a coordinate $t \in P_2$, and a sequence of elementary transformations taking (f, g) to $(0, t)$.

Proof: By Lemma 3 and the famous Abhyankar-Moh-Suzuki Theorem (13) we obtain a proof.

Theorem 1. Let $f, g, h \in P_2$ such that $K[f, g, h] = P_2$. Then the elementary reducibility of the triple (f, g, h) is algorithmically recognizable.¹

Proof: We only need to prove the reducibility of an element in the triple is algorithmically recognizable. By Lemma 4, we may assume that f, g, h are pairwise algebraically independent. Now we are going to recognize the reducibility of an element, say, h of the triple. If $\bar{f}$ and $\bar{g}$ are algebraically independent, then h is reducible if and only if $\bar{h} \in K[\bar{f}, \bar{g}]$. Since $\bar{f}$ and $\bar{g}$ are homogeneous, this can be effectively determined. If $\bar{g} \in K[\bar{f}]$ and $\bar{g} = cf^k$, where c is a nonzero element of K , then h is reducible in (f, g, h) if and only if it is reducible in $(f, g - cf^k, h)$. Since $\deg(f, g - cf^k, h) < \deg(f, g, h)$, we may apply induction in this case.

Now we may assume (f, g) is a $*$ -reduced pair and $\deg(f) < \deg(g)$. Suppose that there exists $G(x, y) \in P_2$ such that $h =$ the highest homogeneous part of $G(f, g)$. Then 2 and 3 give an upper bound k for $\deg_x(G)$ and $\deg_y(G)$. Hence $G(f, g)$ is in the vector space generated by $f^i g^j$, $i, j < k$. The highest homogeneous part of elements in this space can be effectively determined by triangulation.

Now we are going to define four types of (nonelementary) reductions for triples of P_2 ; all of these reductions are compositions of at most four elementary transformations of a special type. These definitions are motivated by the definitions of the four types of nonelementary reductions for automorphisms in $\text{Aut}P_3$ in refs. 9–11.

Definition 1: Let $\theta = (f_1, f_2, f_3)$ be a triple of P_2 such that $\deg f_1 = 2k$, $\deg f_2 = ks$, $s \geq 3$ is an odd number, $2k < \deg f_3 \leq ks$, $f_3 \notin K[f_1, f_2]$. Suppose that there exists $0 \neq \alpha \in K$ such that the elements $g_1 = f_1$, $g_2 = f_2 - \alpha f_3$ satisfy the conditions:

(i) g_1, g_2 is a two-reduced pair and $\deg g_1 = \deg f_1$, $\deg g_2 = \deg f_2$;

(ii) The element f_3 of the triple (g_1, g_2, f_3) is reduced by a triple (g_1, g_2, g_3) with the condition $\deg[g_1, g_3] < \deg g_2 + \deg[g_1, g_2]$. Then we say that θ admits a reduction (g_1, g_2, g_3) of type I.

Definition 2: Let $\theta = (f_1, f_2, f_3)$ be a triple of P_2 such that $\deg f_1 = 2k$, $\deg f_2 = 3k$, $3k/2 < \deg f_3 \leq 2k$, and $\bar{f}_1, \bar{f}_3$ are linearly independent. Suppose that there exist $\alpha, \beta \in K$, where $(\alpha, \beta) \neq (0, 0)$, such that the elements $g_1 = f_1 - \alpha f_3$, $g_2 = f_2 - \beta f_3$ satisfy the conditions i and ii in Definition 1. Then we say that θ admits a reduction (g_1, g_2, g_3) of type II.

Definition 3: Let $\theta = (f_1, f_2, f_3)$ be a triple of P_2 such that $\deg f_1 = 2k$, and either $\deg f_2 = 3k$, $k < \deg f_3 \leq 3k/2$, or $5k/2 < \deg f_2 \leq 3k$, $\deg f_3 = 3k/2$. Suppose that there exist $\alpha, \beta, \gamma \in K$ such that the elements $g_1 = f_1 - \beta f_3$, $g_2 = f_2 - \gamma f_3 - \alpha f_3$ satisfy the conditions:

(i) g_1, g_2 is a two-reduced pair and $\deg g_1 = 2k$, $\deg g_2 = 3k$;

(ii) There exists an element g_3 of the form

$$g_3 = \sigma f_3 + g,$$

where $0 \neq s \in K$, $g \in K[g_1, g_2] \setminus K$, such that $\deg g_3 \leq 3k/2$, $\deg[g_1, g_3] < 3k + \deg[g_1, g_2]$.

