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Analysis of multidimensional difference-of-Gaussians filters in terms of directly observable parameters

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Abstract

The difference-of-Gaussians (DOG) filter is a widely used model for the receptive field of neurons in the retina and lateral geniculate nucleus and is a potential model in general for responses modulated by an excitatory center with an inhibitory surrounding region. A DOG filter is defined by three standard parameters: the center and surround sigmas (which define the variance of the radially symmetric Gaussians) and the balance (which defines the linear combination of the two Gaussians). These parameters are not directly observable and are typically determined by nonlinear parameter estimation methods applied to the frequency response function. DOG filters show both low-pass (optimal response at zero frequency) and band-pass (optimal response at a non-zero frequency) behavior. This paper reformulates the DOG filter in terms of a directly observable parameter, the zero-crossing radius, and two new (but not directly observable) parameters. In the new two-dimensional parameter space, the exact region corresponding to band-pass behavior is determined. A detailed description of the frequency response characteristics of the DOG filter is obtained. It is also found that the directly observable optimal frequency and optimal gain (the ratio of the response at optimal frequency to the response at zero frequency) provide an alternate coordinate system for the band-pass region. Altogether, the DOG filter and its three standard implicit parameters can be determined by three directly observable values. The two-dimensional band-pass region is a potential tool for the analysis of populations of DOG filters (for example, populations of neurons in the retina or lateral geniculate nucleus) because the clustering of points in this parameter space may indicate an underlying organizational principle. This paper concentrates on circular Gaussians, but the results generalize to multidimensional radially symmetric Gaussians and are given as an appendix.

1. INTRODUCTION

By a *difference-of-Gaussians (DOG) filter*, we mean the convolution $L[p] := f * p$, where the *field function* f is a difference of two Gaussians and p is an input function. The Gaussians are assumed to be radially symmetric, centered at the origin, and normalized to unit volume. The field function then depends on three parameters: a *center sigma* σ_c and a *surround sigma* σ_s , which set the variance for each Gaussian, and a dimensionless *balance parameter* c_s for their combination. Our discussion is stated for the case of two-dimensional Gaussians because the applications motivating our study are planar (Section 5), but all results generalize to the case of N -dimensional Gaussians and are summarized in Appendix A.

DOG filters provide the standard model for the two-dimensional receptive fields of neurons in the mammalian retina and lateral geniculate nucleus (LGN), the application motivating this study [1, 2, 4, 5]. More generally, DOG filters provide simple empirical models for fields involving an excitatory center with an inhibitory surrounding region.

The DOG filter parameters c , s , and c_s are not directly measurable and are typically approximated by nonlinear parameter estimation methods applied to the measured frequency response function. However, the DOG filter has radial symmetry and a zero-crossing radius r_0 , where there is a change of sign between the center and surround portions of the field. Applications (such as those of Section 5) may allow direct measurement of the zero-crossing radius. This paper examines the use of such a value with regard to parameter estimation and behavior of the filter.

Section 2 presents the standard DOG field function in terms of the parameters c , s , and c_s and commonly used conditions, which imply the existence of the zero-crossing radius r_0 . Introducing this value as a parameter, the field function can then be formulated in terms of two dimensionless parameters, $c = c/r_0$ and $s = s/r_0$. The balance parameter c_s can be explicitly expressed in terms of c , s . The specific range, designated Region AB, of the new parameters in $\alpha_c^2 \alpha_s^2$ -space can be determined exactly.

Section 3 analyzes the frequency response function in terms of the new parameters c , s . DOG filters may or may not show an optimal frequency s_{LIN} . We find that the subregion of Region AB where an optimal frequency exists, designated Region ABC, can be determined exactly, and that the s_{LIN} -contours partition this subregion. When an optimal frequency exists, an optimal gain g_{LIN} , defined as the ratio of the response at the optimal frequency to the response at zero-frequency, exists as well. We find the g_{LIN} -contours not only partition Region ABC but that there is also a one-to-one correspondence between s_{LIN} , g_{LIN} -values and $\alpha_c^2 \alpha_s^2$ over this subregion. That is, the directly measurable values s_{LIN} , g_{LIN} provide a coordinate system for Region ABC in parameter space.

Altogether, over the parameter subregion where an optimal frequency exists, the DOG filter is completely determined by three directly measurable values: the zero-crossing radius r_0 , the optimal frequency s_{LIN} , and the optimal gain g_{LIN} .

Section 4 analyzes the structure of Region ABC in more detail, in particular, the contours for the balance parameter c_s and the behavior of the optimal frequency s_{LIN} and optimal gain g_{LIN} on such contours, that is, how the frequency and gain vary while the balance remains constant.

Section 5 discusses potential applications of this work. For example, we have determined the structure of the parameter space for linear DOG filters, including regions of existence and nonexistence for optimal frequencies and the structure of these regions in terms of contours for optimal frequency and gain as well as for center-surround balance. This information is relevant for the development, implementation, and study of nonlinear models based on linear DOG filters (the development of such models being the particular motivation for this work). Physiological studies of LGN cells often involve parameter measurements on population samples, and the representation of the customary three parameters in a two-dimensional parameter space, obtained in this work, may aid in determining structures within such populations through clusterings of parameter values under the new representation. The customary parameters $\alpha_c^2 \alpha_s^2$ and the new parameters r_0 , s_{LIN} , g_{LIN} are discussed with respect to potential use in laboratory measurements.

2. FORMULATION

This section states the standard three parameter DOG field function in two dimensions. The three standard parameters are not directly measurable. The directly measurable zero-crossing radius is introduced as a parameter, and the field function is then reformulated in terms of two new parameters with two auxiliary parameters introduced to simplify expressions. Although these parameters are not directly measurable, they provide the basis for the analysis of Section 3, which replaces them by two directly measurable parameters.

A. DOG Field Function

The field function is a difference of two circular Gaussians with unit volumes. In terms of the radial coordinate $r^2 = x_1^2 + x_2^2$:

$$f(r) := \frac{1}{2\pi\sigma_c^2} \exp\left(-\frac{r^2}{2\sigma_c^2}\right) - \frac{\beta_{cs}}{2\pi\sigma_s^2} \exp\left(-\frac{r^2}{2\sigma_s^2}\right) \quad (1)$$

The three parameters defining the filter are the *center sigma* σ_c , the *surround sigma* σ_s , and the dimensionless *balance parameter* β_{cs} . The total volume for f is $1 - \beta_{cs}$, and f is said to be *balanced* for $\beta_{cs} = 1$ (that is, has zero total volume). If the Fourier transform is defined by

$$F(s_1, s_2) := \int_{\mathbb{R} \times \mathbb{R}} f(x_1, x_2) \exp(-2\pi i(s_1 x_1 + s_2 x_2)) dx_1 dx_2 \quad (2)$$

then the Fourier transform, that is, the *frequency response function*, is a function of the radial frequency $s^2 = s_1^2 + s_2^2$:

$$F(s) = \exp(-2\pi^2 \sigma_c^2 s^2) - \beta_{cs} \exp(-2\pi^2 \sigma_s^2 s^2) \quad (3)$$

The parameters σ_c , σ_s , β_{cs} are not directly measurable and are typically obtained by nonlinear parameter estimation methods applied to the measured frequency response function.

