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STABILITY OF THE INTERIOR PROBLEM FOR POLYNOMIAL REGION OF INTEREST

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Abstract

In many practical applications, it is desirable to solve the interior problem of tomography without requiring knowledge of the attenuation function f_a on an open set within the region of interest (ROI). It was proved recently that the interior problem has a unique solution if f_a is assumed to be piecewise polynomial on the ROI. In this paper, we tackle the related question of stability. It is well-known that lambda tomography allows one to stably recover the locations and values of the jumps of f_a inside the ROI from only the local data. Hence, we consider here only the case of a polynomial, rather than piecewise polynomial, f_a on the ROI. Assuming that the degree of the polynomial is known, along with some other fairly mild assumptions on f_a , we prove a stability estimate for the interior problem. Additionally, we prove the following general uniqueness result. If there is an open set U on which f_a is the restriction of a real-analytic function, then f_a is uniquely determined by only the line integrals through U . It turns out that two known uniqueness theorems are corollaries of this result.

1. Introduction

While classic CT theory targets theoretically exact reconstruction of a whole cross-section or volume from nontruncated projections, biomedical applications of CT/micro-CT often focus on small internal regions of interest (ROIs) such as cardiac structures. The conventional CT approach cannot exactly reconstruct an internal ROI only from line integrals through the ROI because this interior problem does not have a unique solution [Nat86]. When applying traditional CT algorithms for interior reconstruction from truncated projections, features outside an interior ROI may create disturbing artifacts overlapping features of interest, rendering the image inaccurate or useless. Over the past decades, lambda tomography was developed as a branch of applied mathematics that recovers gradient-like information in an ROI from local data [FFRS97, FRS92, RK96, YWYW07, Qui07, QO08]. However, lambda tomography did not achieve much popularity in practice because of its nonquantitative nature. That is to say, lambda tomography does not allow one to reconstruct the attenuation function pointwise, but only gives the values and locations of jumps.

In contrast, the interior tomography approach seeks a theoretically exact solution to the interior problem. It was recently shown that the interior problem can be exactly and stably solved assuming a known subregion in an ROI [YYW07, YYWW07, YYW08a, YYW08b, KCND08]. Precise knowledge of a subregion is available if we have air gaps, water, blood, or other quantitative landmarks. More generally, such prior knowledge can be acquired in sequential or multiresolution studies. However, the precise subregion knowledge may be unavailable in many important cases such as perfusion cardiac CT/micro-CT.

Another direction of research in interior tomography is based on compressive sensing (CS). Classic sampling theory states that a signal must be sampled at least twice as fast as its maximum frequency to capture all information. Surprisingly, an emerging theory – compressive sensing – has the power to sample compressible signals at a rate much less than the Nyquist rate and yet allow accurate reconstruction of these signals [CRT06], [Don06]. The main idea of CS is that most signals are sparse in an appropriate domain; that is, a majority of their expansion coefficients are close or equal to zero. Typically, CS starts by taking a limited amount of samples using a least-correlated measurement matrix, and then the signal is exactly recovered with an overwhelming probability from the limited data via the ℓ_1 norm minimization. Since samples are insufficient in the traditional sense, recovering the signal would involve an underdetermined system. That is, many candidate solutions can fit the incomplete dataset. Therefore, some additional constraint must be enforced to select the best candidate. While the classic solution to such an inverse problem is to minimize the ℓ_2 norm, it was shown that finding the candidate with the minimum ℓ_1 norm, which is closely related to the total variation (TV) minimization in a number of imaging cases [ROF92], is the most desirable choice [CRT06, Don06, RBFK11].

Using the CS theory, it was recently proved that exact interior reconstruction is theoretically achievable with an interior ROI-focused scan if the attenuation function is piecewise polynomial on the ROI [YW09, YYJW09, HYW09, YYJW10]. This suggests that interior tomography can be further developed in the CS framework without precise subregion knowledge.

As a complement to the result in [YYJW10] that an attenuation function f_a piecewise polynomial on an ROI, is uniquely determined by Radon data on that ROI, it is desirable to also prove that f_a can also be *stably* reconstructed on its ROI. In other words, small errors in measurement should not result in large reconstruction errors. In the piecewise polynomial framework, algorithms minimizing TV have been devised to solve the interior problem numerically [YYJW10]. The fact that these algorithms work well has empirically demonstrated that the interior problem is indeed stable in the case when f_a is piecewise polynomial on the ROI.

