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Assessment of stretched vortex subgrid-scale models for LES of incompressible inhomogeneous turbulent flow

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Summary

The physical space version of the stretched vortex subgrid scale model [Phys. Fluids **12**, 1810 (2000)] is tested in large eddy simulations (LES) of the turbulent lid driven cubic cavity flow. LES is carried out using a higher order finite-difference method [J. Comput. Phys. **229**, 8802 (2010)]. The effects of different vortex orientation models and subgrid turbulence spectrums are assessed through comparisons of the LES predictions against direct numerical simulations (DNS) [Phys. Fluids **12**, 1363 (2000)]. Three Reynolds numbers 12000, 18000, and 22000 are studied. Good agreement with the DNS data for the mean and fluctuating quantities is observed.

Keywords

Large Eddy Simulation; WENO; High-Order Finite Difference; Driven Cavity; Incompressible flows; Stretched Vortex Subgrid-scale Model

1. Introduction

Over the last decade, use of LES as a means to compute turbulent flows is clearly growing in both the research and industrial CFD communities. This is primarily due to progress in both computational software and hardware, specifically related to high-performance computing, and in the rigor of the mathematical formulation [1]. The ability of LES to deliver high fidelity results for complex turbulent flows, compared to Reynolds-averaged Navier-Stokes (RANS), but at a reduced computational cost, compared to DNS, has resulted in LES making the move from a pure research tool to a viable engineering tool able to tackle industrial turbulent flows. However, complex geometries associated with industrial applications introduce additional challenges such as the lack of a homogeneous direction, required by some sub-grid (SGS) models for averaging, e.g. dynamic Smagorinsky model [2], as well as the possibility for coexistence of laminar, transitional, and turbulent flow regions. A SGS model must now not only be accurate and efficient but should be capable of dynamically adjusting to the changing local conditions. This motivates performance assessments of SGS models suitable for fully inhomogeneous wall-bounded flows.

The two-dimensional lid-driven cavity problem is a classic CFD benchmark problem for testing new numerical methods [3, 4]. The three-dimensional version of this problem, the lid-driven cubical cavity (LDCC) problem, has been studied experimentally [5, 6] and numerically [7, 8, 9, 10, 11, 12, 13]. The LDCC problem provides a simple geometry with clear boundary conditions, yet exhibiting complex flow structures such as Reynolds number dependent, multiple counter rotating recirculating regions, to test the performance of SGS

models under inhomogeneous turbulent flow conditions. At higher Reynolds numbers, LDCC flows are transitional or fully turbulent and LES generally requires an explicit SGS model to facilitate stable and accurate predictions [13]. In Bouffanais *et al.* [12], LES predictions of LDCC flow at $Re = 12000$ were computed using a spectral element method and were found to be in good agreement with the previous DNS data of Leriche and Gavrilakis [9] when using both the dynamic Smagorinsky and dynamic mixed SGS models.

LES of incompressible turbulent flows generally require higher-order (greater than second) numerical methods to minimize numerical dissipation effects [14, 15]. Recently, a number of papers have focused on the development and application of higher-order numerical methods for the simulation of incompressible turbulent flows [16, 17, 15, 18]. In our previous work [13], we employed an overall 5th-order accurate numerical scheme to study LDCC flows. We also reported on the performance of three global coefficient eddy viscosity type SGS models, namely the Smagorinsky model (SM) [19], the constant coefficient Vreman model (VM) [20], and the dynamic Vreman model (DVM) [21]. It was shown that DVM outperformed the other two models both in terms of accuracy and stability, and ascertained the need for the dynamic procedure for high Reynolds number inhomogeneous turbulent flow simulations.

In the present work, we provide a performance analysis of a class of non-eddy viscosity type SGS models, namely the stretched vortex models (SVM) [22, 23]. There are multiple variants of SVM models possible based on the underlying type of vortex orientation models and the form of turbulent spectrum. There are also two averaging procedures associated with dynamic formulation of SVM models. This work attempts to evaluate multiple combinations vortex orientation models, turbulence spectrum and averaging procedure for inhomogeneous turbulent flows using higher order numerical schemes and provides a guideline to obtain better performance from SVM models.

The rest of this paper is organized as follows. In the next section, the governing equations associated with LES of incompressible flows are presented. Following that, the motivation for and a brief but complete description of the SVM variants considered in this study is presented. A complete set of statistical and instantaneous results are presented with side-by-side comparisons to DNS data is then presented followed by a summary of the findings from this study.