B. Conditions

Conditions on the field function typically include:

- (a) the field function f is positive at the origin (holds iff $\beta_{cs} \sigma_c^2 / \sigma_s^2 < 1$);
- (b) a *zero-crossing radius* $r_0 > 0$ exists (holds iff $\exp(\rho_0^2 (\sigma_c^2 - \sigma_s^2) / (2\sigma_c^2 \sigma_s^2)) = \beta_{cs} \sigma_c^2 / \sigma_s^2$);
- (c) the total volume associated with the field function f is nonnegative (holds iff $\beta_{cs} \leq 1$).

For example, these conditions occur in modeling the receptive fields of ON-center LGN cells (Section 4). Fig. 1 shows a cross-section of a field function illustrating the conditions. The zero-crossing radius r_0 is unique. We assume these conditions and note that they are equivalent to the following relations, which describe the exact extent of σ_c , σ_s , β_{cs} -space in which the parameters can lie:

$$0 < \sigma_c < \sigma_s \text{ and } 0 < \beta_{cs} \leq 1 \quad (4)$$

C. New Parameters

This paper is based on the observation that the zero-crossing value ρ_0 can typically be determined by direct measurement. We introduce new parameters α_C , α_S defined by:

$$\alpha_C := \sigma_C / \rho_0 \text{ and } \alpha_S := \alpha_S / \rho_0 \quad (5)$$

By condition (b) on the field function, the balance β_{CS} can be written explicitly in terms of the new parameters as

$$\beta_{CS} = \frac{\alpha_S^2}{\alpha_C^2} \exp\left(\frac{1}{2\alpha_S^2} - \frac{1}{2\alpha_C^2}\right) \quad (6)$$

Since the function $x \exp(1/2x)$ is initially decreasing, then increasing, with an absolute minimum at $x = 1/2$, the condition $\alpha_S^2 > \alpha_C^2$ (from Eq. (4)) combined with a value $\alpha_C^2 > 1/2$ would force the ratio in Eq. (6) to be greater than one, contradicting the condition on β_{CS} . Such considerations give the following properties for the new parameters α_C^2, α_S^2 .

- (a) $0 < \alpha_C^2 \leq \alpha_S^2$;
- (b) $0 < \alpha_C^2 \leq 1/2$;
- (c) for fixed $0 < \alpha_C^2 < 1/2$, the minimum value for β_{CS} occurs at $\alpha_S^2 = 1/2$;
- (d) for fixed $0 < \alpha_C^2 < 1/2$, the boundary value $\beta_{CS} = 1$ occurs twice (at $\alpha_S^2 = \alpha_C^2 < 1/2$ and at some $\alpha_S^2 > 1/2$);
- (e) each β_{CS} -contour has a vertical tangent at $\alpha_S^2 = 1/2$ (the point of maximum value for α_C^2 on the contour).

Fig. 2 shows this parameter region in α_C^2, α_S^2 -space with its corner point at $(1/2, 1/2)$, sample β_{CS} -contours, and lower and upper bounds designated respectively by Curve A and Curve B. The three-dimensional parameter region Eq. (4) has now been reformulated as this two-dimensional region, designated Region AB. Notice Region AB is partitioned by the β_{CS} -contours, that is, each point of the region lies on one and only one contour for $0 < \beta_{CS} < 1$. The boundary curves A and B both correspond to $\beta_{CS} = 1$ with Curve A corresponding to $\alpha_C^2 = \alpha_S^2$. The region and its boundary are defined by the totality of contours, $0 < \beta_{CS} < 1$.

D. Auxiliary Parameters

It is possible to introduce auxiliary parameters μ, ν which provide a coordinate system for the parameter region of Fig. 2, simplify expressions, and prove especially useful in the following section, where the connection with directly measurable parameters is made. The auxiliary parameters are:

$$\begin{aligned} \mu &= \frac{\alpha_S^2}{\alpha_C^2}, \nu = \frac{\alpha_S^2 - \alpha_C^2}{2\alpha_C^2 \alpha_S^2 \ln(\alpha_S^2 / \alpha_C^2)} \text{ or} \\ \alpha_C^2 &= \frac{\mu - 1}{2 \ln(\mu)} \frac{1}{\mu \nu}, \alpha_S^2 = \frac{\mu - 1}{2 \ln(\mu)} \frac{1}{\nu} \text{ with } 1 < \mu < \infty, 1 < \nu < \infty \end{aligned} \quad (7)$$

In terms of these auxiliary parameters:

$$\beta_{CS} = \mu^{1-\nu} \quad (8)$$

The correspondence between the points of Region AB, defined as the totality of α_{CS} -contours with $0 < \alpha_{CS} < 1$, and ordered pairs (μ, ν) can be described as follows: The boundary curves A and B correspond to the bounds $\mu = 1$ and $\nu = 1$, respectively. Fig. 2 shows gridlines corresponding to fixed μ and fixed ν for these auxiliary parameters. The gridlines for fixed μ are rays with slope μ through the origin since $\alpha_s^2 = \mu \alpha_c^2$. On each gridline with fixed ν , α_{CS} decreases monotonically from 1 to 0 as μ increases. This behavior can be seen in Fig. 2 by examining a fixed- ν gridline crossing the α_{CS} -contours. These properties of the gridlines imply a one-to-one correspondence between the points of Region AB and pairs (μ, ν) , that is, the auxiliary parameters μ, ν provide a coordinate system for Region AB. The boundary curves A and B at the limits $\mu = 1$ and $\nu = 1$ have the representations:

$$\text{Curve A: } \alpha_s^2 = \alpha_c^2 \text{ for } \mu = 1 \quad (9)$$

$$\text{Curve B: } \alpha_c^2 = \frac{\mu - 1}{2 \ln(\mu)} \frac{1}{\mu}, \alpha_s^2 = \mu \alpha_c^2 \text{ for } 1 < \mu < \infty, \nu = 1 \quad (10)$$

3. OPTIMAL FREQUENCY AND OPTIMAL GAIN

This section determines the subregion, designated Region ABC, of Region AB where an optimal frequency s_{LIN} exists and thus the subregion where an optimal gain g_{LIN} , which is defined in terms of the optimal frequency, exists. It is observed that, in Region ABC, there is a one-to-one correspondence between the directly measurable values s_{LIN} , g_{LIN} and the parameters α_c^2, α_s^2 which in turn give the balance α_{CS} and, by means of the directly measurable zero-crossing ρ_0 , the center and surround parameters σ_c^2, σ_s^2 . This observation is one of the main results of this paper, and a rigorous derivation for the full N -dimensional case is given in Appendix A. Other relations occur as well, for example, that optimal wavelengths λ_{LIN} must be related to the zero-crossing radius ρ_0 by $\lambda_{LIN} = \rho_0$.

