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On the Evaluation of the Elastic Modulus of Soft Materials Using Beams with Unknown Initial Curvature

Hamed Khatam and K. Ravi-Chandar[†]

Center for Mechanics of Solids, Structures and Materials The University of Texas at Austin
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Abstract

A nonlinear optimization procedure is established to determine the elastic modulus of slender, soft materials using beams with unknown initial curvature in the presence of large rotations. Specifically, the deflection of clamped-free beams under self-weight – measured at different orientations with respect to gravity – is used to determine the modulus of elasticity and the intrinsic curvature in the unloaded state. The approach is validated with experiments on a number of different materials – steel, polyetherimide, rubber and pig skin. Since the loading is limited to self-weight, the strain levels attained in these tests are small enough to assume a linear elastic material behavior. This nondestructive methodology is also applicable to engineered tissues and extremely delicate materials in order to obtain a quick estimate of the material's elastic modulus.

1. Introduction

Developments in the areas of biology and bioengineering over the past few decades have engendered significant interest in understanding, modeling and characterizing the mechanical properties of a wide range of soft materials and structures such as cells, tissues, blood vessels, plant stalks, fruits and vegetables, and processed foods. Motivation for such investigations varies depending on the particular application. In imaging applications related to tissues, elastic properties are crucial to the interpretation of various imaging modalities such as ultrasound, magnetic resonance imaging etc; the difference in the moduli between normal tissue and tumors can be about two orders of magnitude [1]. When these tissues are excited with ultrasound and tissue motion is detected either from Doppler signal or through magnetic resonance imaging, a three dimensional map of the tissue stiffness is obtained. Many biopolymers and gels are under investigation as potential actuators and drug-delivery systems; careful characterization of the stiffness of these materials has been undertaken by many investigators [2,3]. Borschel et al. [4] measured the Young's modulus and failure stress of acellular peripheral nerve. Engelmayer et al. [5] performed a beam bending test to determine the Young's modulus of tissue engineered heart valve biomaterials. Wei et al. [6] examined the mechanical properties of epidermal cells of onion using a specially designed apparatus to pull the cells in tension. Numerous investigators have used indentation methods for the extraction of the mechanical properties by imposing a Hertz contact type model. If the main purpose is to determine the Young's modulus, all of the *in vitro* methods indicated above are much too difficult and error prone.

Gittes et al. [7] determined the bending rigidity of microtubules and actin filaments by monitoring thermal fluctuations; Vernier et al. [8] used hydrodynamic flow to evaluate the bending stiffness of microtubules. However, the analysis used in both of these investigations

[†]Corresponding Author: ravi@utexas.edu.

was based on assuming small deflections and rotations of the beam. While the nonlinear analysis is easily incorporated, the use of thermal fluctuations will only work in very small structures and the use of hydrodynamic flow introduces the need to measure the flow velocity and interpret the drag forces through a model. More recently, Peng and coworkers [11], considered the large deformation formulation to obtain the elastic modulus of soft gels; while they incorporated the effect of transverse shear in generating curvature in addition to the usual Bernoulli-Euler beam theory, the effect of axial forces was ignored rendering the formulation useful only for small deflections. Furthermore, the effect of the initial curvature of the beam was not considered in their formulation. Here we present a very simple, quick and efficient means of determining the Young's modulus of soft materials, which circumvents the many difficulties involved in preparing and clamping small, delicate samples, and eliminates the need for measurements of small forces. In fact, only geometrical measurements and weight measurements— both of which can be done with great accuracy – are required. By examining the shape of a thin filament bent under its own weight, one can extract the bending rigidity and Young's modulus as well as the initial shape in the unloaded configuration directly through an optimization procedure that involves the nonlinear deflection of the beam.

This paper is organized as follows: the system of nonlinear ordinary differential equations that govern the large deflections of a slender curved beam are described in Section 2; the calculation of the bent shape of a clamped-free beam under self-weight loading is discussed for two different orientations of the loading. The optimization procedure for the inverse problem of obtaining the initial shape and the bending rigidity is also discussed in this section. This is followed in Section 3 by examples of validation on beam specimens made of different materials: steel, polyetherimide, rubber, and pig skin. The main conclusions are summarized in Section 4.