2. Governing Equations

The low-pass, spatially filtered incompressible Navier-Stokes equations are presented below in non-dimensional form:

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} \quad (1)$$

$$\frac{\partial \tilde{u}_i}{\partial x_i} = 0,$$

where u_i is the i^{th} component of the velocity vector, p is the pressure, τ_{ij} represents the SGS stress tensor, Re is the Reynolds number, and the symbol $\tilde{\phi}$ represents the filtering operation for any quantity, ϕ . The SGS stress tensor τ_{ij} appearing in the equation (1), represents the difference between the filtered velocity product and the product of the filtered velocities and is given by

$$\tau_{ij} = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j. \quad (2)$$

In the majority of LES studies an explicit SGS model is utilized to approximate the SGS stress tensor, typically using the resolved simulated flow field to estimate an eddy viscosity and either a user-provided or dynamically computed modeling constant. In general, some of the desirable features of a good SGS model include applicability across different types of flows such as free-shear and wall-bounded flows, transitional flows, and separated flows, as well as the ability to stabilize a numerical simulation on grids coarser than DNS. In addition, the models should be free of tunable coefficients, and should possess the ability to be extended for passive scalar mixing and reacting flow problems [24].

3. Stretched Vortex Models

Misra and Pullin [22] proposed a SGS model based on a structure-based Reynolds stress closure. The model assumes that the SGS turbulence can be represented by an ensemble of vortical structures whose orientation and SGS kinetic energy determine the SGS stress as shown below:

$$\tau_{ij} = K(\delta_{ij} - e_i^v e_j^v). \quad (3)$$

In the above equation the unit vector e represents the axis of the subgrid vortex and K represents the subgrid kinetic energy computed using the relation,

$$K = \int_{k_c}^{\infty} E(k) dk, \quad (4)$$

where $E(k)$ represents the energy spectrum and k_c represents the cut-off wave number computed as $k_c = \pi/\Delta$ with Δ being the filter width.

3.1. Vortex Orientation Models

In order to compute the SGS model, the vortex orientation vector e has to be described first. Misra and Pullin [22] proposed three different types of vortex orientation models. We describe two of them here:

3.1.1. e^3 - e^2 Model—This model was obtained based on the idea that the subgrid structure responds to the supergrid strain at a fast time scale [26]. Let $\tilde{S}_{ij} = \frac{1}{2}(\partial \tilde{u}_i / \partial x_j + \partial \tilde{u}_j / \partial x_i)$ represent the filtered strain rate tensor. Let the eigenvectors of $\tilde{S}_{ij}$ be e^1, e^2 and e^3 with corresponding eigenvalues λ_1, λ_2 and λ_3 such that $\lambda_1 < \lambda_2 < \lambda_3$. Then it is assumed that a fraction of the local subgrid structures align with the e^3 vectors and the remaining structures align with e^2 vector. Hence, the expression for the subgrid stress tensor is given by:

$$\tau_{ij} = K [\lambda(\delta_{ij} - e_i^3 e_j^3) + (1 - \lambda)(\delta_{ij} - e_i^2 e_j^2)], \quad (5)$$

where

$$\lambda = \frac{\lambda_3}{|\lambda_2| + \lambda_3}.$$

3.1.2. e^3 - ω Model—In this orientation model it is assumed that a fraction of the local subgrid structures are aligned with the e^3 vectors and the remaining structures are aligned

with the unit resolved vorticity vector, $\mathbf{e}$. Hence, the expression for the subgrid stress tensor is given by:

$$\tau_{ij} = K [\mu (\delta_{ij} - e_i^3 e_j^3) + (1 - \mu) (\delta_{ij} - e_i^\omega e_j^\omega)], \quad (6)$$

where $\mathbf{e}$ is the vorticity vector and

$$\mu = \frac{\lambda_3}{|\boldsymbol{\omega}| + \lambda_3}.$$

3.2. Subgrid kinetic energy

In order to compute the subgrid kinetic energy, for use in equation (3), the form of the energy spectrum must be provided. For our studies on LDCC, we employ two energy spectrums.