In terms of the parameters of Eq. (5), the frequency response function Eq. (3) becomes

$$F(s) = \exp\left(-2\alpha_c^2 \pi^2 \rho_0^2 s^2\right) - \beta_{CS} \exp\left(-2\alpha_s^2 \pi^2 \rho_0^2 s^2\right) \quad (11)$$

Examination of the derivative gives two cases (examples of each are shown in Fig. 3):

- (a) $F(s) < 0$ for $s > 0$ (holds iff $\alpha_s^2 \beta_{CS} < \alpha_c^2$);
- (b) $F(s)$ has a unique root on $s > 0$ with a maximum for F (holds iff $\alpha_s^2 \beta_{CS} \geq \alpha_c^2$).

In both cases, we have $F(0) = 1 - \beta_{CS} > 0$ and $F(\infty) = 0$, that is, a DOG filter can be either low-pass (case (a)) or band-pass (case (b)). In the second case, the maximum defines the *optimal frequency* s_{LIN} given by

$$(\pi \rho_0 s_{LIN})^2 = \frac{\ln(\alpha_s^2 \beta_{CS} / \alpha_c^2)}{2(\alpha_s^2 - \alpha_c^2)} = \nu(2 - \nu) \mu \left(\frac{\ln(\mu)}{\mu - 1} \right)^2 \quad (12)$$

The representation in auxiliary parameters shows that optimal frequencies exist over the space defined by $1 < \mu < \infty$, that is, over the right half of Region AB. The boundary curve at $\nu = 2$ corresponds to $s_{LIN} = 0$ and is designated Curve C. Its parametric representation is:

$$\text{Curve C: } \alpha_C^2 = \frac{\mu-1}{4 \ln(\mu)} \frac{1}{\mu}, \alpha_S^2 = \mu \alpha_C^2 \text{ for } 1 < \mu < \infty, \nu=2 \quad (13)$$

Fig. 4 shows this region, designated Region ABC, where optimal frequencies s_{LIN} exist. Examination of Eq. (12) shows that, for fixed μ gridlines, s_{LIN} strictly decreases as α_C^2 increases (that is, moving from Curve B to Curve C) and, for fixed α_C^2 gridlines, s_{LIN} strictly decreases as μ increases. The maximum for s_{LIN} thus occurs at the right corner of Region ABC, where the limiting value is one. Consequently, optimal frequencies s_{LIN} satisfy the bounds:

$$0 \leq \pi \rho_0 s_{\text{LIN}} \leq 1 \quad (14)$$

That is, optimal wavelengths s_{LIN} for a DOG filter must be related to the zero-crossing radius ρ_0 by $s_{\text{LIN}} = \rho_0$. A parametric representation for s_{LIN} -contours can be obtained by solving Eq. (12) for $\alpha_C^2 = (\mu)$ with $1 < \mu < \mu_0$, where μ_0 is the boundary value at $s_{\text{LIN}} = 1$:

$$\begin{aligned} \nu &= 1 + \left(1 - \frac{1}{\mu} \left(\frac{\mu-1}{\ln(\mu)} \right)^2 (\pi \rho_0 s_{\text{LIN}})^2 \right)^{1/2} \\ (\pi \rho_0 s_{\text{LIN}})^2 &= \mu_0 \left(\frac{\ln(\mu_0)}{\mu_0-1} \right)^2 \end{aligned} \quad (15)$$

and combining this result with Eq. (7). Fig. 4 shows Region ABC with sample s_{LIN} -contours. Notice the s_{LIN} -contours, $0 < s_{\text{LIN}} < 1$, partition Region ABC.

The *optimal gain* g_{LIN} is defined as the ratio of the maximum response to the zero-frequency response:

$$g_{\text{LIN}} := F(s_{\text{LIN}}) / F(0) \quad (16)$$

The optimal gain is thus defined on the same Region ABC, defined by $1 < \mu < 2$, as s_{LIN} . The results above for s_{LIN} provide an exact result in terms of the auxiliary parameters:

$$g_{\text{LIN}} = \exp \left(- \frac{\ln(\alpha_S^2 / \alpha_C^2) - \ln(\beta_{\text{CS}})}{\alpha_S^2 / \alpha_C^2 - 1} \right) \frac{1 - \alpha_C^2 / \alpha_S^2}{1 - \beta_{\text{CS}}} = \exp \left((\nu - 2) \frac{\ln(\mu)}{\mu - 1} \right) \frac{\mu - 1}{\mu - \mu^{2-\nu}} \quad (17)$$

Fig. 5 shows Region ABC with sample g_{LIN} -contours. Examination of Eq. (12) shows that, for fixed μ gridlines, $g_{\text{LIN}} = +\infty$ on the boundary $\alpha_C^2 = 1$ (Curve B) and strictly decreases to $g_{\text{LIN}} = 1$ at $\alpha_C^2 = 2$ (Curve C). Along fixed α_C^2 gridlines, g_{LIN} strictly decreases from the value $\exp(-2)/(-1) > 1$ at $\mu = 1$ (Curve A) to one as μ increases. This behavior can be seen in Fig. 5, where g_{LIN} -contours are shown together with gridlines for fixed μ and α_C^2 values. Notice the g_{LIN} -contours, $0 < g_{\text{LIN}} < +\infty$, partition Region ABC.

Fig. 6 shows the result when the s_{LIN} - and g_{LIN} -contours from Figs. 4 and 5 are combined. The contours appear to form a coordinate grid for Region ABC, that is, there is a one-to-one correspondence between parameter pairs (α_C^2, α_S^2) and the pairs $(s_{\text{LIN}}, g_{\text{LIN}})$. Appendix A proves that the optimal frequency and gain do indeed provide such a coordinate system. Since both s_{LIN} and g_{LIN} are directly observable, we have a representation for the parameter space of the model in terms of directly observable parameters.

Fig. 7 is the converse to Fig. 6, showing α_C^2, α_S^2 -contours in terms of coordinates $\log_{10}(s_{\text{LIN}}), (\log_{10}(g_{\text{LIN}}))^2$. The thick solid line is a boundary curve corresponding to Curve A in Fig.

6. The dashed lines are contours for $\alpha_c^2 = 0.15, 0.20, 0.25, 0.30, 0.35$. For $0 < \alpha_c^2 \leq 0.25$, the contours extend to the origin ($\log_{10}(g_{LIN}) = 0$), but for $0.25 < \alpha_c^2 < 0.5$, the contours meet the boundary curve at nonzero values of $\log_{10}(g_{LIN})$, behavior which is just visible on the plot. The solid lines are contours for $\alpha_s^2 = 0.45$ (unmarked and just visible) and for $\alpha_s^2 = 0.5, 1.0, 2.0$. For $\alpha_s^2 \geq 0.5$, the contours extend to infinity ($\log_{10}(g_{LIN}) = \infty$), but for $0 < \alpha_s^2 < 0.5$, the contours meet the boundary curve at finite values of $\log_{10}(g_{LIN})$, behavior which is just visible in the plot.