In this paper, we make a step towards confirming this empirical evidence. More precisely, the result we prove assumes that f_a is polynomial, rather than piecewise polynomial, on the ROI. We view this result as being the first step in proving stability for the piecewise polynomial case. A generalization to the case in which f_a is piecewise polynomial on the ROI is likely to require methods of lambda tomography. These methods allow one to stably compute the size and location of discontinuities of a function and its derivatives from only the local data [RK96, FBH⁺01].

We prove stability under the assumption that the support of f_a is a disk centered at the origin, and the ROI a smaller disk, which is centered at the origin as well. Stability in the more general case when the ROI is a disk centered at some point $x_0 \in \text{supp } f_a$ follows by enlarging $\text{supp } f_a$ to a ball also centered at x_0 . It is interesting to note a difference between the estimates obtained here and in [CNDK08]. The one in [CNDK08] gets progressively weaker as one gets further away from the known subregion and closer to the edge of the ROI. The estimates obtained here are not based on a known subregion, but rather on the assumption that f_a is polynomial on the ROI. As a result, our estimates are uniform. Intuitively, it is clear that the closer one gets to the boundary of the ROI, the less stable the recovery of f_a should be. Hence, one can conjecture that the estimates obtained here are not the best possible and could be improved. This can be the subject of future research.

The other main result of this paper is a uniqueness theorem, which is stronger than those of [CNDK08] and [YYJW10]. We prove in Theorem 8.1 that a function is uniquely determined by its Radon transform on any open set where the function is real-analytic. (In this paper, we use the loose terminology "Radon transform on an open set" to denote the collection of integrals along lines through the open set.) This is an interesting result, because it seems to be a common root of the piecewise polynomial and known subregion approaches. In fact, the uniqueness results of both [CNDK08] and [YYJW10] are corollaries of Theorem 8.1.

All of the results of this paper are formulated using relatively minimal assumptions on the smoothness of f_d outside of the ROI. We make use of the Sobolev spaces for our smoothness conditions, and we begin the paper with a section on these spaces and their norms. Though we formulate our results in terms of functions that may not even be continuous, we perform operations in our proofs such as differentiation, restriction to lines, and evaluation at points. We reconcile this discrepancy by working with smooth approximations of our functions, which then lead to results about the original functions themselves.

We first present our stability result and its proof, and then turn to our uniqueness result. We prove auxiliary results in appendices at the end of the paper.

2. Classes of Functions and Norms

In this section, we will define the classes of functions that we will use in stating and proving stability. Define $B_R := \{x \in \mathbb{R}^2 : |x| < R\}$ and $Z_R := \{(p, \alpha) \in \mathbb{R} \times S^1 : |p| < R\}$. Define $H_0^s(B_R)$ by the closure of $C_0^\infty(B_R)$ in the norm

$$\|f\|_s := \left[\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |\tilde{f}(\xi)|^2 (1+|\xi|^2)^s d\xi \right]^{1/2}, \quad (2.1)$$

where $\tilde{f}$ denotes the Fourier transform of f . Also define $H_0^s(Z_R)$ by the closure of $C_0^\infty(Z_R)$ in the norm

$$\|g\|_s := \left[\frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}} |\tilde{g}(\lambda, \alpha)|^2 (1+\lambda^2)^s d\lambda d\alpha \right]^{1/2}, \quad (2.2)$$

where the Fourier transform is being taken with respect to the first variable only. See [Nat86] for these classical definitions. There is a relationship between the two norms defined above.

If $f \in H_0^s(B_R)$, then there is a constant C_s such that

$$\|Rf\|_{s+1/2} \leq C_s \|f\|_s, \quad (2.3)$$

([RK96], eq. (2.4.3)), where Rf is the Radon transform of f . As a consequence, $f \in H_0^s(B_R)$ implies $Rf \in H_0^{s+1/2}(Z_R)$. Next we define the following pseudo-norm, which in particular satisfies the triangle inequality. If $s > 1/2$, $R > 0$, and $g \in H_0^s(Z_{R'})$ for $R' > R$, let

$$\|g\|_R^* := \sup_{0 \leq p \leq R} \left(\frac{1}{2\pi} \int_0^{2\pi} |g(p, \alpha)|^2 d\alpha \right)^{1/2}. \quad (2.4)$$

We must verify that the above quantity is well-defined. By ([Fol84], eq. (8.54)), we have that for each fixed α and p ,

$$|g(p, \alpha)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\tilde{g}(\lambda, \alpha)| d\lambda \leq c_s \left[\int_{\mathbb{R}} |\tilde{g}(\lambda, \alpha)|^2 (1+\lambda^2)^s d\lambda \right]^{1/2}, \quad (2.5)$$

for $s > 1/2$. It follows by integrating in α that

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |g(p, \alpha)|^2 d\alpha \right)^{1/2} \leq c_s \|g\|_s. \quad (2.6)$$