3.2.1. Lundgren Spectrum—In this approach the subgrid structures are modeled as a distribution of stretched vortices that are approximate solutions of the Navier-Stokes equations [27], and the form of the energy spectrum is given by:

$$E(k) = K_0 \varepsilon^{2/3} k^{-5/3} \exp\left(-\frac{2k^2 \nu}{3|\tilde{a}|}\right), \quad (7)$$

where K_0 represents the Kolmogorov prefactor, ε denotes the local cell averaged dissipation, and $\tilde{a} = \tilde{S}_{ij} e_i^v e_j^v$ represents the axial strain along the subgrid vortex axis. The application of this form of the energy spectrum for turbulent flows can be found in the studies of Kosovic *et al.* [24], Hill *et al.* [28], Lombardini *et al.* [29], and Chung and Pullin [30]. The subgrid kinetic energy can be computed by substituting equation (7) into equation (4) and can be shown to be [30]:

$$K = \frac{1}{2} K_0 \varepsilon^{2/3} \left(\frac{2\nu}{3|\tilde{a}|}\right)^{1/3} \Gamma\left[-\frac{1}{3}, k_c^2 \left(\frac{2\nu}{3|\tilde{a}|}\right)\right], \quad (8)$$

where Γ represents the incomplete gamma function.

3.2.2. Kolmogorov Spectrum—The subgrid kinetic energy can also be computed by assuming a Kolmogorov form for the energy spectrum. This approach can be seen in LES studies of Misra and Pullin [22], and Voelkl *et al.* [23]. The form of the energy spectrum using a sharp viscous cut-off is given by

$$E(k) = \begin{cases} K_0 \varepsilon^{2/3} k^{-5/3}, & k_c < k < J\eta^{-1} \\ 0, & k > J\eta^{-1}, \end{cases} \quad (9)$$

where $\eta = (\nu/\varepsilon)^{1/4}$ is the local Kolmogorov length scale and J is a cut-off parameter. More details about this spectrum can be found in Voelkl *et al.* [23]. The subgrid kinetic energy based on the Kolmogorov spectrum can be shown to be [23]

$$K = \frac{3K_0 \varepsilon^{2/3}}{2k_c^{2/3}} \left[1 - \left(\frac{k_c \eta}{J} \right)^{2/3} \right], \quad k_c < J\eta^{-1} \quad (10)$$

$$= 0, \quad k_c > J\eta^{-1}.$$

Note that this model ensures that the subgrid stress reduces to zero when the flow is locally resolved, which is a desirable property of SGS models. To determine whether the flow is locally resolved or not, the following non-linear equation must be solved:

$$1 - \Pi_1 X^4 + 3\Pi_2 \Pi_3 X^4 \left[1 - (X/J)^{2/3} \right] = 0, \quad (11)$$

where

$$X = k_c \eta, \quad \Pi_1 = \frac{2\tilde{S}_{ij}\tilde{S}_{ij}}{k_c^4 \nu^2}$$

$$\Pi_2 = \frac{K_0 \varepsilon^{2/3}}{k_c^{8/3} \nu^2}, \quad \Pi_3 = \frac{\tilde{S}_{ij}(\delta_{ij} - e_i^v e_j^v)}{2k_c^2 \nu}.$$

If $X > J$, the flow is estimated to be locally resolved and the subgrid kinetic energy is set to be zero. More information about the derivation of above method and an analysis of the non-linear equation (11) can be found in [23].

3.3. Spectrum coefficient estimation

In order to provide closure to the SGS model formulation described above, the group factor $\kappa_0^{2/3}$ that appears in the spectrum needs to be computed using the resolved field. Voelkl *et al.* [23] provided a physical space version of the stretched vortex model, where the group factor $\kappa_0^{2/3}$ is related to the averaged second-order velocity structure function. The spherical averaged estimate of the velocity structure function is given by:

$$\langle F_2^s \rangle = 4 \int_0^{k_c} E(k) \left[1 - \frac{\sin kr}{kr} \right] dk, \quad (12)$$

and the circular averaged estimate is given by

$$\langle F_2^c \rangle = \frac{2}{\pi} \int_{\phi=0}^{2\pi} \int_{k=0}^{k_c} E(k) \left[1 - J_0 \left(kr \sqrt{1 - \sin^2 \psi \cos^2 \phi} \right) \right] dk d\phi. \quad (13)$$

In the above equations $F_2 = \overline{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^2}$ is the velocity structure function, ψ is the angle between the normal of the circular plane parallel to the wall and the axis of the subgrid vortex, $\mathbf{x} = \mathbf{x} - \mathbf{x}'$ and $r^2 = (|\mathbf{x}|)^2 - (\mathbf{x} \cdot \mathbf{e})^2$. When using spherical averaging the $\langle F_2^s \rangle$ in equation (12) can be computed using a six point stencil [28] or a twenty seven point stencil [30]. For circular averaging the average of F_2^c , in equation (13), can be computed using a four point stencil [23] in the plane selected for the averaging. Once the averaged velocity structure function is known, the integral relations can be evaluated to estimate the group factor $K_0^{2/3}$. More descriptions of the circular and spherical averaging methods, and a computationally efficient method to solve the above relations can be found in Voelkl *et al.* [23].

