Complex-analyticity of harmonic maps and strong rigidity of compact Kähler manifolds

(holomorphic maps/biholomorphism/Kähler metrics/negative curvature/bounded symmetric domains)

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ABSTRACT A harmonic map f between two compact Kähler manifolds is shown to be either holomorphic or conjugate holomorphic under a suitable negativity condition on the curvature of the image manifold and a condition on the rank of df . As a consequence, a compact Kähler manifold of dimension ≥ 2 that is of the same homotopy type as a compact Kähler manifold with suitable negative curvature condition or as a compact quotient of an irreducible classical bounded symmetric domain must be either biholomorphic or conjugate biholomorphic to it.

Mostow (1) proved the strong rigidity of locally symmetric Riemannian manifolds of nonpositive curvature not admitting any closed one- or two-dimensional geodesic submanifolds as locally direct factors. In particular, two compact quotients of the ball of complex dimension ≥ 2 with isomorphic fundamental groups are either biholomorphic or conjugate biholomorphic. S. T. Yau (personal communication) conjectured that this phenomenon of strong rigidity should hold also for compact Kähler manifolds of complex dimension ≥ 2 with negative sectional curvature. That is, any two such manifolds are biholomorphic or conjugate biholomorphic if they are of the same homotopy type. We confirm this conjecture if the curvature tensor is strongly negative according to the following definition.

Definition: The curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ of a Kähler manifold is said to be *strongly negative* (respectively *strongly seminegative*) if at every point

$$\sum_{\alpha, \beta, \gamma, \delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}} \xi^{\alpha\bar{\beta}} \bar{\xi}^{\gamma\bar{\delta}} > 0 \text{ (respectively } \geq 0 \text{)}$$

for any nonzero matrix $(\xi^{\alpha\bar{\beta}})$ of the form $\xi^{\alpha\bar{\beta}} = A^{\alpha\bar{\alpha}} \bar{B}^{\beta\bar{\beta}} - C^{\alpha\bar{\alpha}} \bar{D}^{\beta\bar{\beta}}$.

The curvature tensor of the ball is strongly negative. So is the curvature tensor of the compact Kähler surface recently constructed by Mostow and Siu,* which has negative sectional curvature and whose universal covering is not biholomorphic to the ball.

THEOREM 1. Suppose M and N are compact Kähler manifolds and the curvature tensor of M is strongly negative. Suppose $f: N \rightarrow M$ is a harmonic map and the rank over $\mathbf{R}$ of the differential df of f is at least 4 at some point of N . Then f is either holomorphic or conjugate holomorphic.

The following theorem is a consequence of Theorem 1 and the fundamental result of Eells and Sampson on harmonic maps (2).

THEOREM 2. If M is a compact Kähler manifold of complex dimension ≥ 2 whose curvature tensor is strongly negative, then any compact Kähler manifold of the same homotopy type as M must be either biholomorphic or conjugate biholomorphic to it.

Another consequence is the following theorem on representing homology classes by complex-analytic subvarieties. Unfortunately it is still far from being able to prove the Hodge conjecture even for the case of the compact quotients of the ball.

THEOREM 3. Suppose M is a compact Kähler manifold whose curvature tensor is strongly negative. Then for $k \geq 2$ an element of $H_{2k}(M, \mathbf{Z})$ can be represented by a complex-analytic subvariety of M if it can be represented by the continuous image of a compact Kähler manifold.

The curvature tensor of an irreducible classical bounded symmetric domain is not strongly negative, but is only strongly seminegative. By using the explicit expression of the curvature tensor, we obtain the following theorems for classical bounded symmetric domains.

THEOREM 4. Suppose N is a compact Kähler manifold and M is a compact quotient of a bounded symmetric domain of type $I_{m,n}$ ($m, n \geq 2$), II_n ($n \geq 3$), III_n ($n \geq 2$), or IV_n ($n \geq 3$) (ref. 3). Suppose $f: N \rightarrow M$ is a harmonic map that is a submersion at some point of N . Then f is either holomorphic or conjugate holomorphic.

A consequence of Theorem 4 is the following.

THEOREM 5. If M is a compact quotient of a bounded symmetric domain of type $I_{m,n}$ ($m, n \geq 2$), II_n ($n \geq 3$), III_n ($n \geq 2$), or IV_n ($n \geq 3$), then any compact Kähler manifold of the same homotopy type as M must be either biholomorphic or conjugate biholomorphic to it.

Theorems 1 and 4 are proved by a Bochner type argument. The usual technique of proving properties of a harmonic map f is to obtain a Bochner type formula by considering the Laplacian of the pointwise square norm of df . For the complex manifold case the pointwise square norm of df is replaced by the pointwise square norm of $\bar{\partial}f$ in this technique (4). The metric tensor of the image manifold and the inverse matrix of the metric tensor of the domain manifold appear in the pointwise square norm of $\bar{\partial}f$. Hence in this Bochner type formula an expression involving the difference of the curvature tensors of the domain manifold and the image manifold appears. This prevents one from drawing any conclusion when both mani-

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* Mostow, G. D. & Siu, Y.-T. (1979) "A compact Kähler surface of negative curvature not covered by the ball," unpublished manuscript.

folds have negative curvature. One key technique in our proof is to replace the pointwise square norm of $\bar{\partial}f$ by the contraction of $\bar{\partial}f \wedge \bar{\partial}\bar{f}$ with the metric tensor of the image manifold and to replace the Laplacian by $\partial\bar{\partial}$. This method enables us to get rid of the curvature tensor of the domain manifold in the Bocher type formula.

Details will appear elsewhere.

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