2. Large Deflection Analysis of Curved Beams

2.1. Governing Equations

Let us consider a slender beam, of arbitrary cross-section; for simplicity, we will assume that the cross-section contains a plane of symmetry and that forces applied on the beam will be restricted to lie in this plane and will result in a two-dimensional problem. We will also assume that the extensibility of the beam is negligible; we will, however, allow the tension to vary along the length of the beam. These assumptions are not necessary since the general theory of combined tension, bending and torsion of slender beams can also be worked out completely, but the restriction will simplify our presentation in this manuscript and furthermore this restricted formulation is sufficient for the applications suggested here for the determination of the modulus. Let (x, y) represent the global Cartesian coordinate system. The position vector of any point can be written by the vector $\mathbf{r}$

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} \quad (1)$$

where $(\mathbf{i}, \mathbf{j})$ are the unit vectors in the (x, y) directions and s is the arc length parameter. From now on, we will imply the s dependence after first occurrence. The unit vectors defined in the local coordinate system are $(\mathbf{t}(s), \mathbf{n}(s))$ where $\mathbf{t}$ is the unit tangent vector and $\mathbf{n}$ is the unit normal vector. Let the angle between the tangent vector and the x direction be denoted by $\theta(s)$. From the geometry, we have the curvature :

$$\frac{dx}{ds} = \cos \theta, \quad \frac{dy}{ds} = \sin \theta, \quad \kappa = \frac{d\theta}{ds} \quad (2)$$

Let the applied loading be given as a vector-valued function $\mathbf{q}(s) = q_t \mathbf{t} + q_n \mathbf{n}$, including the possibility of distributed forces as well as concentrated forces; it has dimensions of force per unit length. At any cross section, the resultant of the internal forces may be expressed as a set of forces resolved along the tangent and normal directions ($T(s)$ and $N(s)$ respectively) along with a moment about the z -axis, $M(s)$; these resultants are indicated in Figure 1. Equilibrium equations for a beam under such conditions were first written down by Kirchoff in 1859; it is given in Love's book [9] in the following form (reduced here for the two-dimensional case):

$$\begin{aligned} \frac{dT}{ds} + T\kappa + q_n &= 0 \\ \frac{dN}{ds} - N\kappa + q_t &= 0 \\ \frac{dM}{ds} + N &= 0 \end{aligned} \quad (3)$$

For typical slender beams, the flexibility is typically orders of magnitude larger than the extensibility and hence the axial stretching will be ignored; thus, while the tension T is allowed to vary along the length of the beam, its effect in producing an elongation of the beam is neglected. Furthermore, for such slender beams, the contribution of the shear force N to the deflection of the beam is typically two orders of magnitude smaller than the deflection due to the bending moment and hence, the effect of transverse shear force on the deflection will also be neglected. Both these simplifications result in dictating that the slender rod will indeed deform according to the Bernoulli-Euler beam theory (that plane sections remain plane and rotate about axis of the bending moment). This provides the relationship between the curvature at any point and the corresponding bending moment (about the z -axis)

$$EI\kappa(s) = M(s) \quad (4)$$

where E is the modulus of elasticity or Young's modulus and I is the second moment of the cross sectional area. EI is called the flexural rigidity in the engineering literature and is denoted by B in Love's book; the biological literature has retained Love's notation. Note that possible variation in cross-sectional dimensions along the beam affects the flexural rigidity and this can be easily incorporated into Eq.(4). Initially curved beams are included in this analysis if $\kappa(s)$ is interpreted as the change in curvature: $\kappa(s) - \kappa_0(s)$, where $\kappa_0(s)$ is the initial curvature that could arise from residual stress due to misfit, growth, lamination etc. Equations (2) through (4) constitute seven equations for the unknown quantities x, y, θ, N, T and M . The bending moment may be eliminated between Eqs. (3) and (4), resulting in six equations in six unknowns; this system of first-order differential equations requires the specification of appropriate boundary conditions. For a beam of finite length, boundary conditions must be specified at each support point; such supports for beams may be fixed, free, or a simple support; other supports on elastic restraints are also possible. A fixed support prescribes the displacement and rotation of the beam to take on specific values