Because the bound in (2.6) is independent of p , the pseudo-norm introduced in (2.4) is well-defined. Also, from (2.6), we see that the $\|\cdot\|_s$ norm dominates the $\|\cdot\|_R^*$ pseudo-norm:

$$\|g\|_R^* \leq c_s \|g\|_s, \quad s > \frac{1}{2}. \quad (2.7)$$

3. Setting of the problem and formulation of stability

Suppose we are given an attenuation function $f_a: \mathbb{R}^2 \rightarrow \mathbb{R}$ that is supported in B_{R_2} . Moreover, we are only interested in the behavior of f_a on an ROI, which we will assume to be the disk B_{R_1} for some $R_1 < R_2$. Also, let us assume that f_a restricted to B_{R_1} is a polynomial $P_a(x_1, x_2)$ of degree at most N . In this setting, we will prove stability for the interior problem.

In formulating a general stability result, we would like to make as few smoothness assumptions as possible. Let us fix $s > 0$. Suppose $f_a \in H_0^s(B_{R_2})$ for some $s > 0$. By (2.3), we find that $Rf_a \in H_0^{s+1/2}(Z_{R_2})$. In view of this fact, (2.3) and (2.7) imply that $\|Rf_a\|_{R_1}^* < \infty$.

Hence we can assume that the data h measured on Z_{R_1} also satisfies $\|h\|_{R_1}^* < \infty$. Here h is the collection of line integrals of f_a , measured (possibly with some error) for all $(p, \alpha) \in Z_{R_1}$. Next we define a class $S_{\mathcal{M}}$ of functions that adequately fit the data. For $\varepsilon, M > 0$, let $S_{\mathcal{M}}$ be the class of reconstructed functions $f_r \in H_0^s(B_{R_2})$ (note that s is the same as above) such that

- i. f_r restricted to B_{R_1} is a polynomial $P_r(x_1, x_2)$ of degree at most N ,
- ii. $\|f_r\|_s \leq M$
- iii. $\|Rf_r - h\|_{R_1}^* \leq \varepsilon$.

To clarify our notation, the subscript “ a ” in f_a stands for the “actual” attenuation function, and the subscript “ r ” in f_r stands for the “reconstructed” attenuation function. Having defined $S_{\mathcal{M}}$, we may now formulate our stability result:

Theorem 3.1

Let $s > 0$ and let $f_a \in H_0^s(B_{R_2})$ be an attenuation function, with $\|f_a\|_s \leq M$ and $f|_{B_{R_1}} = P_a(x_1, x_2)$, where P_a is a polynomial of degree at most N . Suppose that the Radon data h measured on Z_{R_1} satisfy $\|h\|_{R_1}^* < \infty$, and that the error in measurement is small: $\|h - Rf_a\|_{R_1}^* \leq \varepsilon$, for some $\varepsilon > 0$. Let f_r be a reconstruction satisfying the conditions imposed by $f_r \in S_{\mathcal{M}}$. Then

there is a constant A , depending only on R_1, R_2, s, N , and an exponent $\omega > 0$, depending only on R_1, R_2 , such that

$$|P_a(x) - P_r(x)| \leq AM \left(\frac{\varepsilon}{M} \right)^\omega \quad (3.1)$$

for all $x \in B_{R_1}$.

Proof—As a first step towards proving this theorem, we introduce the "error" functions $f := f_r - f_a$ and $P := P_r - P_a$. Note that $f \in H_0^s(B_{R_2})$, and that f also inherits the following properties from f_r and f_a :

- i. f restricted to B_{R_1} is the polynomial $P(x_1, x_2)$ of degree at most N ,
- ii. $\|f\|_s \leq 2M$,
- iii. $\|Rf\|_{R_1}^* \leq \|Rf_r - h\|_{R_1}^* + \|h - Rf_a\|_{R_1}^* \leq 2\varepsilon$.

As it is difficult to work directly with the function f due to our fairly minimal assumptions on its smoothness outside of the ROI, we will instead work with smooth approximations of f . We state Theorem 3.2 below, which is the result we need, but with the stronger assumption that $f \in C_0^\infty$. We will prove Theorem 3.1 by using the result of Theorem 3.2, and will prove the latter theorem afterwards.