4. Numerical Method

Higher-order numerical methods (third-order and higher) have the potential to more efficiently resolve flow features than lower-order methods [31]. This is especially the case for several space dimensions and for long time integration. In this paper, the high-order method described in Shetty *et al.* [13], which has been shown to be able to capture turbulence, is adopted. Below a brief description of the numerical method is presented.

The computational domain is discretized using a staggered mesh that allows accurate imposition of the tangential wall velocity. For efficient near-wall resolution smooth grid stretching function is employed in directions normal to walls to cluster nodes near the wall. The convective terms appearing in the LES equations are discretized using a fifth-order WENO scheme [33]. The viscous terms are discretized using a fourth-order central difference scheme [31]. The boundary conditions are imposed using three layers of ghost nodes and the ghost nodes are computed using Stoke's boundary conditions [34]. The fractional-step method described in Zhang *et al.* [18], which treats both the non-linear convective terms and the viscous terms explicitly is employed. The resulting decoupled equations for velocity are advanced in time using a low-storage third-order Runge-Kutta scheme [17], which involves four calculation stages for each simulation time step. A Poisson equation for the pressure is derived based on the incompressibility condition and is solved at each stage of the Runge-Kutta method using the MUDPACK solver [35].

5. Results

The SGS models are tested by performing the LES of the LDCC and comparing predictions to the direct numerical simulation (DNS) data of Leriche *et al.* [9], where a complete description of the problem, as well as additional details regarding the DNS study, can be found. LES predictions for the same case at $Re = 12000$ were carried out by Bouffanais *et al.* using the dynamic Smagorinsky and mixed SGS models [11]. Shetty *et al.* [13] performed LES for the same problem for $Re = 12000, 18000$ and 22000 using the Smagorinsky (SM), Vreman (VM) and dynamic Vreman models (DVM).

In this study, the previously described stretched vortex models were used to conduct LES. The total non-dimensional simulation time was $t^* = 4500$ with the last 3500 time units utilized for collecting flow statistics. The geometry of the computational domain is shown in Fig. 1. Steady motion of the top wall induces flow, while the other walls remain stationary. The velocity of the top wall is assumed to be of the form [9]:

$$u(x, 2h, z) = U_0 \left(1 - (x/h - 1)^{18}\right)^2 \left(1 - (z/h - 1)^{18}\right)^2 \quad (14)$$

where U_0 is the maximum velocity. Results for $Re = \frac{U_0(2h)}{\nu} = 12000, 18000$, and 22000 are presented in what follows.

5.1. $Re = 12000$

As a first test, the effect of different averaging procedures was studied. Simulations were performed using spherical and circular averaging procedures. The results from spherical averaging (not shown here) were found to be unsatisfactory and discrepancies were observed between the LES and DNS results for the mean flow fields. This tends to confirm the comments made by Voelkl *et al.* [23] that the spherical averaging procedure might be better suited for free flows but not for wall bounded flows due to the large shear combined with the suppression of turbulence in the sublayer near the wall region. Hence, only the

circular averaging procedure employed for all the subsequent simulations. Voelkl *et al.*[23] also point out that the circular averaging procedure could be used for both wall bounded and free flows. For a given node inside the domain, the plane for circular averaging was chosen such that it is parallel to the closest wall boundary containing the node. Once the plane is fixed the circular averaging is performed using the four points surrounding the node of interest.

Comparisons between LES and DNS for mean velocity profiles are shown in figure 2. The figure 2(a) shows predictions obtained using the Lundgren (LG) spectrum and the figure 2(b) shows predictions using the Kolmogorov (KG) spectrum. These spectrums are studied using the $e^3 - e^2$ and $e^3 - e^1$ orientation models. Good agreement with the DNS results is observed for the four possible combinations. There is no observable differences among the four combinations chosen for the mean velocity field components.

Figure 3 shows the comparison of the root mean square (rms) of the velocity components to the DNS predictions. The figures 3(a) and 3(b) show good agreement for $e^3 - e^2$ with LG and KG spectrum in the near-wall regions. However, slight over predictions can be seen in the center of the graph. Slight over predictions in the near-wall region can also be observed for the $e^3 - e^1$ orientation models. In figure 3(c), all the models capture the double-peak behavior near the bottom wall, with slightly better performance with the $e^3 - e^2$ orientation models. A slight over prediction near the top wall can be observed in this figure for all the models studied here. Figure 3(d) shows similar predictions by all models here with an under prediction in the peak near the bottom wall. The comparisons for velocity cross correlations are shown in figure 4. The model predictions are similar in figure 4(a) for all the models except for the slight overshoot in the $e^3 - e^1$ with the KG spectrum. The models are not able to capture the double dip behavior towards the bottom wall in figure 4(b).