$$x = \hat{x}, y = \hat{y}, \theta = \hat{\theta} \quad (5)$$

where the quantities with the $\hat{}$ are specified quantities. A free end imposes no constraints on the displacement or rotation and hence all the resultant forces must vanish

$$T=0, N=0, M=0 \quad (6)$$

A simple support is one that prevents beam displacement but not rotations; hence

$$x = \hat{x}, y = \hat{y}, M=0 \quad (7)$$

Note that the zero moment conditions should be replaced in terms of zero curvature in light of Eq. (4). It is also important to note that while the displacements are in the global x - y coordinate system, the forces, N and T are in the local (tangent-normal) coordinate system. This is particularly crucial in mixed boundary conditions where the displacement is prevented in one direction but free to occur in a perpendicular direction. For example, for a simple support on a roller that permits free displacement in the x -direction, but not the y -direction, the boundary condition is

$$T \cos \theta - N \sin \theta = 0, \quad y = 0, \quad M = 0 \quad (8)$$

but, θ is itself unknown. We will now specialize these equations to the case of a beam of length L , supported at the two end points; to simplify the equations, we will normalize all length quantities by L and all force quantities appropriately as follows

$$\begin{aligned} \bar{x} &= x/L, \quad \bar{y} = y/L, \quad \bar{s} = s/L, \quad \bar{\kappa} = \kappa L \\ \bar{N} &= \frac{NL^2}{EI}, \quad \bar{T} = \frac{TL^2}{EI}, \quad \bar{q}_t = \frac{q_t L^3}{EI}, \quad \bar{q}_n = \frac{q_n L^3}{EI} \end{aligned} \quad (9)$$

With this normalization, the arc length parameter takes on values from zero to one. Introducing this normalization in Eqs. (1) through (4) and then dropping the over bars, results in the following system of normalized equations

$$\begin{aligned} x' &= \cos \theta \\ y' &= \sin \theta \\ \theta' &= \kappa \\ \kappa' &= -N + \kappa_0' \\ N' &= -T\kappa - q_n \\ T' &= N\kappa - q_t \end{aligned} \quad (10)$$

where the prime denotes differentiation with respect to s . For a straight beam, $\kappa_0' = 0$. This system of first order differential equations subjected to appropriate boundary conditions selected from Eqs (5) through (8) completes the formulation of the problem.

2.2. Example Solutions

Now we consider a beam that is loaded by its own weight, but with different boundary conditions. Letting q_0 be the weight per unit length of the beam acting in the negative y -direction; the distributed loading on the beam is

$$\mathbf{q}(s) = -q_0 \mathbf{j} = (-q_0 \sin \theta) \mathbf{t} + (-q_0 \cos \theta) \mathbf{n} \quad (11)$$

First, consider that the initially curved beam is cantilevered – fixed at $s = 0$ with a slope and freely hanging at $s = 1$. Thus the boundary conditions are

$$\begin{aligned} x(0) &= 0 & y(0) &= 0 & \theta(0) &= \theta_0(0) \\ T(1) &= 0 & N(1) &= 0 & \kappa(1) &= \kappa_0(1) \end{aligned} \quad (12)$$

If the beam has a nonnegligible initial curvature, ignoring the initial shape can lead to substantial error in the beam's deformed shape. A straight beam with no initial curvature is a special case of the above boundary conditions when the slope at $s = 0$ and the curvature at $s = 1$ are both zero. Eqs. (10) and (11) are now to be solved subjected to the boundary conditions in Eqs. (12). This represents a straightforward two-point boundary value problem and is readily solved numerically with many different techniques such as the shooting methods; we obtained the solution using a MATLAB routine.

The general solution to the deformed shape of a straight beam under self-weight, with $\phi(0) = 0^\circ$, is given in Figure 2 for $q_0 L^3/EI = 0.1, 1, 10$, and 100 . The self-weight loading appropriately normalized by the beam length and bending rigidity is varied over three orders of magnitude to demonstrate the importance of self-weight in bending of beams: clearly, for materials with a large bending rigidity, large deflections can arise in very long beams. On the other hand, for materials with a very small bending rigidity, even short beams might hang almost vertically. The corresponding variation of the tension and shear force along the length of the beam is shown in Figure 3 for the cases $q_0 L^3/EI = 10, 100$. Note that for this boundary condition, the tension must go to zero at the fixed end, while the shear force is equal to the normalized weight of the beam ($q_0 L$).