Let K be a C_0^∞ -function supported in the unit ball centered at the origin, with $K(x) \geq 0$ for all x and $\int_{\mathbb{R}^2} K(x) dx = 1$. For $\eta > 0$, define also

$$K_\eta(x) := \frac{1}{\eta^2} K(x/\eta). \quad (3.2)$$

Thus, $\{K_\eta\}_{\eta>0}$ is a normalized family of kernels. Define $f_\eta := f * K_\eta$. This way, $f_\eta \in C_0^\infty(B_{R_2+\eta})$ (note the slightly expanded support). Also, it is easy to check that f_η is a polynomial P_η on $B_{R_1-\eta}$. Moreover, P_η has the same degree as P . Next, we will verify that f_η satisfies the same bound in the Sobolev norm as does f . By the positivity of the kernel K_η , it follows easily that $\|K_\eta\|_1 = 1$, and thus $\|f_\eta\|_s \leq \|f\|_s \leq 2M$. By construction, the functions f_η tend to f in the Sobolev norm $\|\cdot\|_s$. Thus, if we fix an $\varepsilon_0 > 0$, then we can choose a sufficiently small η so that

$$\|f_\eta - f\|_s \leq \varepsilon_0. \quad (3.3)$$

By equations (2.3), (2.7), and (3.3), we have

$$\|Rf_\eta - Rf\|_{R_1-\eta}^* \leq c_s \|Rf_\eta - Rf\|_{s+1/2} \leq c_s C_s \|f_\eta - f\|_s \leq c_s C_s \varepsilon_0. \quad (3.4)$$

Moreover, from property (iii) of f above, and from the definition of $\|\cdot\|_{R_1-\eta}^*$ in (2.4), it is clear that

$$\|Rf\|_{R_1-\eta}^* \leq \|Rf\|_{R_1}^* \leq 2\varepsilon. \quad (3.5)$$

By (3.4) and (3.5), we find

$$\|Rf_\eta\|_{R_1-\eta}^* \leq \|Rf_\eta - Rf\|_{R_1-\eta}^* + \|Rf\|_{R_1-\eta}^* \leq 2\varepsilon + 2C\varepsilon_0, \quad (3.6)$$

where we have renamed $c_s C_s$ by $2C$. From the above discussion, we may conclude that the function f satisfies the conditions of the following theorem, with R_1 replaced by $R_1 -$, R_2 replaced by $R_2 +$, and ε replaced by $\varepsilon + C\varepsilon_0$:

Theorem 3.2

Let $0 < R_1 < R_2$ and let $f \in C_0^\infty(B_{R_2})$. Suppose further that f satisfies the following three criteria for some $M, N, \varepsilon > 0$:

- i. f restricted to B_{R_1} is the polynomial $P(x_1, x_2)$ of degree at most N ,
- ii. $\|f\|_s \leq 2M$,
- iii. $\|Rf\|_{R_1}^* \leq 2\varepsilon$.

There is a constant A depending only on R_1, R_2, s, N , and an exponent $\omega > 0$, depending only on R_1, R_2 , such that

$$|P(x)| \leq AM \left(\frac{\varepsilon}{M} \right)^\omega \quad (3.7)$$

for all $x \in B_{R_1}$.

Applying Theorem 3.2 to f with the modifications in R_1, R_2 , and ε discussed above, we obtain that there are positive constants A and ω such that

$$|P_\eta(x)| \leq A_\eta M \left(\frac{\varepsilon + C\varepsilon_0}{M} \right)^{\omega_\eta} \quad (3.8)$$

for all $x \in B_{R_1-}$. By decreasing ω if necessary, we can suppose that $|P_\eta(x) - P(x)| \leq \varepsilon_0$ for all $x \in B_{R_1-}$. Thus,

$$|P(x)| \leq |P_\eta(x)| + |P(x) - P_\eta(x)| \leq A_\eta M \left(\frac{\varepsilon + C\varepsilon_0}{M} \right)^{\omega_\eta} + \varepsilon_0 \quad (3.9)$$

for all $x \in B_{R_1-}$. From our calculation of A and ω in the proof of Theorem 3.2 and in Appendix C, we see that A and ω depend continuously on ε , and thus tend to A and ω , respectively, as ε tends to 0. Thus, letting $\varepsilon_0 \rightarrow 0$ in (3.9), we find that

$$|P(x)| \leq AM \left(\frac{\varepsilon + C\varepsilon_0}{M} \right)^\omega + \varepsilon_0 \quad (3.10)$$

for all $x \in B_{R_1}$. Since (3.10) holds for all $\varepsilon_0 > 0$, we can let ε_0 tend to 0, which leads to

$$|P(x)| \leq AM \left(\frac{\varepsilon}{M} \right)^\omega, \quad (3.11)$$

which proves Theorem 3.1.