Based on the LES results for the $Re = 12000$ case, there is no significant difference between orientation models for mean quantities and only a slight benefit can be observed with $e^3 - e^2$ models for the velocity fluctuations. Therefore, we choose to perform LES of LDCC at the higher Reynolds numbers using the $e^3 - e^1$ models with the two selected spectrums to better understand the influence of Reynolds number on the choice of turbulence spectrum model.

5.2. $Re = 18000$

For the $Re = 18000$ case, Shetty *et al.*[13] were not able to obtain a stable simulation using the constant coefficient Smagorinsky model. The VM model was also unstable at resolutions less than 80^3 . Only the DVM model was able to provide a stable solution at a resolution of 64^3 .

In the present simulations, stable solutions were achieved at 64^3 for both the Lundgren and Kolmogorov spectrums using the $e^3 - e^2$ orientation model, which indicates that the dynamic procedure associated with the SVM model is effective in providing sufficient energy transfer to stabilize the calculation while providing good predictions at lower resolutions. Simulation results at a resolution of 80^3 are also presented to understand the low and high resolution behavior of these SGS models. Figure 5 shows good agreement between the LES predictions and the DNS data [10] for mean velocity components. There is no observable difference in the prediction of mean velocity components with the selected spectrums. It also shows that LES predictions at 64^3 and 80^3 resolutions are identical indicating adequate resolution for the mean profiles.

The rms fluctuations for the velocity components along the x -direction are shown in figure 6. The plots also illustrate the influence of resolution on the predictions of LES statistics. Figure 6(a) compares the rms of the U velocity component along the x -direction. Slight over

predictions in the peak towards the wall are evident and these predictions can be seen to improve with resolution. The peaks also move towards their correct position (as in DNS) with increase in resolution. Figure 6(b) shows the same quantity along the y -direction. The predictions using both spectrums are similar. The KG spectrum appears to be able to capture the kink in the curve near the bottom wall. This behavior is more pronounced at lower resolution but becomes more accurate as the resolution increases. The rms for the V velocity component along the x -direction are shown in figure 6(c). Slight over predictions near the central region can be seen, similar to $Re = 12000$ case, however, this improves with resolution. In figure 6(d), the peak of rms for V is under predicted near the bottom wall but the peak moves towards the correct position (as in DNS) with increase in resolution. Figure 7 shows comparisons of cross-correlations of velocity components. Good agreement can be observed and improvement in the agreement with resolution can be seen. A closer inspection of low resolution prediction reveals better predictions with the Kolmogorov spectrum in comparison to the Lundgren spectrum for the same resolution.

5.3. $Re = 22000$

For the high Reynolds number $Re = 22000$ case, Shetty *et al.*[13] observed that stable simulations could not be obtained using VM model at 80^3 . Also a stable simulation with DVM was not possible at 64^3 resolution, and 80^3 resolution was needed to keep the simulation stable.

We observed that the selected SVM models did result in a stable solution even at a resolution of 64^3 confirming the superior stability of the SVM models over the eddy viscosity models tested by Shetty *et al.*[13]. We believe this is due to the dynamic estimation of the spectrum coefficients in the SVM model formulation, which in turn allows the sub-grid kinetic energy to change and adjust the sub-grid energy transfer to keep the overall simulation stable on coarser grid.

Figure 8 shows the comparison of mean velocity profiles against the DNS data [10] at the same Reynolds number. The figure shows that the result from the 80^3 simulations and from the 64^3 for the mean profiles are identical. Good agreement between DNS and LES can also be seen in this figure for mean velocity components. Again, there is no observable difference for the mean velocity components due to the spectrums selected at both the resolutions.

The comparison of rms of velocity fluctuations with the DNS data are shown in figure 9. The low resolution predictions shows significant over shoot for most of the quantities. The high resolution simulation can be seen to be able to capture the features accurately. In figure 9(d), the peak is under predicted and this behavior has been consistent for the SVM models across the Reynolds numbers examined in the present study. The velocity cross-correlation comparisons are shown in figure 10. Again, the increase in resolution can be seen to improve the results significantly. Also the low resolution prediction using the Kolmogorov spectrum is shown to be better than with Lundgren spectrum, consistent with our earlier observation for $Re = 18000$ case.