As a second example, we consider the same cantilever beam, with no initial curvature, but held vertically at $s = 0$: $\phi(0) = \pi/2$. This problem was considered long ago by Greenhill [10] who considered the stability of growing trees; Greenhill, however, assumed that the deflection of the beam was small and showed that if the length is $L > 7.84(EI/q)^{1/3}$ the vertical shape is unstable and other bent shapes can be assumed. Using the equations described above, and modifying the boundary condition at the fixed end to be

$$x(0) = 0, y(0) = 0, \theta(0) = \pi/2 \quad (13)$$

we find the equilibrium shapes of the bent rod for values of $q_0 L^3/EI > 7.84$. These are displayed in Figure 4, for different values of $q_0 L^3/EI$.

The upshot of this analysis is that one simply has to observe the deflected shape; then given other geometrical parameters (length and cross-sectional dimensions) and the weight of the beam, the modulus can be determined directly by solving this nonlinear inverse problem through an optimization procedure as described in the next section.

2.3. Optimization procedure to find the best estimate for the elastic modulus

We use the solution to the system of ordinary differential equations in Eqs. (10) subjected to the loading in Eqs.(11) and the boundary conditions in Eqs. (12) and compare it to the measured shape through a least-squared-error minimization procedure in order to determine the unknown initial curvature and bending rigidity of the beam. First, we consider an initially straight beam. The deflected shape of the beam is obtained from a photograph of the deflected beam; this is accomplished by digitization and image processing to obtain the coordinates, $\mathbf{r}_m(s)$, representing the midline of the deflected beam. We define the square-error function:

$$f(E) = (\mathbf{r}_c(s;E) - \mathbf{r}_m(s)) \cdot (\mathbf{r}_c(s;E) - \mathbf{r}_m(s)) \quad (14)$$

where $\mathbf{r}_c(s;E)$ is the calculated deflection that depends on the unknown parameter E . By making multiple measurements of the beam shape at a large number of positions along the beam, and minimizing the squared-error function in Eq.(14) a best estimate of the elastic modulus can be obtained.

Next, we consider beams that possess intrinsic curvature in the unloaded state that must be taken into account in determining the modulus. In this case, an additional step is necessary: either an image of the beam is obtained in an unloaded condition¹ to determine its initial shape or alternatively, this initial curvature may be considered as an unknown and estimated using the minimization procedure if we measure the response of the beam in more than one

¹We note that this may be possible if the beam stiffness in two orthogonal planes is significantly different or if the beam is placed in a neutrally buoyant state in a fluid.

orientation (for example by rotating the fixed-end or placing the specimen in an upside down orientation, effectively reversing the loading direction). In the case of multiple measured responses at different orientations, the appropriate solution can be obtained by prescribing the actual orientation and loading; then, minimization of the total error function (summation of errors for all orientations), results in the best estimate for the averaged elastic modulus, as well as the curvature represented in terms of a polynomial of suitable order, based on all measurements. Note that even for a single measured shape more than one comparison can be performed if another point along the beam is selected as the starting point. The only difference is that the new beam now possesses a shortened length and modified initial slope. This method can be particularly useful when the initial boundary condition of the whole beam is not well-controlled.

It is worth noting that at any time at which an image is taken, the assumption is that the material is linearly elastic. However, this is not a necessary condition. A different relation between stress and strain with more than one unknown parameter can be assumed and a slightly modified system of ODEs can be obtained with some manipulation starting from Eq (4).

3. Demonstration of Measurement of Young's Modulus

In order to demonstrate the ease with which Young's modulus of soft materials may be measured, we performed measurements on specimens of rubber, and pig skin. However, we first demonstrate the validity and reliability of the method for two materials with linear behavior in low strain range at room temperature: steel and polyetherimide.