We now turn to proving Theorem 3.2. We begin by expressing f and Rf in terms of Fourier series $f(r) = \sum_k f_k(r) e^{ik\theta}$ and $Rf(p, \theta) = \sum_k g_k(p) e^{ik\theta}$. Here and in the following we use the notation $(\cos \theta, \sin \theta)$. The coefficients $f_k(r)$ and $g_k(p)$ are related by the classical formulas (4.11) and (4.12). The key consequence of (4.12) is the relation (6.2), which is repeated here for convenience:

$$P_k(r) + \frac{1}{\pi} \int_{p_0}^{R_2} g'_k(p) \frac{T_k(\frac{p}{r})}{\sqrt{p^2 - r^2}} dp = \frac{-1}{\pi} \int_r^{p_0} g'_k(p) \frac{T_k(\frac{p}{r})}{\sqrt{p^2 - r^2}} dp, \quad (3.12)$$

where T_k is the k th Chebyshev polynomial, $P_k(r)$ is the k th Fourier coefficient of $P(r)$, and $p_0 \in (0, R_1)$ is suitably chosen. The idea of the proof is to analytically continue the left side of (3.12) from the region $r < p_0$ to the region $r > R_2$. This continuation is done in two different ways: via the upper and lower half-planes of $\mathbb{C}$. The square root takes opposite signs on the two sides of the real half-axis $r > R_2$. Thus, adding these two analytic continuations cancels the integral terms and allows us to recover the polynomial P_k on $r > R_2$. Using Nevanlinna's principle (Theorem 5.1), we obtain bounds on the results of analytic continuation, and thus on P_k . Applying the Lagrange interpolation formula yields a bound on P_k on the interval $[0, R_1]$, which in turn yields the desired bound on $P(x)$.

4. Analysis of Fourier coefficients of f and Rf

Proof of Theorem 3.2

To reduce our task to a one-dimensional problem, let us introduce the functions

$f_k(r) := \frac{1}{2\pi} \int_0^{2\pi} f(r\Theta) e^{-ik\theta} d\theta$ and $g_k(p) := \frac{1}{2\pi} \int_0^{2\pi} Rf(p, \alpha) e^{-ik\alpha} d\alpha$, for $-N \leq k \leq N$. We make the following observations about $f_k(r)$ and $g_k(p)$. Let us first translate the bound $\|f\|_s \leq 2M$ into a statement about $f_k(r)$. As will become apparent later (see (5.15)), it is convenient to bound the following quantity:

$$\begin{aligned} \int_0^{R_2} |f_k(r)|^2 r dr &= \int_0^{R_2} \left| \frac{1}{2\pi} \int_0^{2\pi} f(r\Theta) e^{-ik\theta} d\theta \right|^2 r dr \\ &\leq \int_0^{R_2} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(r\Theta)|^2 d\theta \right) \left(\frac{1}{2\pi} \int_0^{2\pi} |e^{-ik\theta}|^2 d\theta \right) r dr \\ &= \frac{1}{2\pi} \int_0^{R_2} \int_0^{2\pi} |f(r\Theta)|^2 d\theta r dr \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} |f(x)|^2 dx = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} |\tilde{f}(\xi)|^2 d\xi \leq \frac{1}{2\pi} \|f\|_s^2 \leq \frac{2}{\pi} M^2. \end{aligned} \quad (4.1)$$

Note that we used the Parseval-Plancherel theorem in the last line of (4.1). It follows from (4.1) that

$$\left(\int_0^{R_2} |f_k(r)|^2 r dr \right)^{1/2} \leq \sqrt{\frac{2}{\pi}} M. \quad (4.2)$$

To get a bound on $g_k(p)$ for $|p| \leq R_1$, we use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
|g_k(p)| &= \left| \frac{1}{2\pi} \int_0^{2\pi} Rf(p, \alpha) e^{-ik\alpha} d\alpha \right| \\
&\leq \left(\frac{1}{2\pi} \int_0^{2\pi} |Rf(p, \alpha)|^2 d\alpha \right)^{1/2} \left(\frac{1}{2\pi} \int_0^{2\pi} |e^{-ik\alpha}|^2 d\alpha \right)^{1/2} \quad (4.3) \\
&\leq \|Rf\|_{R_1}^* \leq 2\varepsilon, \quad |p| \leq R_1.
\end{aligned}$$