4. OPTIMAL PARAMETERS ALONG BALANCE CONTOURS

This section gives further information about the structure of Region ABC in parameter space by describing the behavior of optimal gain and optimal frequency along balance contours. The optimal gain is found to vary monotonically, while the optimal frequency attains a maximum value on the contour.

A. Optimal Gain on a Balance Contour

By use of Eq. (8) and Eq. (17), the optimal gain can be written as

$$g_{LIN} = \exp\left(-\frac{\ln(\beta_{CS} \mu)}{\mu - 1}\right) \frac{1 - 1/\mu}{1 - \beta_{CS}} \quad (18)$$

Each β_{CS} -contour enters Region ABC at the point on Curve C where $\mu = 1/\beta_{CS}$. As μ increases, consideration of the logarithmic derivative of Eq. (18) shows that g_{LIN} is strictly increasing with μ along the contour and has the end point values:

$$g_{LIN} = 1 \text{ at } \mu = 1/\beta_{CS} \text{ and } g_{LIN} \rightarrow \frac{1}{1 - \beta_{CS}} \text{ as } \mu \rightarrow \infty \quad (19)$$

B. Optimal Frequency on a Balance Contour

By use of Eq. (8) and (12), the optimal frequency can be written as

$$(\pi \rho_0 s_{LIN})^2 = (\ln(\mu))^2 - (\ln(\beta_{CS}))^2 \frac{\mu}{(\mu - 1)^2} \quad (20)$$

Again, each β_{CS} -contour enters Region ABC at the point on Curve C where $\mu = 1/\beta_{CS}$ (and $s_{LIN} = 0$), but consideration of the derivative of Eq. (20) shows that s_{LIN} is strictly increasing with μ along the contour to a maximum, then strictly decreases. The endpoint values are

$$\pi \rho_0 s_{LIN} = 0 \text{ at } \mu = 1/\beta_{CS} \text{ and } \pi \rho_0 s_{LIN} \rightarrow 0 \text{ as } \mu \rightarrow \infty \quad (21)$$

The intermediate point $\mu = \mu_{CS}$ gives the maximum value s_{CS} of s_{LIN} on the β_{CS} -contour (with corresponding gain g_{CS} given by Eq. (18) as the corresponding value of g_{LIN} at that point). These values are given by:

$$(\ln(\mu_{CS}))^2 - (\ln(\beta_{CS}))^2 = 2 \frac{\mu_{CS} - 1}{\mu_{CS} + 1} \ln(\mu_{CS}) \quad (22)$$

$$(\pi \rho_0 s_{CS})^2 = 2 \frac{\mu_{CS}}{\mu_{CS}^2 - 1} \ln(\mu_{CS}) \quad (23)$$

As s_{CS} varies, it defines corresponding values $\mu = \mu_{CS}$, which in turn define a curve of points where the maximum optimal frequency occurs on a s_{CS} -contour, designated Curve D. Fig. 8 shows Curve D together with s_{CS} -contours and $\rho_0 s_{LIN}$ -contours. The maximum optimal frequency on a balance contour occurs at the point on the contour that is tangent to an optimal frequency contour, and these points of tangency form an alternate description of Curve D.

5. POTENTIAL APPLICATIONS

This section describes potential applications of this study to various instances where linear DOG filters occur.

A. Potential Applications for Nonlinear Models

A two-dimensional linear DOG filter with its three parameters C , s , s_{CS} is the standard descriptive model for receptive fields of ON-center retinal ganglion cells, as well as LGN cells which provide input to the visual cortex [1, 2, 4, 5]. The linear DOG filter is consequently the natural starting point for nonlinear models which describe nonlinear aspects of cell response such as gain control and saturation. One approach to understanding the behavior of a nonlinear model is to describe its frequency response characteristics, that is, its response to sinusoidal stimuli. The nonlinear frequency response will differ in various ways from the linear frequency response, and a thorough description of the linear frequency response provides a useful baseline for comparison with the nonlinear response.

This paper provides such a baseline study of the frequency response for linear DOG filters and thus has potential interest for nonlinear modeling efforts. We have provided changes of parameters which reduce the three-dimensional parameter space of C , s , s_{CS} to the two-dimensional parameter space of Fig. 2 and have provided the detailed structure of that reduced parameter space with regard to contours for the balance parameter s_{CS} , the determination of subregions where an optimal frequency response does and does not exist, contours for such characteristics of the frequency response as optimal frequency and the corresponding optimal gain, and a discussion of the behavior of the optimal frequency and gain along balance contours (Section 4).

Such knowledge of the parameter space also provides potential assistance in selecting numerical values for parameters in simulations. For example, if the balance s_{CS} has been fixed for study purposes, then Section 4 shows that the optimal gain g_{LIN} cannot be arbitrarily specified, but must lie in the range $1 < g_{LIN} < 1/(1 - s_{CS})$. Furthermore, for a specified g_{LIN} -value in this range, there will be a unique point on the s_{CS} -contour, that is, a unique value for the ratio $s/C = s_{LIN}/C$, where that gain occurs.

B. Potential Applications for the Analysis of Populations of LGN Cells

Experimental studies of ON-center LGN or retinal ganglion cells typically involve measuring frequency response functions for large numbers of such cells with corresponding estimates of C , s , s_{CS} . These values can be converted directly to points in the two-dimensional parameter space of Fig. 2 by the intersection of the contour for s_{CS} with the ray with slope $\mu = (s/C)^2 = (s_{LIN}/C)^2$. Clusterings of points based on parameter values from a population of cells would then suggest some structure or organizing principle for the population. Examples of population studies from which such receptive field parameters can be inferred are [7–10, 13, 16, 18, 20] for μ and [3, 6, 14, 16, 19, 21] for g_{LIN} . Conversely,

this analysis also shows that some combinations of experimentally measured parameters will not be observed, for example, points outside Region AB will not be observed at all, while measurements involving a frequency response curve with an optimal frequency will produce points only in Region ABC, not in the region bounded by curves A and C (Fig. 4).

As an example of a hypothesis that might be tested in this way, notice that for a two-dimensional DOG filter, the simplest stimulus is a homogeneous pattern of amplitude A , and the response pattern is a homogeneous field of amplitude $(1 - c_s) A$. Typically ON-center LGN cells have a nonzero response to homogeneous patterns, implying that $0 < c_s < 1$ [3,8,9,12,13,15–18]. The balance parameter thus controls the simplest response of the cell. One might then hypothesize that the balance for an LGN cell is set early as the visual cortex develops and remains fixed as synchronization of a group of cells continues during development. This would imply that the point α_c^2, α_s^2 describing the parameters of the cell remains on a particular c_s -contour as the cell develops, eventually attaining values suitable for providing input to the mature cortex.