3.1. Steel

The first example consists of a validation of the robustness of the method for a linear elastic material with negligible time or strain-rate effects. A stainless steel feeler gauge (thickness = 0.076 mm, width = 12.27 mm, free length = 208.4 mm) was selected. The factory cut shape and constant thickness minimizes uncertainties regarding the variable width and thickness. The beam was clamped at one end to an aluminum block and left free at the other end; the beam was loaded by its self-weight corresponding to the problems described in Section 2.2. The high strength of the material ensures that the forced clamping does not alter the initial curvature of the beam. The reliability of a fully clamped fixed end condition is important in a validation test, since it ensures that the correct value for $y(0)$ is prescribed in Eq.(12). The beam had an initial curvature; this curvature was determined by capturing an image of the beam $(x_0(s), y_0(s))$ when held in an unloaded state (see Figure 5(a)). As remarked earlier, this method is usable if the bending stiffness in one direction has a much higher stiffness than the orthogonal direction. It was determined that a polynomial can be fitted to the measured initial shape. Then the beam holding clamp was installed on a goniometric positioner to orient the specimen at precise angles relative to gravity. A simple pendulum was used to provide the reference vertical direction. Three different still images were taken with the specimen oriented at $\theta(0) = 30^\circ, 60^\circ, 90^\circ$ (see Figure 5(b, c, and d)). The images were digitized to obtain the initial and deflected shapes $(x(s), y(s))$. The best estimate for the elastic modulus was obtained through the optimization procedure described in Section 2.3. Figure 5e shows the digitized initial shape (blue dashed line), the deflected shapes (black symbols) for the three orientations, and the best estimated shapes (red lines) for the corresponding orientation. The estimated elastic modulus is 210.0 GPa (SD = 0.6%) based on all three orientations and also three different values for the degree of the polynomial representing the initial curvature. The obtained elastic modulus was not very sensitive to the degree of the polynomial used to fit the initial curvature as long as it was between three and five.

3.2. Polyetherimide

A polyetherimide strip was cut to the shape of a beam with the following dimensions: thickness = 0.254 mm, width = 8.0 mm, free length = 140.5 mm. Since the glass transition temperature, T_g , for this material is around 216°C, the beam showed very little time dependent shape change at room temperature; hence, we shall consider the material to exhibit its short time elastic modulus. The condition of clamped-free beam under selfweight loading was implemented as described above for the steel. The initial curved shape of the specimen was measured by imaging the specimen in the unloaded state; however, capturing such an image might be challenging in some cases such as biological tissues. Therefore, the alternative method of estimating the initial shape described in Section 2.3 was chosen for this problem as a demonstration. Two images of the beam were taken with $\theta(0) = 0^\circ$, one each for the beam positioned with the initial curvature facing upward and downward – this corresponds to loading the beam twice, but in opposite directions. Each image was taken one hour after releasing the free end in order to minimize any time dependence. The optimization procedure was used to determine both the modulus of elasticity and the initial curvature. Figure 6a shows the digitized deflected shapes (black symbols), and the best estimated shapes (red lines) for the two loading orientations. The best estimated average elastic modulus was $E = 2.775$ GPa and corresponds well with the glassy modulus for this material. The initial curved shape that was obtained as part of the optimization procedure is shown in Figure 6b by the blue dash line and compared to the measured initial curved shape (black symbols). Clearly, there is good agreement between the two, demonstrating that the method proposed could be used to determine the bending rigidity as well as the initial curved shape of the beam.

3.3. Rubber

In the 3rd example, we consider a commercial natural rubber². While this material exhibits a time dependent viscoelastic behavior, at room temperature, the material is well above its glass transition temperature, T_g , and is in a rubbery state with very long relaxation times. Therefore, an instantaneous elastic response may be assumed for the short term response and used to estimate the instantaneous modulus of the rubber. Furthermore, for the small strains encountered in the bending problem, we shall assume a linear response for the rubber with a time dependent elastic modulus $E(t)$. The condition of clamped-free beam under self-weight loading was implemented as described above for the steel and polyetherimide, but this time, the beam was held in the vertical orientation. The initial shape and the deformed shape of the beam at 4 different times after releasing the free end from a vertical position are superimposed into a single image in Figure 7a to show the time dependence of the deflection at 3, 10, 60 and 1200 minutes after release. The deflected shapes clearly indicate the increasing compliance of the material. The average elastic modulus at each time step was determined through the optimization procedure described in Section 2.3. Comparison of the measured deflected shape (black symbols) and the calculated shape (red lines) is shown in Figure 7b. The variation of elastic modulus with time is plotted in Figure 7c. This time dependence appears to exhibit a power law behavior that is typical for rubbers and elastomers above T_g [13], although a limiting ‘long-time’ modulus may be observed if relaxation tests are performed for longer duration.