Moreover, by using (2.3), (2.7), and the fact that $|g_k(p)| \leq \|Rf\|_{R_2}^*$, $|p| \leq R_2$, which can be proved as in (4.3), we obtain

$$|g_k(p)| \leq \|Rf\|_{R_2}^* \leq c_s \|Rf\|_{s+1/2} \leq c_s C_s \|f\|_s \leq 2c_s C_s M =: K_s M, \quad (4.4)$$

for all $|p| \leq R_2$. Since f is polynomial on B_{R_1} , we will see that the functions $f_k(r)$ take a special form for $0 \leq r \leq R_1$. Let us write $P(x_1, x_2)$ in polar form. Let $x_1 = r \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $x_2 = r \frac{e^{i\theta} - e^{-i\theta}}{2i}$. Making this substitution, we see that there exist polynomials $P_k(r)$ such that

$$P(r\Theta) = \sum_{k=-N}^N P_k(r) e^{ik\theta}. \quad (4.5)$$

It follows from their definition that the polynomials $P_k(r)$ satisfy $\deg P_k \leq N$. Thus, we can write

$$P_k(r) = \sum_{j=0}^N c_{j,k} r^j. \quad (4.6)$$

Hence, from (4.5) and the definition of $f_k(r)$,

$$f_k(r) = \frac{1}{2\pi} \int_0^{2\pi} f(r\Theta) e^{-ik\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} P(r\Theta) e^{-ik\theta} d\theta = P_k(r), \quad 0 \leq r \leq R_1. \quad (4.7)$$

It follows from (4.7) and the last line of (4.1) that

$$\|P\|_{L_2(B_{R_1})} \leq \left(\int_0^{2\pi} \int_{\mathbb{R}} |f(r\Theta)|^2 r dr d\theta \right)^{1/2} \leq 2M. \quad (4.8)$$

Because the space of polynomials of degree at most N on B_{R_1} is finite dimensional, all norms on that space are equivalent. In particular, the supremum norm is equivalent to the L_2 norm. Thus, (4.8) implies that there is a constant $C_N > 0$ such that

$$\sup_{x \in B_{R_1}} |P(x)| \leq C_N \cdot 2M \quad (4.9)$$

It follows by the second half of (4.7) that for $0 \leq r \leq R_1$,

$$|P_k(r)| \leq \sup_{x \in B_{R_1}} |P(x)| \leq 2C_N M. \quad (4.10)$$

Now, ([RK96], eqs. (2.2.35), (2.2.37)) gives us two useful formulas connecting f_k and g_k :

$$g_k(p) = 2 \int_p^\infty f_k(r) \frac{T_{|k|}(\frac{p}{r})}{\sqrt{1 - \frac{p^2}{r^2}}} dr, \quad (4.11)$$

and

$$f_k(r) = -\frac{1}{\pi} \int_r^\infty g'_k(p) \frac{T_{|k|}(\frac{p}{r})}{\sqrt{p^2 - r^2}} dp, \quad (4.12)$$

where $T_k(x)$ is the Chebyshev polynomial of order k . See also ([Nat86], eq. (2.18)). As a first step in dealing with (4.12), we will find a bound for $g'_k(p)$ in the next section.

5. Bound for $g'_k(p)$

We first state Nevanlinna's Principle, which is a very useful result for bounding analytic functions; we will apply it twice in this paper. See ([DNCK06], Appendix B) for a sketch of its proof and [DNCK06], [CNDK08] for more examples of its use.

Theorem 5.1

Let f be a function analytic in the domain Ω and continuous on its closure $\bar{\Omega}$. Suppose that $|f(z)| \leq M$ for $z \in \bar{\Omega}$, and that there is a segment B of the boundary $\partial\Omega$ such that $|f(z)| \leq M$ for $z \in B$. If $u(z)$ is the harmonic function in Ω that satisfies $u(z) = 1$ for $z \in B$ and $u(z) = 0$ for $z \in \partial\Omega \setminus B$, then

$$|f(z)| \leq M \left(\frac{\varepsilon}{M} \right)^{\omega(z)}, \quad z \in \Omega \quad (5.1)$$

Note that by the maximum principle for harmonic functions, we have

$$0 < \omega(z) < 1 \text{ for } z \in \Omega. \quad (5.2)$$

Steps to obtaining a bound for $g'_k(p)$ —Let us fix $k \geq 0$. For most of the paper, we will work with only this case because the cases for positive and negative k are symmetric (as can be seen from the $|k|$ in (4.11) and (4.12)). Also, let us fix any $\varepsilon \in (0, R_1/9)$. For example, we can take $\varepsilon = R_1/18$. The reason for the requirement $\varepsilon \in (0, R_1/9)$ will become clear later (see the paragraph following equation (6.20) below). We will bound $g'_k(p)$ on the interval $(3\varepsilon, R_1 - 3\varepsilon)$ in the following three steps (See Figure 1):

1. Analytically continue $g_k(p)$ from $(\varepsilon, R_1 - \varepsilon)$ to the disk $D \subset \mathbb{C}$ with the latter interval as a diameter.
2. Apply Nevanlinna's principle to the function $g_k(z)$ on D^+ and D^- (the upper and lower half-disks of D).