6. Flow Structure and Effect of Reynolds Number

Statistical comparisons of profiles between LES predictions and DNS data provide a practical means of assessment for SGS models but the mean and the root mean square fluctuations of the velocity field, and the structure comparisons provide additional important insight into overall model behavior especially with regard to predicting dynamic flow physics events in practical problems. Therefore, comparisons of flow features predictions of SVM using the Kolmogorov and Lundgren spectrums with the mean and the rms from DNS

data for all the three Reynolds numbers are presented in this section. Figure 11, figure 12 and figure 13 show contour plot of mean velocity U in $x - y(z = 0)$ plane (symmetry plane) for $Re=12000$, 18000 and 22000 respectively. Similar plots for mean velocity component V are shown in figure 14, figure 15 and 16. It can be noticed that the LES results successfully reproduce all the existing rotating structures at the correct locations. The regions with positive and negative velocity fields (shown with solid and dotted lines, respectively) and their shapes have been captured quite accurately. The results clearly capture influence of Reynolds number on flow features like the wall jet becoming thinner and straighter. This forces the vortex at the lower right corner (in V plots) to become shorter and thinner.

For the LDCC, it is well known that the bulk of the turbulence is generated as a result of the impact of the wall jet onto the bottom wall, and the rms velocity values reach their maximum near the bottom corner[10]. To further assess the ability of SGS models, we compare the *rms* contours from the LES predictions with DNS results near high turbulent regions. The *rms* fluctuation of the u component of velocity comparison is shown in figure 17 for the $Re=18000$. A similar comparison for the *rms* fluctuation of the component of velocity is shown in figure figure 18. The LES predictions can be seen to compare well with the DNS results. The position and the spread of the peak *rms* quantities have been captured accurately. A slight better prediction using Kolmogorov spectrum can be seen. The comparisons for other Reynolds numbers are not shows, as the agreement is very similar to that presented here.

The two-dimensional projected averaged resolved velocity vectors are shown in figure 19 and figure 20, similar to that reported in Bouffanais *et al.*[11]. The projected velocity in the plane $z/h = 0$ for the three selected Reynolds numbers are shown in figure 19 and a similar plot for the plane $y/h = -0.9388$ is shown in figure 20. The presence of the core central primary vortex and secondary corner vortices can be clearly seen in these pictures. In figure 20 the interface between the core vortex and the secondary vortices can be observed and also the change in the interface pattern with Reynolds number can be seen. The vector magnitude contour shows a double elliptical pattern in the core vortex for the $Re = 12000$ but this behavior vanishes with increase in Reynolds number.

Figure 21 shows the P_{22} production term computed using following equation [9]

$$P_{ij} = - \langle u_i u_k \rangle \frac{\partial U_j}{\partial x_k} - \langle u_j u_k \rangle \frac{\partial U_i}{\partial x_k} \quad (15)$$

Only the P_{22} component of the above tensor is computed and reported as it is the largest of the turbulence producing terms [9, 11] for the LDCC problem (figure 21). For the $Re = 12000$ case we can observe a double elliptical shape of the P_{22} contour indicated two separated elliptical jets, similar to that observed by Bouffanais *et al.*[11]. This separated jet behavior becomes less pronounced with increase in Reynolds number, consistent with the result reported in Shetty *et al.*[13].

The mean SGS dissipation provided by the SVM model is estimated using the relation[29]:

$$\langle \varepsilon_{sgs} \rangle = - \left(\langle \tau_{ij}^{sgs} \tilde{S}_{ij} \rangle - \langle \tau_{ij}^{sgs} \rangle \langle \tilde{S}_{ij} \rangle \right), \quad (16)$$

where $\langle \cdot \rangle$ represents an ensemble average. The plot of the mean SGS dissipation is shown in figure 22 and figure 23. From the figure 22, it can be observed that most of the SGS dissipation is restricted to the bottom of the downstream wall where the wall jet impacts on

the wall. Figure 23 shows the contour plot of the mean SGS dissipation in a plane parallel to the bottom wall ($y/h = -0.88$). It shows the extent of the region where the SGS model is active, and it can be observed that the region where SGS dissipation is provided expands with the increase in Reynolds number.