3.4. Porcine Skin

Skin is the subject of our 4th example. Skin is a connective tissue but with one significant difference from other connective tissues: it is a protective barrier with one side always subjected to the outside environment. The general practice in tissue testing is to place the

²Abrasion-resistant natural gum rubber obtained from McMaster-Carr (www.mcmaster.com)

sample inside a body temperature hydration bath for maintaining the physiological temperature and also avoiding excessive drying. However, the compromise in this process is that the sample will absorb some water and may exhibit a non-native response. Obviously, skin in physiological condition is not fully immersed inside a liquid and the *in-vivo* mechanical properties may vary from those obtained from an overhydrated skin sample [14]. Another important factor in determining the mechanical response of soft tissues is the strain-rate dependence of the response. Although the response is expected to become less sensitive to strain rate at very low strain rates [15], this requires very long loading and immersing time which inevitably leads to an overhydrated sample.

We performed the beam bending test on pig skin obtained from the local abattoir and tested within 24 hours post-mortem. Samples from the abdominal area were cut along the longitudinal axis to uniform width (5 mm) after removing the subcutaneous layer. The measured density was between 1.100 and 1.160 g/cm³, and skin thickness varied between 1.25 and 1.75 mm which is in agreement with the commonly expected values [12]. All bending tests were performed in air and at room temperature. In order to have comparable responses with similar viscous effects, each image was taken at 30 seconds after releasing the free-end. The best estimate for the average elastic modulus at each instance was obtained through the procedure described in Section 2.3.

Three different samples were tested; two were from adjacent regions in the abdominal area while the third sample came from the same area, but from a different animal: for samples number 1 and 2, the initial curvature was determined by placing the sample inside a saline bath and taking the image immediately. Since the density of the saline solution can be adjusted to the density of the skin sample resulting in an almost buoyant sample, this technique allows obtaining the initial shape of the sample under a no-load condition. The time necessary to capture an image in such condition is very small – typically a few seconds – and as a result this procedure avoids over-hydration of the sample. In the case of third sample, the initial curvature was not measured, but inferred from measurements of deflection at different orientations. The clamped boundary condition for samples 1 and 3 was achieved by gluing the samples. However, for sample 2, we applied another technique in which the slender beam was placed at its longitudinal center on a horizontal narrow rod (see Figure 8). This technique is completely nondestructive; by fully eliminating gluing and clamping, it produces a freely hanging beam with two segments of roughly the same size and weight, with zero curvature at the support point and effectively simulating two beams with clamped-free boundary conditions. Due to the fact that the sample does not undergo any type of external alteration such as clamping or gluing, this technique is particularly suited to obtain and monitor the estimated elastic properties of engineered tissues during culture. Since the sample possessed an initial curvature, two sets of responses are obtained by placing the beam with the initial curvature both up and down. After this, the condition of clamped-free beam under self-weight loading was implemented as described above for the other materials and the following images were taken: sample 1 at $\theta(0) = 0^\circ$ with the initial curvature up and at $\theta(0) = 90^\circ$; sample 2 at $\theta(0) = 0^\circ$ with the initial curvature up and down; and sample 3 at $\theta(0) = 0^\circ$ and the initial curvature both up and down, and $\theta(0) = 90^\circ$. The optimization procedure described in Section 2.3 was used to obtain the best estimates for the modulus in the case of specimens 1 and 2, and both the modulus and the initial shape for specimen 3. A summary of the results is shown in Table 1. The best estimate for the modulus from the three samples is in the range of 320 to 345 kPa. These measurements indicate that a consistent estimate of the modulus can be obtained even for initially arbitrarily curved skin specimens.