3. Use Cauchy's integral formula to express $g_k(p)$, $p \in (3, R_1 - 3)$ as an integral over C , the circle with diameter $(2, R_1 - 2)$. We can differentiate this integral and use part (2) to obtain a bound for $g'_k(p)$.

Before proceeding with these steps, we use (4.7), (4.11), and (4.6) to arrive at the following expression for $g_k(p)$:

$$g_k(p) = \left[\sum_{j=0}^N 2c_{j,k} \int_p^{R_1} r^j \frac{T_k\left(\frac{p}{r}\right)}{\sqrt{1-\frac{p^2}{r^2}}} dr \right] + 2 \int_{R_1}^{R_2} f_k(r) \frac{T_k\left(\frac{p}{r}\right)}{\sqrt{1-\frac{p^2}{r^2}}} dr \quad (5.3)$$

Step 1: Analytic Continuation of $g_k(p)$ —Note that the first integral in (5.3) has p in both a limit of integration and the integrand. This property is inconvenient for analytic continuation, and we get rid of it by the change of variables $u = p/r$. We have

$$\begin{aligned} 2c_{j,k} \int_p^{R_1} r^j \frac{T_k\left(\frac{p}{r}\right)}{\sqrt{1-\frac{p^2}{r^2}}} dr &= 2c_{j,k} \int_1^{\frac{p}{R_1}} \left(\frac{p}{u}\right)^j \frac{T_k(u)}{\sqrt{1-u^2}} \frac{-p}{u^2} du \\ &= 2c_{j,k} p^{j+1} \int_{\frac{p}{R_1}}^1 \frac{T_k(u)}{u^{j+2} \sqrt{1-u^2}} du. \end{aligned} \quad (5.4)$$

For convenience, we will scale the last quantity of (5.4) by a factor of R_1 in the variable p . We define $\delta_1 := p/R_1$ and a function $I_{j,k}(p)$ on $p \in (\delta_1, 1 - \delta_1)$ by

$$\begin{aligned} I_{j,k}(p) &:= 2c_{j,k} (R_1 p)^{j+1} \int_{\frac{p}{R_1}}^1 \frac{T_k(u)}{u^{j+2} \sqrt{1-u^2}} du \\ &= 2c_{j,k} R_1^{j+1} p^{j+1} \left(\int_{\delta_1}^{1-\delta_1} \frac{T_k(u)}{u^{j+2} \sqrt{1-u^2}} du + \int_{\frac{p}{R_1}}^{\delta_1} \frac{T_k(u)}{u^{j+2} \sqrt{1-u^2}} du \right). \end{aligned} \quad (5.5)$$

From (5.3), (5.4), and (5.5), we see that

$$g_k(p) = \sum_{j=0}^N I_{j,k} \left(\frac{p}{R_1} \right) + 2 \int_{R_1}^{R_2} f_k(r) \frac{T_k\left(\frac{p}{r}\right)}{\sqrt{1-\frac{p^2}{r^2}}} dr. \quad (5.6)$$

Let D_1 be the disk D scaled by a factor of $1/R_1$. Then, D_1 is the disk with diameter $(\delta_1, 1 - \delta_1)$, and is a subset of the unit disk. We seek to analytically continue $I_{j,k}$ into D_1 , which, as we see from (5.6), will contribute to continuing $g_k(p)$ into D . Note that the principal branch of the square root is analytic inside the unit disk because it has branch cuts along $(-\infty, -1]$ and $[1, \infty)$. Thus, we can define the analytic continuation of $I_{j,k}$ for $w \in D_1$ by

$$I_{j,k}(w) := 2R_1^{j+1} c_{j,k} \left(w^{j+1} \int_w^{1-\delta_1} \frac{T_k(u)}{u^{j+2} \sqrt{1-u^2}} du + w^{j+1} \int_{1-\delta_1}^1 \frac{T_k(u)}{u^{j+2} \sqrt{1-u^2}} du \right), \quad (5.7)$$

where the first integral is taken over any contour in D_1 connecting w to $1 - \delta_1$ and the square roots are taken in the principal sense. Also, taking the principal branch of the square root in the second integral of (5.3), we can simply replace the p with a z in that integral to find its analytic continuation to D . Thus, we can now write the formula for the analytic continuation of $g_k(p)$ into D .

$$g_k(z) := \sum_{j=0}^N I_{j,k} \left(\frac{z}{R_1} \right) + 2 \int_{R_1}^{R_2} f_k(r) \frac{T_k \left(\frac{z}{r} \right)}{\sqrt{1 - \frac{z^2}{r^2}}} dr, z \in D \quad (5.8)$$