If the balance is fixed, any further change in the cell can only involve adjustments of the optimal frequency s_{LIN} and optimal gain g_{LIN} . Since g_{LIN} varies monotonically along the balance contour, an optimizing principle would be more likely to involve the optimal frequency. If, for example, the optimal frequency were maximized, there would be a clustering of observed points along Curve D in Region ABC (Fig. 8). One could then test this hypothesis by looking for a clustering of observed parameter points in the neighborhood of this curve.

C. Considerations with Respect to Experimental Procedure

It has been noted that the parameters $\theta_0, s_{LIN}, g_{LIN}$ are directly observable, while the parameters c_c, s, c_s are obtained indirectly, which suggests that the first set may be preferable as a goal for experimental measurement. However, laboratory procedure involves a number of considerations. Rather than suggest that directly observable parameters are automatically preferable, we provide our estimate of the experimental considerations on each side.

Experimenters 1 and 2 are both taking measurements on LGN cells, taking care to work in the linear response regime where the linear DOG filter applies.

Experimenter 1 will measure the customary parameters c_c, s, c_s . This involves measuring the responses to drifting sinusoidal gratings at, say, a dozen spatial frequencies, taking care to bracket the optimal frequency. Notice that no localization of the receptive field is required. A nonlinear parameter estimation technique is applied, which produces an estimate of c_c, s, c_s .

Experimenter 2 will measure the proposed parameters $\theta_0, s_{LIN}, g_{LIN}$. Experimental techniques, such as reverse correlation [11], can be used to determine the size of the cell's excitatory center and the corresponding parameter θ_0 . Then responses to sinusoidal gratings sufficient to determine the optimal frequency s_{LIN} must be measured. The response to a homogeneous stimulus must be measured and combined with the optimal frequency response to determine g_{LIN} .

We suggest that the approach experimenters choose should depend on which set of parameter values – frequency- or space-domain – they are interested in characterizing.

6. SUMMARY

The standard parameters defining a DOG filter are the center sigma σ_C , the surround sigma σ_S , and the dimensionless balance parameter σ_{CS} . The parameters σ_C , σ_S , σ_{CS} are not directly measurable and are typically obtained by nonlinear parameter estimation methods applied to the measured frequency response function. We have shown that the DOG filter can be reformulated in terms of the zero-crossing radius ρ_0 , the optimal frequency s_{LIN} , and the optimal gain g_{LIN} , which are directly measurable in some applications where the filter is used as a model. Further results of the analysis include the identification of the exact region of α_C^2, α_S^2 -space ($\sigma_C = \rho_0 / \alpha_C$, $\sigma_S = \rho_0 / \alpha_S$) where an optimal frequency s_{LIN} can exist, designated Region ABC, and an exact bound for these optimal frequencies, namely, $0 \leq \pi \rho_0 s_{LIN} \leq 1$. In particular, there is a one-to-one correspondence between the ordered pairs (α_C^2, α_S^2) and the values (s_{LIN}, g_{LIN}) , that is, the contour lines for the optimal frequency and gain form a coordinate system for Region ABC. Potential applications of these results include:

- (1) A reduction of the three-parameter linear DOG filter to an equivalent two-parameter representation and a detailed description of its frequency response characteristics in that reduced parameter space. This description provides a baseline for comparison of the linear filter with nonlinear models extending the linear filter.
- (2) Assistance in selecting parameter values for numerical models involving DOG filters. For example, if the balance σ_{CS} has been fixed for study purposes, then Section 4 shows that the optimal gain g_{LIN} has a corresponding specific range and cannot be arbitrarily specified. Furthermore, for a specified g_{LIN} -value in this range, there will be a unique point on the σ_{CS} -contour where that gain occurs.
- (3) Use of Region ABC and its structure (e.g. σ_{CS} -contours) for the examination of populations of structures modeled by DOG filters, which may identify an organizational principle of the population through clusterings of points.

The motivating application for this study involved 2-dimensional filters, but all results generalize to N -dimensional filters and are stated in Appendix A.

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APPENDIX A. THE N-DIMENSIONAL CASE

This appendix generalizes the discussion of DOG filters to the N -dimensional case. In particular, it characterizes the parameter subregion in α_C^2, α_S^2 -space where an optimal frequency exists and proves that the optimal frequency s_{LIN} and optimal gain g_{LIN} provide a coordinate system for this subregion. The N -dimensional DOG filter is thus completely described by the zero-crossing radius ρ_0 and optimal frequency and gain, s_{LIN} and g_{LIN} , when these quantities exist.

A. DOG Field Function

The field function is a difference of two Gaussians with unit volumes. In terms of the radial coordinate $r^2 = x_1^2 + x_2^2 + \dots + x_N^2$:

$$f(r) := \frac{1}{(2\pi)^{N/2} \sigma_C^N} \exp\left(-\frac{r^2}{2\sigma_C^2}\right) - \frac{\beta_{CS}}{(2\pi)^{N/2} \sigma_S^N} \exp\left(-\frac{r^2}{2\sigma_S^2}\right) \quad (24)$$

The three parameters defining the filter are the *center sigma* σ_C , the *surround sigma* σ_S , and the dimensionless *balance parameter* β_{CS} . The total volume for f is $1 - \beta_{CS}$, and f is said to be *balanced* for $\beta_{CS} = 1$. If the Fourier transform is defined by

$$F(s_1, \dots, s_N) := \int_{\mathbb{R}^N \times \mathbb{R}^N} f(x_1, \dots, x_N) \exp(-2\pi i(s_1 x_1 + \dots + s_N x_N)) dx_1 \dots dx_N \quad (25)$$

then the Fourier transform, that is, the *frequency response function*, is a function of the radial frequency $s^2 = s_1^2 + s_2^2 + \dots + s_N^2$:

$$F(s) = \exp(-2\pi^2 \sigma_C^2 s^2) - \beta_{CS} \exp(-2\pi^2 \sigma_S^2 s^2) \quad (26)$$

B. Conditions

We assume the following conditions:

- (a) the field function f is positive at the origin (holds iff $\beta_{CS} \sigma_C^N / \sigma_S^N < 1$);
- (b) a *zero-crossing radius* $r = r_0 > 0$ exists (holds iff $\exp(\rho_0^2 (\sigma_C^2 - \sigma_S^2) / (2\sigma_C^2 \sigma_S^2)) = \beta_{CS} \sigma_C^N / \sigma_S^N$);
- (c) the total volume associated with the field function f is nonnegative (holds iff $\beta_{CS} \leq 1$).

In condition (b), the root $r = r_0$ is unique. These conditions are equivalent to the following conditions on parameters σ_C , σ_S , β_{CS} :

- (a) $0 < \sigma_C < \sigma_S$;
- (b) $0 < \beta_{CS} \leq 1$.