If $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ and $\deg g_3 < k + \deg[g_1, g_2]$, then we say that θ admits a reduction (g_1, g_2, g_3) of type III. On the other hand, if there exists $0 \neq \mu \in K$ such that $\deg(g_2 - \mu g_3) \leq 2k$, then we say that θ admits a reduction $(g_1, g_2 - \mu g_3, g_3)$ of type IV.

Definition 4: A simple triple in P_2 is defined inductively as follows: A triple $\{f, g, h\} \subset P_2$ is a simple triple if $\deg(f, g, h) = 2$ and $K[f, g, h] = P_2$. A triple β of $\deg \beta = k > 2$ is a simple triple, if there exists an elementary reduction or a reduction of one of the types I–IV to β , and after such a reduction, the resulting triple γ of $\deg \gamma < k$ is simple.

The above definition of a simple triple of P_2 is motivated by the definition of a simple automorphism in $\text{Aut}P_3$ in refs. 9–11.

¹In formulation of algorithmic results, we always assume that the ground field K is constructive.

Theorem 2. A triple $\{f, g, h\} \subset P_2$ is tame if and only if it is simple.

Proof: It is obvious that a simple triple is a tame one according to Definition 4. To prove the converse, first note that by Lemma 2, we may assume that f, g, h are pairwise algebraically independent for a tame triple (f, g, h) of P_2 . Then the proof of theorem 1 in ref. 9 shows that (f, g, h) is either elementarily reducible, or admits a reduction of one of the types I–IV.

Theorem 3. Let $f, g, h \in P_2$ such that $K[f, g, h] = P_2$. Then the tameness of the triple (f, g, h) is algorithmically recognizable.

Proof: By Theorem 2, we only need to prove that the simple reducibility of (f, g, h) is algorithmically recognizable. The elementary reducibility of (f, g, h) is algorithmically recognizable by Theorem 1, if (f, g, h) admits a reduction of one of types I–IV, then the proof is similar to the proof of theorem 3 in ref. 9.

Remark 1: If $p \in P_3$ is irreducible and $p(f, g, h) = 0$, then we do not know whether the hypothesis of the above theorem implies that p is a coordinate of P_3 . The famous embedding conjecture of Abhyankar-Sathaye claims that in this case p is a coordinate. See, for instance, ref. 16. On the other hand, Lemma 5 below shows that p is a tame coordinate if and only if (f, g, h) is a tame triple.

Lemma 5. Let $\theta = \{f_1, f_2, f_3\} \subset P_2$ be a tame triple with $\deg(f_i) \leq 1$ for some $i \in \{1, 2, 3\}$. Then θ admits an elementary reduction.

Proof: It is easy to see from Definitions 1–3 that in this case θ does not admit a reduction of types I–IV.

Now we consider coordinates of $P_3 = K[x, y, z]$. If one of the coordinates of $\theta = (f, g, h) \in \text{Aut}P_3$ is wild, then obviously θ is a wild automorphism. But even if θ is a wild automorphism, it does not necessarily imply that all coordinates of the θ are wild. For example, the Nagata automorphism $\sigma = (x - 2wy - w^2z, y + wz, z)$ of P_3 , where $w = y^2 + xz$, is wild (9–11), but the coordinate z of σ is obviously tame.

Let $p \in P_3$ be a coordinate,

$$\phi : P_3 \rightarrow P_2 := K[u, v], \quad [4]$$

a surjective homomorphism with $\text{Ker}(\phi) = pP_3$ and $\phi(x, y, z) = (x(u, v), y(u, v), z(u, v))$. Then the triple $(x(u, v), y(u, v), z(u, v))$ gives a faithful parametrization of $p(x, y, z)$: $p(x(u, v), y(u, v), z(u, v)) = 0$ and $K[x(u, v), y(u, v), z(u, v)] = K[u, v]$.

Lemma 6. Let $p \in P_3$ be a coordinate of P_3 and let $(x(u, v), y(u, v), z(u, v))$ be a faithful parametrization of the p . Then the coordinate p is tame if and only if the triple $(x(u, v), y(u, v), z(u, v))$ is tame.

Proof: Put $\hat{x} = x(u, v)$, $\hat{y} = y(u, v)$, $\hat{z} = z(u, v)$. Assume that p is a tame coordinate and let (p, q, r) be a tame automorphism. Then

$$(x, y, z) \Rightarrow (p, q, r).$$

Since $p(\hat{x}, \hat{y}, \hat{z}) = 0$, the same sequence of elementary transformations produces

$$(\hat{x}, \hat{y}, \hat{z}) \Rightarrow (0, f, g),$$

where $f = q(\hat{x}, \hat{y}, \hat{z})$, $g = r(\hat{x}, \hat{y}, \hat{z})$. By surjectivity of the ϕ from 4, we have $K[u, v] = F[f, g]$. It follows that (f, g) is an automorphism of $P_2 = K[u, v]$. By Jung's theorem (see ref. 6) $(a, b) \Rightarrow (u, v)$. Obviously,

$$(\hat{x}, \hat{y}, \hat{z}) \Rightarrow (0, u, v),$$

and the parametrization $(\hat{x}, \hat{y}, \hat{z})$ is tame.