In order to compare this value of the modulus with other test conditions, uniaxial tension tests were also performed at two different loading rates on two adjacent samples. The

response at the first cycle was used to determine an elastic modulus comparable to the bending test results. These results and the corresponding loading rates are also given in Table 1. The difference between the modulus obtained from the tension test (continuous elongation of the sample) and that from the beam bending test is notable; the bending test yields values that are about one half of the tension test values. In uniaxial tension test of biological tissues, the loading and unloading branches are different and each possesses a different tangent modulus near the initial undeformed shape. The actual equilibrium state of the material is expected to be within a range determined by these two branches. Furthermore, measurement of the modulus in the tension test at very low load levels is quite challenging due to lack of resolution of both the force and the elongation, resulting in potentially large errors in the estimated modulus. In the bending tests, it was observed that the obtained elastic modulus from bending test is always smaller than the initial elastic modulus extracted from the loading branch of the uniaxial tension test even when the tension test was performed in a low strain rate of 0.0001/s. The unloading branch was unreliable due to residual stretch left in the specimen.

The bending of a beam under self-weight also serves to demonstrate the influence of moisture content on the elastic modulus of skin. Dehydration starts as soon as a sample is placed in air and can be readily monitored by measuring the weight of the sample. However, accomplishing this in a uniaxial test is not particularly easy; since each weight measurement requires the sample to be un-mounted from the machine determining the mechanical properties as a function of moisture content is not particularly a feasible option for biological tissues (preconditioning, vague zero-stress-strain state, time-temperature-history dependence, etc will influence the results significantly). The capability of the beam bending test in capturing the response of a beam at a very short period of time and with minimum sample and testing apparatus preparation efforts makes it a powerful method in these situations. In this case, the images of the sample bending under self-weight in three different orientations were captured at three different times (6 minutes apart); its weight was measured at the end of each time interval. The sample was placed in a load-free position when not imaging to prevent the sample from continuous creep. Each image was taken after 30 s from the free-end release. Therefore, for each of the three time instants, the weight and the beam response at three different orientations were obtained. Images of the deflected shapes at times $t_1 = 0$ s, $t_2 = 360$ s, and $t_3 = 720$ s, for two different orientations – $\theta(0) = 0^\circ$ (with the initial curvature up) and $\theta(0) = 90^\circ$ – are shown in Figure 9a and 9b. The best estimates of the initial shapes are also shown by the blue dashed lines. The initial shape became more curved from t_1 to t_2 , while the change was smaller from t_2 to t_3 . These images show clearly that as the specimen dehydrates, it gets stiffer and the bending deflection gets smaller. Based on these three deflected shapes, the average elastic modulus as well as the initial shape was estimated through the optimization procedure described in Section 2.3. It should be noted that because of the 30 s waiting time and also re-orientation time, there was an approximately 60 s time gap between two differently oriented images. Since, as the results of this study suggests, dehydration dramatically affects the sample stiffness, the calculated elastic modulus and initial shape are average values. A comparison of the measured and calculated shapes for the two orientations and three different dehydration levels is shown in Figures 8c and 8d. The optimal estimates of the modulus as a function of the dehydration time is shown in Table 2; the variation of the total weight of the specimen is also shown in this table. The strong influence of water content on the elastic modulus is readily apparent: while the variation in weight is approximately 3%, the stiffness increases by a factor of five. This clearly demonstrates the sensitivity of the mechanical response to moisture content and the fact that in order to obtain mechanical properties close to the in-vivo behavior, the effect of each actor from the sample's native physiological environment should be evaluated individually.

4. Conclusion

A method of evaluating the elastic modulus (the initial tangent modulus) of soft materials through a simple beam bending test with the assumption of small strain but large rotation has been presented. This is accomplished through a nonlinear optimization scheme. Furthermore, it is shown that although slender samples are seldom completely straight, no additional information about the native shape of a curved beam (unstressed geometry) is necessary since the initial curvature can also be estimated by the nonlinear optimization procedure.