C. New Parameters

Define three new parameters: the zero-crossing radius r_0 and σ_C , σ_S given by

$$\alpha_C := \sigma_C / \rho_0 \text{ and } \alpha_S := \sigma_S / \rho_0 \quad (27)$$

An immediate consequence is an explicit representation for the balance β_{CS} :

$$\beta_{CS} = \frac{\alpha_S^N}{\alpha_C^N} \exp\left(\frac{1}{2\alpha_C^2} - \frac{1}{2\alpha_S^2}\right) \quad (28)$$

Since the function $x^{N/2} \exp(1/2x)$ is initially decreasing, then increasing, with an absolute minimum at $x = 1/N$, the condition $\alpha_S^2 > \alpha_C^2$ combined with a value $\alpha_C^2 > 1/N$ would force the

ratio in Eq. (28) to be greater than one. Such considerations give the following conditions on the parameters:

- (a) $0 < \alpha_C^2 \leq \alpha_S^2$;
- (b) $0 < \alpha_C^2 \leq 1/N$;
- (c) for fixed $0 < \alpha_C^2 < 1/N$, the minimum value for β_{CS} occurs at $\alpha_S^2 = 1/N$;
- (d) for fixed $0 < \alpha_C^2 < 1/N$, the boundary value $\beta_{CS} = 1$ occurs twice (at $\alpha_S^2 = \alpha_C^2 < 1/N$ and at some $\alpha_S^2 > 1/N$);
- (e) each β_{CS} -contour has a vertical tangent at $\alpha_S^2 = 1/N$ (the point of maximum value for α_C^2 on the contour).

This parameter region in α_C^2, α_S^2 -space is similar to that shown in Fig. 2 except the corner point is now at $(\alpha_C^2, \alpha_S^2) = (1/N, 1/N)$. The lower and upper bounds are designated respectively by Curve A_N and Curve B_N , and the enclosed open and connected set is designated Region $A_N B_N$. Notice Region $A_N B_N$ is partitioned by the β_{CS} -contours, that is, each point of the region lies on one and only one contour for $0 < \beta_{CS} < 1$, and the region is thus defined by the totality of contours, $0 < \beta_{CS} < 1$. The boundary curves A_N and B_N both correspond to $\beta_{CS} = 1$.

D. Alternate Coordinate Systems for Region $A_N B_N$

For (α_C^2, α_S^2) in Region $A_N B_N$, define auxiliary parameters μ, ν by

$$\mu := \alpha_S^2 / \alpha_C^2 \text{ and } \nu := \frac{\alpha_S^2 - \alpha_C^2}{N \alpha_C^2 \alpha_S^2 \ln(\alpha_S^2 / \alpha_C^2)} \quad (29)$$

The range $1 < \mu < \infty$ is immediate from the extent of the region. In terms of these auxiliary parameters,

$$\beta_{CS} = \mu^{(1-\nu)N/2} \quad (30)$$

and since Region $A_N B_N$ is defined as the totality of points with $0 < \beta_{CS} < 1$, the range of values $1 < \mu < \infty$ directly follows. The map Eq. (29) is one-to-one and onto with the explicit inverse for (μ, ν) in $1 < \mu < \infty, 1 < \nu < \infty$:

$$\alpha_C^2 = \frac{\mu - 1}{N \ln(\mu)} \frac{1}{\mu \nu} \text{ and } \alpha_S^2 = \frac{\mu - 1}{N \ln(\mu)} \frac{1}{\nu} \quad (31)$$

The μ - and ν -contours provide an alternate coordinate system for Region $A_N B_N$, and the boundary curves A_N and B_N correspond to the bounds $\mu = 1$ and $\nu = 1$, respectively. The gridlines for fixed μ are rays with slope μ through the origin. On each gridline with fixed μ ,

β_{CS} decreases monotonically from 1 to 0 as μ increases. Since $\nu = 1 - \frac{2}{N} \ln(\beta_{CS}) / \ln(\mu)$, it is now evident that μ and β_{CS} can be taken as coordinates for Region $A_N B_N$. These coordinates will be useful in deriving the Theorem below. For (μ, β_{CS}) in $1 < \mu < \infty, 0 < \beta_{CS} < 1$, the representation is:

$$\alpha_C^2 = \frac{\mu - 1}{N\mu} \frac{1}{\ln(\mu) - 2\ln(\beta_{CS})/N} \text{ and } \alpha_S^2 = \frac{\mu - 1}{N} \frac{1}{\ln(\mu) - 2\ln(\beta_{CS})/N} \quad (32)$$

E. Optimal Frequency and Optimal Gain

The Fourier transform $F(s)$ in Eq. (26) is either strictly decreasing or has a unique maximum at a nonzero *optimal frequency* s_{LIN} . The *optimal gain* g_{LIN} is defined as the ratio of the maximum response to the zero-frequency response, $g_{LIN} := F(s_{LIN})/F(0)$. Both quantities can be found explicitly, and it is sufficient to work in terms of the auxiliary parameters:

$$\begin{aligned} (\pi\rho_0 s_{LIN})^2 &= \left(\frac{N}{2}\right)^2 \nu \left(\frac{2}{N} + 1 - \nu\right) \mu \left(\frac{\ln(\mu)}{\mu - 1}\right)^2 \\ \ln(g_{LIN}) &= \frac{N}{2} \left(\nu - 1 - \frac{2}{N}\right) \frac{\ln(\mu)}{\mu - 1} + \ln\left(\frac{1 - 1/\mu}{1 - \mu^{(1-\nu)N/2}}\right) \end{aligned} \quad (33)$$

From the μ, ν -representation, it is immediate that the subregion of existence for s_{LIN} , and

thus g_{LIN} , is bounded by $\mu = 1$ (Curve A_N), $\nu = 1$ (Curve B_N), and $\nu = 1 + \frac{2}{N}$, defining Curve C_N . Notice Curve C_N satisfies $\beta_{CS} = \mu^{-1}$. This subregion will be designated Region $A_N B_N C_N$. Equivalent descriptions in terms of μ, β_{CS} -coordinates or in terms of μ, β_{CS} -coordinates for Region $A_N B_N C_N$ are

$$\left\{(\mu, \nu) : 1 < \mu < \infty, 1 < \nu < 1 + \frac{2}{N}\right\} \text{ and } \left\{(\mu, \beta_{CS}) : 1 < \mu < \infty, \mu^{-1} < \beta_{CS} < 1\right\} \quad (34)$$

The following lemma gives information about the optimal frequency and gain on their region of existence.