Now assume that

$$(\hat{x}, \hat{y}, \hat{z}) \Rightarrow (0, u, v).$$

The same elementary transformations give

$$(x, y, z) \Rightarrow (p_1, q_1, r_1) =: \theta$$

such that $\phi(p_1) = 0$. Since $\text{Ker}(\phi) = pP_3$, then $p_1 = pt$, where $t \in P_3$. We have

$$\theta^{-1}(p_1) = x = \theta^{-1}(p)\theta^{-1}(t).$$

Since $\theta^{-1}(p)$ is also a coordinate, we obtain that $\deg(\theta^{-1}(t)) = 0$ and $0 \neq t \in K$. Since (p_1, q_1, r_1) is a tame automorphism, p_1 is a tame coordinate; so is p .

Recall that the well-known Nagata automorphism $\sigma \in \text{Aut}P_3$ is defined as follows (see ref. 8):

$$\sigma(x) = f = x - 2wy - w^2z, \quad \sigma(y) = g = y + wz,$$

$$\sigma(z) = h = z, \quad w = y^2 + xz,$$

Theorem 4. The coordinates f and g of the Nagata automorphism σ are wild.

Proof: Note that

$$\sigma^{-1} = (x + 2(y^2 + xz)y - (y^2 + xz)^2z, y - (y^2 + xz)z, z).$$

Since $\sigma\sigma^{-1} = \text{id}$, then by putting $x = 0, y = u, z = v$ in σ^{-1} we get a parametrization

$$(2u^3 - u^4v, u - u^2v, v)$$

of the f ; and by putting $x = u, y = 0, z = v$ we get a parametrization

$$(u - u^2v^3, -uv^2, v)$$

of the g . It is easy to see that the highest homogeneous parts of the elements in both triples are pairwise algebraically independent, and the highest homogeneous part of an element is not contained in the subalgebra generated by the highest homogeneous part of the other two elements in the same triple, hence both triples do not admit elementary reduction. Moreover, both triples do not admit any reduction of one of types I–IV by Lemma 3 since $\deg(v) = 1$. Therefore, both triples are not simple. By Theorem 2, both f and g are wild coordinates of P_3 .

Lemma 7. Let p be a coordinate of the algebra P_n and let ϕ be a tame automorphism in $\text{Aut}P_n$. Then p is a tame coordinate if and only if $\phi(p)$ is a tame coordinate.

Proof: The conclusion is obvious by the definition of a tame coordinate.

Theorem 5. Let $\theta = (f, g, z) \in \text{Aut}P_3$. If θ is a wild automorphism, then both coordinates f and g are wild.

Proof: Put $\theta^{-1} = (p, q, z)$. Then the homomorphism ϕ (see 4) defined by

$$\phi(x, y, z) = (p(0, u, v), q(0, u, v, v) = \xi$$

gives a faithful parametrization of the f . If ξ is wild, then f is wild by Lemma 6. Suppose that ξ is tame. By Lemma 5, ξ is elementarily reducible. Hence there exists a sequence of elementary automorphisms of $P_3 := K[x, y, z]$ preserving z such that

$$\xi = (p(0, u, v), q(0, u, v), v) \Rightarrow (0, u, v).$$

The same sequence produces

$$(p, q, z) \Rightarrow (p_1, q_1, z).$$

It follows that $\psi(p_1) = 0$, where $\psi: P_3 \rightarrow K[u, v]$ is the surjective homomorphism defined by $\psi(x) = 0, \psi(y) = u, \psi(z) = v$. Then $p_1 = xt$ and as in the proof of Lemma 5 we can take $p_1 = x$. Therefore, both automorphisms (p_1, q_1, z) and $(p, q, z) = \theta^{-1}$ are

tame. Consequently, θ is also tame. The contradiction completes the proof.

Remark 2: As a consequence, a wild coordinate of $K[z][x, y]$ is always a wild coordinate of $P_3 = K[x, y, z]$, hence by Drensky and Yu (1, 2) (where all wild coordinates of $K[z][x, y]$ are effectively classified), we obtain many wild coordinates of P_3 .

Theorem 6. Let $\theta = (f, g, h)$ be a wild automorphism in $\text{Aut}P_3$. Then at least two coordinates among the coordinates f, g, h of θ are wild coordinates of P_3 .