Next, we compute the total turbulent kinetic energy as sum of the resolved turbulent kinetic energy (K) and the subgrid kinetic energy (k). We use the following relations [29]:

$$\langle K \rangle = \frac{1}{2} \langle u'_i u'_i \rangle, \quad \langle k \rangle = \frac{1}{2} \langle \tau_{ii}^{sgs} \rangle. \quad (17)$$

Figure 24 shows the region of the cavity for total turbulent kinetic energy values greater than $0.003U_0^2$. This also indicates the region of the cavity for which the flow is turbulent. As can be observed in the figure, at lower Reynolds number the turbulent region is located at the center of the downstream wall spreading to the bottom wall. With the increase in the Reynolds number this region spreads towards the sidewalls and the bottom corners.

Finally we measure the inhomogeneity of the turbulence using the parameter [11]:

$$\langle \delta \rangle = 2 \frac{\partial^2 \langle u'_i u'_j \rangle}{\partial x_i \partial x_j}. \quad (18)$$

For homogeneous flows we obtain $\langle \delta \rangle = 0$ as the derivatives of the Reynolds stress

components $\langle u'_i u'_j \rangle$ are zero [11]. Figure 25(a) shows the region of the cavity that is inhomogeneous according to the criteria $|\langle \delta \rangle|_{max} > 1/100$. As can be observed in figure 25(a), the top portion of the downstream wall where the wall jet is formed due to the action of the moving wall, some inhomogeneity is observed. The majority of the inhomogeneous flow region is located at the bottom of the downstream wall where the primary wall jet impinges on the wall. Similar inhomogeneous regions of lower magnitude can be observed at the bottom of the upstream wall, due to the impingement of secondary wall jet. Finally, a small region of inhomogeneity can be observed on the upstream part of the top moving wall due to the impingement of the tertiary wall jet. These observations are consistent with that reported by Bouffanaï et al. [11] for $Re = 12000$ case. Figure 25(b) and figure 25(c) show the inhomogeneous part of the cavity for Reynolds number 18000 and 22000, respectively. We can observe that the overall volume of the cavity with homogeneous flow increases with an increase in Reynolds number. Also, the inhomogeneous region created by the impingement of tertiary jet on the upstream of the lid increases with Reynolds number. Thus, we can conclude that the turbulent region of the LDCC flows are completely inhomogeneous making the need for an accurate SGS model for inhomogeneous flow that much more important.

7. Cost of Computation

In this section we provide cost of computation associated with each of the SVM models studied here. The first step of SVM model is to compute the eigenvalues and the eigenvectors of the strain tensor S . Our experience showed that the cost associated with utilizing standard packages such as LAPACK [36] to perform this step could become comparable to the total cost. This cost can be significantly decreased by utilizing the fact that the matrix S is a 3×3 real symmetric matrix and hence using analytical techniques to compute eigenvalues and the vectors, for example following approaches described in Smith [37] and Anastassakis [38]. Another factor that can contribute to the total cost is the

evaluation of the incomplete Gamma function $\Gamma[-1/3, 2k_c^2\nu/(3|\tilde{u}|)]$ associated with the LG spectrum using standard packages, for example SLATEC [39]. However noticing that we need only $[-1/3, x]$, that is a single-variable function, and that the Gamma function has exponential trend we can utilize techniques such as look-up table or interpolation functions etc to reduce the associated computational time. Along with these approaches, the techniques described in Voelkl *et al.*[23] to reduce the cost of root finding associated with the KG spectrum and the integral evaluation required for the circular averaging are utilized.

Table I shows the time in seconds per iteration for the SVM models reported here. These testings were done on an Intel(R) Core(TM) i7 CPU 860, 2.8 GHz, four core linux machine. The results are presented at two resolutions 64^3 and 80^3 , respectively. It can be immediately noticed that the cost of computation doubles from resolution of 64^3 to 80^3 which is understandable as the nodes have been almost doubled. Also the difference in computational cost due to different vortex orientation models is insignificant for both the spectrums. It was observed that the cost of computation with KG spectrum without the root-finding part is very close to that with LG spectrum, indicating the difference in the time of computation is almost due to the root-finding part of KG procedure. So it can be concluded that the root finding cost constitute 9% of the total time at 64^3 and 6.8% at 80^3 resolution. This indicates the effectiveness of the root-solving procedure of the KG spectrum with cut-off implementation. It was found that on an average root-finding was performed on 61.14% and 38.56% of the nodes for 64^3 and 80^3 runs respectively. This shows that with increase in resolution more nodes falls under resolved fields and hence do not require SGS action.