Lemma. Let $\mu, 1 < \mu < \infty$, be fixed and consider the corresponding ray R passing through Region $A_N B_N C_N$. Then:

- (1a) If $N = 1$, then $\max_R (\pi\rho_0 s_{LIN})^2 = \frac{9\mu}{16} \left(\frac{\ln(\mu)}{\mu - 1}\right)^2$, occurring at $\nu = 3/2$. In this case, $\sup_{\text{Region } A_N B_N C_N} (\pi\rho_0 s_{LIN})^2 = 9/16$, occurring at $(\alpha_C^2, \alpha_S^2) = (2/3, 2/3)$.
- (1b) If $N > 1$, then $\sup_R (\pi\rho_0 s_{LIN})^2 = \frac{N\mu}{2} \left(\frac{\ln(\mu)}{\mu - 1}\right)^2$, occurring at $\nu = 1$. In this case, $\sup_{\text{Region } A_N B_N C_N} (\pi\rho_0 s_{LIN})^2 = N/2$, occurring at $(\alpha_C^2, \alpha_S^2) = (1/N, 1/N)$.
- (2) For $N \geq 1$, $\ln(g_{LIN})$ is strictly increasing from 0 to $\ln(g_{LIN})$ on the ray R as β_{CS} decreases from $1 + 2/N$ to 1 (and as μ increases from μ^{-1} to 1).

Proof. For result (1), the value of $(\pi\rho_0 s_{LIN})^2$ when μ is fixed depends on the quadratic $\nu(2/N + 1 - \nu)$. When $N = 1$ and the region is described by $1 < \nu < 3$, the quadratic has a maximum at $\nu = 3/2$. When $N > 1$ and the region is described by $1 < \nu < 1 + 2/N$, the quadratic increases as ν decreases. These observations give the maximum value on the ray, and the supremum occurs as $\mu \rightarrow 1$. For result (2), $\mu^{(1-\nu)N/2}$ increases as ν decreases to 1, and it follows from Eq. (33) that $\ln(g_{LIN})$ is strictly increasing, that is, that g_{LIN} increases

from $g_{LIN} = 1$ at $\nu = 1 + \frac{2}{N}$ to $g_{LIN} = \frac{1}{N}$ at $\nu = 1$.

The bound on the optimal frequency in Part (1) of the Lemma can be restated as a bound on the optimal wavelength λ_{LIN} in N dimensions, namely, $\lambda_{\text{LIN}} \leq 3\lambda_0/4$ for $N = 1$ and $\lambda_{\text{LIN}} \leq (N/2)^{1/2}\lambda_0$ for $N \geq 2$. The $s_{\text{LIN}}, g_{\text{LIN}}$ -contours for Region $A_N B_N C_N$ are similar to Fig. 6 for $N > 1$ but are somewhat more involved for $N = 1$. This case is illustrated in Fig. 9, which shows Region $A_1 B_1 C_1$ as a subregion of Region $A_1 B_1$ with contours for $\rho_0 s_{\text{LIN}} = 0.60, 0.65, 0.70, 0.725$ and $g_{\text{LIN}} = 1.25, 2.0, 4.0, 8.0$. The dashed line is $\mu = 3/2$, where $\rho_0 s_{\text{LIN}}$ is a maximum for fixed μ and where the $\rho_0 s_{\text{LIN}}$ -contour is tangent to the corresponding ray through the origin, as is evident in the plot.

We now state our main result, namely, that different pairs $(s_{\text{LIN}}, g_{\text{LIN}})$ correspond to different points in Region $A_N B_N C_N$, that is, that the optimal frequency s_{LIN} and optimal gain g_{LIN} provide a coordinate system for Region $A_N B_N C_N$.

Theorem. The map given by Eq. (33) is one-to-one on Region $A_N B_N C_N$.

Proof. Assume Eq. (33) maps two distinct points $(\mu_1, \beta_1), (\mu_2, \beta_2)$ in the region described by Eq. (34) to a point $(s_{\text{LIN}}, g_{\text{LIN}})$. If $\mu_1 = \mu_2$, that is, the points are on a ray intersecting the g_{LIN} -contour, then result (2) of the Lemma forces $\beta_1 = \beta_2$. Consequently, the two points lie on two rays intersecting the g_{LIN} -contour. The following Claim then implies that the corresponding s_{LIN} -values cannot be equal, and the Theorem follows.

Claim. $\frac{d}{d\mu}(\pi\rho_0 s_{\text{LIN}})^2 < 0$ on each g_{LIN} -contour.

Proof. It will be useful to start with coordinates μ, β_{CS} on the region defined by Eq. (34) and change to x, y with domain $D = \{(x, y) : 0 < x < \infty, 0 < y < x\}$ by

$$x := \ln(\mu), y := \ln(1/\beta_{\text{CS}}) \quad (35)$$

Then Eq. (33) becomes

$$\begin{aligned} (\pi\rho_0 s_{\text{LIN}})^2 &= \left(\frac{N}{2}x + y\right)(x - y) \frac{e^x}{(e^x - 1)^2} \\ \ln(g_{\text{LIN}}) &= \frac{y - x}{e^x - 1} + \ln(1 - e^{-x}) - \ln(1 - e^{-y}) \end{aligned} \quad (36)$$

For g_{LIN} fixed, that is, on a g_{LIN} -contour, logarithmic differentiation of Eq. (36) gives

$$\frac{dy}{dx} = \frac{x - y}{e^x - e^y} \frac{e^y - 1}{e^x - 1} e^x \quad (37)$$

For s_{LIN} on this g_{LIN} -contour, logarithmic differentiation of Eq. (36) gives

$$\frac{1}{(\pi\rho_0 s_{\text{LIN}})^2} \frac{d}{dx} (\pi\rho_0 s_{\text{LIN}})^2 = \frac{N + 2 \frac{dy}{dx}}{Nx + 2y} + \frac{1 - \frac{dy}{dx}}{x - y} - \frac{e^x + 1}{e^x - 1} \quad (38)$$

that is,

$$\frac{1}{(\pi\rho_0 s_{\text{LIN}})^2} \frac{d}{dx} (\pi\rho_0 s_{\text{LIN}})^2 = \frac{2Nx + (2 - N)y}{(Nx + 2y)(x - y)} - \frac{e^x + 1}{e^x - 1} + \frac{(2 - N)x - 4y}{(Nx + 2y)(x - y)} \frac{dy}{dx} \quad (39)$$

and combining Eqs. (37) and (39) gives

$$\frac{d}{dx}(\pi\rho_0 s_{\text{LIN}})^2 = \frac{-(\pi\rho_0 s_{\text{LIN}})^2 e^{x+y/2} E}{(Nx+2y)(e^x - e^y)(e^x - 1)} \quad (40)$$

where E is

$$\begin{aligned} E := & e^{x-y/2} [(Nx+2y)(x-y) - (Nx+2y) - N(x-y)] - \\ & e^{y/2} [2(x-y)^2 - (Nx+2y) - N(x-y)] + \\ & e^{-y/2} [2(x-y)^2 + (Nx+2y) + N(x-y)] - \\ & e^{-x+y/2} [(Nx+2y)(x-y) + (Nx+2y) + N(x-y)] \end{aligned} \quad (41)$$