Proof: Suppose that h is a tame coordinate and let $\phi \in \text{Aut}P_3$ be a tame automorphism with $\phi(h) = z$. Then $\theta_1 = \phi\theta = (\phi(f), \phi(g), z)$ is also a wild automorphism. By Theorem 5, the coordinates $\phi(f)$ and $\phi(g)$ are wild. Now the wildness of both coordinates f and g follows from Lemma 7.

The conclusion of the main theorem follows from Theorems 4–6.

Some Open Problems

Recall that an element q of the free associative algebra $A_n := K\langle x_1, \dots, x_n \rangle$ is called a primitive element if there exists an automorphism $\phi \in \text{Aut}A_n$ such that $\phi(q) = x_1$. Tame and wild automorphisms and primitive elements of A_n can also be defined naturally.

Problem 1. Let $p \in P_3$ be a coordinate. Can p be lifted to a primitive element p' of the free associative algebra A_3 ? In other words, does there exist a primitive element $p' \in A_3$ such that the natural homomorphism (the abelianization) from A_3 to P_3 takes p' to p ?

Remark 3: If the answer to Problem 1 is yes for some wild coordinate p of P_3 , then obviously p' is a wild primitive element of A_3 , hence there exist wild primitive elements and wild automorphisms of A_3 that would settle a famous conjecture of Cohn (17) affirmatively. On the other hand, if the answer is no for some coordinate q of P_3 , then we would obtain a new proof of the Nagata conjecture without using the previous results of Shestakov and Umirbaev (9–11).

Suppose that $f_1, \dots, f_k \in P_n$ and $K[f_1, \dots, f_k] = P_n$. Then obviously $n \leq k$. The following two problems are naturally raised.

Problem 2. Does there exist an automorphism $\phi \in \text{Aut}P_k$ such that

$$\phi(f_1, \dots, f_k) = (0, \dots, 0, x_1, \dots, x_n)? \quad [5]$$

Problem 3. Does there exist a tame automorphism $\phi \in \text{Aut}P_k$ such that

$$\phi(f_1, \dots, f_k) = (0, \dots, 0, x_1, \dots, x_n)? \quad [6]$$

Note that Problem 2 is the algebraic formulation of the problem of rectifiability of an embedding in affine algebraic geometry (see ref. 8).

If $k = n$, then Problem 2 becomes the problem of recognizing automorphisms of P_n and was solved by van den Essen (16) by the Gröbner basis method; and Problem 3 becomes the Nagata conjecture for P_n .

The case $k = 2, n = 1$ of both Problems 2 and 3 have positive solutions if $\text{Char}(K) = 0$ (that is the well-known Abhyankar-Moh Theorem) and a negative solution if $\text{Char}(K) > 0$ (13).

The case $k \geq 2n + 2$ of Problem 2 was solved positively in refs. 18–21 (see also refs. 12 and 22–24). In fact, the automorphisms obtained in refs. 12 and 18–24 are all tame, hence also give the positive solutions of the case $k \geq 2n + 2$ of Problem 3.

The case $n = k = 3$ of Problem 3 was solved negatively by Shestakov and Umirbaev (9–11). Note that some other examples considered in refs. 1, 2, and 25 are also wild by refs. 9–11.

Also Theorems 4 and 5 give the answer to the case $n = 3, k = 2$ of Problem 3 negatively.

It is natural to raise the following.

Problem 4. Let $p \in P_3$ be a given coordinate. Does there exist an algorithm to determine whether p is tame or wild?

This problem is closely related to the following question: for a given polynomial $p \in P_3$, how do we effectively determine whether p is a coordinate? Moreover, if p is a coordinate, how do we effectively construct an automorphism $\phi \in \text{Aut}P_3$ such that $\phi(p) = x$?

Theorem 7. Let p be a given coordinate of P_3 and let $\theta = (f, g, h)$ be a given automorphism in $\text{Aut}P_3$ taking p to x . Then there exists an algorithm to determine whether the coordinate p is tame or wild.

Proof: Suppose $\theta = (f, g, h)$. Then the homomorphism ϕ (see 4) defined by

$$(\phi(x), \phi(y), \phi(z)) = (f(0, u, v), g(0, u, v), h(0, u, v)) = \xi$$

gives a faithful parametrization of p . By Lemma 6, ξ is a wild triple if and only if p is a wild coordinate of P_3 . Then apply Theorem 3.

Finally, in view of Theorems 5 and 6, we may propose the following.

Problem 5. Is $\text{Aut}K[x, y, z]$ essentially generated by $\text{Aut}K[z][x, y]$? Or, more precisely, is it true that any automorphism in $\text{Aut}K[x, y, z]$ can be decomposed as a product of automorphisms fixing z and linear automorphisms?

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