8. Conclusions

In this study we assessed the ability of non-eddy viscosity type stretched vortex models for completely inhomogeneous flows using higher order numerical methods. We tested the two averaging procedures and found that the circular averaging procedure yields better SGS model performance compared to the spherical averaging procedure. This outcome is consistent with the remarks and results provided by Voelkl *et al.*[23] and elucidates the need of circular averaging for LES of inhomogeneous turbulent flows. We found that among the two sub-grid vortex orientation models, e^3 - e^2 model found to give better performance compared to the e^3 - model for rms of velocity components. On the other hand, Misra *et al.* [22] observed very little influence of orientation models for homogeneous flows. Also our analysis of the assumed turbulence spectrum in the SVM model formulation revealed that Kolmogorov spectrum with cut-off produced better results compared to the Lundgren spectrum at low resolutions, justifying the cost of root-finding problem and the complexity of implementation. For the range of the Reynolds numbers tested the SVM models were found to have superior stability compared to the eddy-viscosity type of models studied in Shetty *et al.*[13], and provide good agreement with the DNS results. Finally we also showed that the turbulent LDCC flows are inhomogeneous verifying that the assessment was in fact performed for inhomogeneous turbulent conditions.

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Figure 1.
Geometry of lid-driven cubic cavity problem under study.

Figure 2. Comparisons of mean velocity profiles between DNS and LES results for $Re = 12000$

Figure 3.
Comparisons of rms u and v velocity profiles between DNS and LES results for $Re = 12000$.

Figure 4. Comparisons of Reynolds shear stress profile between DNS and LES results for $Re = 12000$

Figure 5. Comparisons of mean velocity profiles between DNS and LES results for $Re = 18000$

Figure 6.
Comparisons of rms u and v velocity profiles between DNS and LES results for $Re = 18000$.

Figure 7. Comparisons of Reynolds shear stress profile between DNS and LES results for $Re = 18000$

Figure 8. Comparisons of mean velocity profiles between DNS and LES results for $Re = 22000$

Figure 9.
Comparisons of rms u and v velocity profiles between DNS and LES results for $Re = 22000$.

Figure 10. Comparisons of Reynolds shear stress profile between DNS and LES results for $Re = 22000$

Figure 11.

Re=12000; Mean Velocity $U = -0.2U_0$ and $1.0U_0$; 100 intervals, dashed negative lines (color online).

Figure 12.

Re=18000; Mean Velocity $U = -0.2U_0$ and $1.0U_0$; 100 intervals, dashed negative lines (color online).

Figure 13.

Re=22000; Mean Velocity $U = -0.2U_0$ and $1.0U_0$; 100 intervals, dashed negative lines (color online).

Figure 14.
 $Re=12000$; Mean Velocity $V = -0.6U_0$ to $0.1U_0$; 100 intervals, dashed negative lines
 (color online).

Figure 15.

Re=18000; Mean Velocity $V = -0.6U_0$ to $0.1U_0$; 100 intervals, dashed negative lines (color online).

Figure 16.
 $Re=22000$; Mean Velocity $V = -0.6 U_0$ to $0.1 U_0$; 100 intervals, dashed negative lines
 (color online).

Figure 17.
 $Re=18000$; Rms Velocity $u=0$ to $0.0845 U_0$; 66 intervals (color online).

Figure 18.

Re=18000; Rms Velocity $v=0$ to 0.16; $65 U_0$ intervals (color online).

Figure 19.

Two dimensional averaged resolved velocity vectors colored by velocity magnitude($z/h = 0$)
 $= 0.0$ to $1.0 U_0$; 50 intervals.

Figure 20.

Two dimensional averaged resolved velocity vectors colored by velocity magnitude($y/h = -0.9388$) = 0.0 to $0.2U_0$; 50 intervals.

Figure 21.

P_{22} Production term = $-0.025U_0^3/2h$ to $0.075U_0^3/2h$, 50 intervals, dashed negative lines, using KG spectrum (color online).

Figure 22.

SGS dissipation ($z/h=0.0$) = 0 to $0.003\nu U_0^2/(2h)^2$; 50 intervals; KG spectrum.

Figure 23.

SGS dissipation ($y/h = -0.88$ to $0.003\nu U_0^2/(2h)^2$; 50 intervals; KG spectrum).

Figure 24.
Turbulent kinetic energy $k = 0.003 U^2$, using KG spectrum (color online).

Figure 25. Inhomogeneous turbulent region of the cavity ($|||_{max}| > 1/100$), using KG spectrum (color online).

Table I

Time(in seconds)/iteration on 4 processors using different SGS models.

(a) KG.		
Model	64 ³	80 ³
e ³ - e ²	0.76	1.56
e ³ -	0.76	1.57

(b) LG.		
Model	64 ³	80 ³
e ³ - e ²	0.70	1.46
e ³ -	0.70	1.45