By the conditions on domain D , the quantity E in Eq. (40) is multiplied by a strictly negative quantity. Consequently, since $d/dx = \mu d/d\mu$, the Claim follows if E can be shown positive on the domain D . On this domain, we can set $y := x - ax$ with $0 < a < 1$, $0 < x < \infty$, and rewrite E as

$$E = 4(N + Na + 2 - 2a)x \sinh(x/2) ((ax/2) \cosh(ax/2) - \sinh(ax/2)) + 2(N+2)ax^2 (\cosh(x/2) \sinh(ax/2) - a \sinh(x/2) \cosh(ax/2)) \quad (42)$$

Notice the inequalities:

$$\begin{aligned} f(z) &:= z \cosh(z) - \sinh(z) > 0 \text{ for } z > 0 \\ g(a, z) &:= \cosh(z) \sinh(az) - a \sinh(z) \cosh(az) > 0 \text{ for } z > 0, 0 < a < 1 \end{aligned} \quad (43)$$

These inequalities follow from observing that $f(0) = g(a, 0) = 0$ and that $df/dz, dg/dz$ are both positive for the stated ranges. It follows that E is positive in Eq. (42), and the Claim is proved.

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Fig. 1.

Diagram illustrating cross-sections of three DOG field functions (1, 2, and 3) with different values of the balance (α_C) parameter. Curve (1): $\alpha_C^2=0.2459$, $\alpha_S^2=0.9837$ ($\mu = 4.0$, $\sigma = 1.1$, $\alpha_C = 0.871$, strong bandpass). Curve (2): $\alpha_C^2=0.2164$, $\alpha_S^2=0.8656$ ($\mu = 4.0$, $\sigma = 1.25$, $\alpha_C = 0.707$, moderate bandpass). Curve (3): $\alpha_C^2=0.1288$, $\alpha_S^2=0.5152$ ($\mu = 4.0$, $\sigma = 2.1$, $\alpha_C = 0.218$, low pass).

Fig. 2.

Diagram illustrating the reformulation and graphical representation of the DOG filter typically parameterized using α_C , α_S , and β_{CS} , reduced to a two-dimensional parameter space using α_C^2, α_S^2 -coordinates. The lower and upper bounds of Region AB are designated respectively by Curve A and Curve B (thick blue lines). Region AB is partitioned by the β_{CS} -contours (thin blue lines) such that each point of the region lies on one and only one β_{CS} -contour for $0 < \beta_{CS} < 1$. The boundary curves A and B both correspond to $\beta_{CS} = 1$. The region and its boundary are defined by the totality of contours, $0 < \beta_{CS} < 1$. There is also a one-to-one correspondence between the points of Region AB and ordered pairs of auxiliary parameters (μ, v) , shown as dashed lines, where the boundary curves A and B correspond to the bounds $\mu = 1$ and $v = 1$, respectively. The gridlines for fixed μ are rays with slope μ through the origin since $\alpha_S^2 = \mu \alpha_C^2$. On each gridline with fixed μ , β_{CS} decreases monotonically from 1 to 0 as μ increases. Points labeled (1), (2) and (3) correspond to three DOG filters whose space- and frequency-domain representations are illustrated in Figs. 1 and 3, respectively.

Fig. 3.

Diagram illustrating the frequency response functions of the DOG filters (1), (2), and (3), whose space-domain cross-sections appear in Fig. 1. Note the progression from strongly band-pass (1) to low-pass (3) frequency response as the value of the balance parameter (β_{CS}) decreases. The maximum of each band-pass case defines the optimal linear frequency s_{LIN} . Notice the optimal frequencies satisfy the bound $0 \leq s_{LIN} \leq 1$. Parameter values are given in Fig. 1.

Fig. 4.

Diagram illustrating normalized optimal-frequency contours ($\alpha_{0S_{LIN}}$; thin pink lines) of DOG filters within Region ABC. Curve C (thick pink line) indicates a boundary condition where the optimal frequency becomes zero, and the frequency response function becomes low-pass.

Fig. 5.

Diagram showing Region ABC with sample g_{LIN} -contours (green lines). Notice that the g_{LIN} -contours partition Region ABC. For a fixed value of μ , g_{LIN} ranges from infinity on the boundary $\alpha_C^2 = 1$ (Curve B) and strictly decreases to unity at $\alpha_C^2 = 2$ (Curve C). For a fixed value of ν , g_{LIN} strictly decreases from the value $\exp(-2)/(-1) > 1$ at $\mu = 1$ (Curve A) to one as μ increases.

Fig. 6.

Diagram illustrating the combination of the s_{LIN} - and g_{LIN} -contours from Figs. 4 and 5. The contours form a coordinate grid for Region ABC, that is, there is a one-to-one correspondence between parameter pairs (α_C^2, α_S^2) and the pairs (s_{LIN}, g_{LIN}) , which are directly observable.

Fig. 7.

Converse to Fig. 6 showing α_c^2, α_s^2 – contours in coordinates $\log_{10}(g_{\text{LIN}}), (\pi \rho_{s_{\text{LIN}}})^2$. The thick solid line is a boundary curve corresponding to Curve A in Fig. 6. The dashed lines are contours for $\alpha_c^2=0.15, 0.20, 0.25, 0.30, 0.35$. For $0 < \alpha_c^2 \leq 0.25$, the contours extend to the origin ($\log_{10}(g_{\text{LIN}}) = 0$), but for $0.25 < \alpha_c^2 < 0.5$, the contours meet the boundary curve at nonzero values of $\log_{10}(g_{\text{LIN}})$, behavior which is just visible on the plot. The solid lines are contours for $\alpha_s^2=0.45$ (unmarked and just visible) and for $\alpha_s^2=0.5, 1.0, 2.0$. For $\alpha_s^2 \geq 0.5$, the contours extend to infinity ($\log_{10}(g_{\text{LIN}}) = \infty$), but for $0 < \alpha_s^2 < 0.5$, the contours meet the boundary curve at finite values of $\log_{10}(g_{\text{LIN}})$, behavior which is just visible in the plot.

Fig. 8.

Diagram illustrating Curve D together with α_{CS} -contours and $\alpha_{0s_{LIN}}$ -contours. Note that the maximum optimal frequency on a given balance contour occurs at the point on the contour that is tangent to an optimal frequency contour.

Fig. 9.

Optimal frequency and gain contours for dimension $N=1$. The plot shows Region $A_1B_1C_1$ as a subregion of Region A_1B_1 with contours for $\alpha_{S_{LIN}} = 0.60, 0.65, 0.70, 0.725$ and $g_{LIN} = 1.25, 2.0, 4.0, 8.0$, and a balance contour for $\beta_{CS} = 0.85$. The optimal frequency is bounded by $0 < \alpha_{S_{LIN}} < 0.75$. The dashed line is $\alpha_{S_{LIN}} = 3/2$, where $\alpha_{S_{LIN}}$ is a maximum for fixed μ and where the $\alpha_{S_{LIN}}$ -contour is tangent to the corresponding ray through the origin, as is evident in the plot.