The response of a fixed-free beam under self-weight is represented by a system of ODE which can be solved by prescribing the appropriate boundary conditions. For a particular beam, the calculated response with different values of elastic modulus is compared to the measured response and the best estimated elastic modulus is obtained through minimizing the error between calculated and measured deflected shapes. The measured response is obtained from digitization of an image taken from the beam loaded under self-weight. The only other required data are the geometry and density of the beam. The verification examples on steel and polyetherimide beams demonstrated that the error can be controlled to remain within 1%.

In the case of biological materials, the beam bending test can be performed by gluing a sample at one end to produce a fixed boundary condition. If gluing is not possible or desired, simply placing a slender sample at its longitudinal center on a narrow rod will produce two hanging beams. This avoids clamping and grip issues common in tension tests of soft materials such as gels and delicate tissues. In addition, by eliminating the necessity of clamping and gluing, this technique is well suited for nondestructively monitoring the elasticity and consistency of delicate engineered tissues during culture. Since some biological tissues possess a curvature in their native state, an estimate for this initial curvature can also be obtained from the optimization procedure provided that the response of the beam in more than one orientation is captured. The ease of mounting and unmounting the sample makes this method particularly suitable in evaluating the effect of various testing and environmental factors such as time dependence and water content.

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Figure 1.
Bending of a slender beam in the x - y plane.

Figure 2. Bending of a beam under self-weight; the normalized loading $q_0 L^3 / EI$ is varied over three orders of magnitude.

Figure 3. Variation of the tension (solid lines) and shear force (dashed lines) along the length of the beam for $q_0 L^3 / EI = 10, 100$. Note that the tension goes to zero at the fixed end and the shear force is equal to the weight of the beam ($q_0 L$).

Figure 4. Bending of a beam under self-weight; the normalized loading $q_0 L^3/EI$ is varied over four orders of magnitude; one end is fixed vertically and the other end is left free.

Figure 5.

The initial shape (a) and the deflated shapes (b-d) of a steel beam at three different orientations, $\theta = 30^\circ, 60^\circ, 90^\circ$. In (b)-(d), the blue line scale bar indicates 1 cm. (e) The best estimated deflection shape of the steel beam (with $E=210$ GPa) is shown by the red lines and compared with the measured response at $\theta = 30^\circ, 60^\circ, 90^\circ$ orientation (black symbols). The measured initial curved shape is also shown by the blue dashed line.

Fig 6.

(a) Comparison of measured response (black symbols) and the calculated response (red line) for a polyetherimide beam held with the initial curvature up and then down. The best guesses for the initial shape are also shown in both directions by the blue dashed and dash-dot lines. (b) Comparison of the measured initial shape and the best estimate for the initial shape obtained from the optimization process.

Figure 7.

(a) The continuously changing deflection of a rubber beam at 3, 10, 60 and 1200 minutes after releasing the free end. The initial shape is shown at 90 degree; however, it was captured in an unstressed state. (b) The digitized shapes represented as discrete points vs the calculated shapes with the best estimated average elastic modulus, the initial shape is shown in blue color; (c) the time variation of the optimized average elastic moduli.

Figure 8.

The superimposed image of a pig skin sample at two different orientations while the sample is simply placed on a rod. This technique eliminates necessity of gluing or clamping hence it is especially suitable when the sample is desired to stay unaltered and undamaged for other studies.

Figure 9.

The superimposed images from a pig skin sample at three different times; the weight at each instant was 1.52, 1.49, and 1.47 g. Each image was taken 30 s after releasing the free-end to equalize the viscous effects for all tests. The estimated initial curvature at the different times is shown by the blue dashed lines. The blue scale bar is 10 mm long.

Table 1

The low-strain range elastic modulus of pig skin from beam bending test and uniaxial tension test at room temperature.

Sample number	Thickness <i>mm</i>	<i>E</i> Bending Test <i>kPa</i>	<i>E</i> Uniaxial Tension Test <i>kPa</i>	Loading rate for tension test <i>s</i> ⁻¹
1	1.5	345	890	1%
2	1.625	320	-	-
3	1.375	340	736	0.01%

Table 2

The effect of moisture content on the elastic modulus of pig skin.

Time <i>s</i>	Weight <i>g</i>	E <i>kPa</i>
0	1.52	340
360	1.49	785
720	1.47	1500