Step 2: Applying Nevanlina's Principle to $g_k(z)$ —Now, we have that $g_k(z)$ is an analytic function in D . Recall that D^+ and D^- are the upper and lower half-disks of D (see Figure 1). We will apply Nevanlina's principle to $g_k(z)$ in the domains D^+ and D^- . Let us focus first on D^+ . In view of the bound (4.3), we take B (see Theorem 5.1) to be the segment $(-R_1, R_1)$. It remains to find an M that bounds $g_k(z)$ on the remainder of the boundary of D^+ . First, we find a bound for $I_{j,k}(w)$ on $\overline{D}_1$. It will be useful to note that when $u \in \overline{D}_1$, we have $|1 - u^2| = |1 - u||1 + u| \geq \delta_1 |\operatorname{Re}(1 + u)| \geq \delta_1$, and so

$$\left| \sqrt{1 - u^2} \right| = |1 - u^2|^{\frac{1}{2}} \geq \sqrt{\delta_1}. \quad (5.9)$$

We write

$$w^{j+1} \int_w^{1-\delta_1} \frac{T_k(u)}{u^{j+2} \sqrt{1-u^2}} du = w^{j+1} \int_{\Gamma} \frac{T_k(u)}{u^{j+2} \sqrt{1-u^2}} du + w^{j+1} \int_{|w|}^{1-\delta_1} \frac{T_k(u)}{u^{j+2} \sqrt{1-u^2}} du \quad (5.10)$$

$$=: I_1 + I_2,$$

where Γ is the contour that goes from w to the real axis along an arc of constant modulus (see Figure 2).

Using (5.9) and Proposition 9.2 of Appendix A with $R = 1$, we have

$$|I_1| \leq |w|^{j+1} \int_{\Gamma} \frac{|T_k(u)|}{|u|^{j+2} \sqrt{1-u^2}} du \leq \frac{|w|^{j+1} (1 + \sqrt{2})^k |\Gamma|}{|w|^{j+2} \sqrt{\delta_1}} = \frac{(1 + \sqrt{2})^k |w| |\arg w|}{|w| \sqrt{\delta_1}} \leq \frac{\pi (1 + \sqrt{2})^k}{2 \sqrt{\delta_1}}. \quad (5.11)$$

Also, (9.2) in Appendix A and (5.9) give us

$$|I_2| \leq |w|^{j+1} \int_{|w|}^{1-\delta_1} \frac{|T_k(u)|}{u^{j+2} \sqrt{1-u^2}} du \leq \frac{|w|^{j+1}}{\sqrt{\delta_1}} \int_{|w|}^{1-\delta_1} \frac{du}{u^{j+2}} \quad (5.12)$$

$$= \frac{|w|^{j+1}}{(j+1) \sqrt{\delta_1}} \left[\frac{1}{u^{j+1}} \right]_{|w|}^{1-\delta_1} \leq \frac{1}{(j+1) \sqrt{\delta_1}} \leq \frac{1}{\sqrt{\delta_1}}$$

To bound the second summand of the RHS of (5.7), we use (9.2) again and note that $|w| \leq 1 - \delta_1$ for $w \in \overline{D}_1$ to obtain

$$\left| w^{j+1} \int_{1-\delta_1}^1 \frac{T_k(u)}{u^{j+2} \sqrt{1-u^2}} du \right| \leq \frac{|w|^{j+1}}{(1-\delta_1)^{j+2}} \int_{1-\delta_1}^1 \frac{du}{\sqrt{1-u^2}} \leq \frac{\pi}{2(1-\delta_1)}. \quad (5.13)$$

Combining (5.7), (5.10), (5.11), (5.12), (5.13), (4.10), and Proposition 9.8 in Appendix D, we find that for $w \in \overline{D}_1$,

$$|I_{j,k}(w)| \leq 2R_1(R_1^j |c_{j,k}|) \left(\left| w^{j+1} \int_w^{1-\delta_1} \frac{T_k(u)}{u^{j+2} \sqrt{1-u^2}} du \right| + \left| w^{j+1} \int_{1-\delta_1}^1 \frac{T_k(u)}{u^{j+2} \sqrt{1-u^2}} du \right| \right) \leq 4R_1 M C_N (N+1)(2N)^N \left(\frac{\pi(1+\sqrt{2})^k}{2\sqrt{\delta_1}} + \frac{1}{\sqrt{\delta_1}} + \frac{\pi}{2(1-\delta_1)} \right) \quad (5.14)$$

Note that this result does not depend on j . Also, note that the above expression does not make sense for the important special case $N=0$. However, if $N=0$, then $j=k=0$, and (4.10) becomes $|c_{0,0}| \leq 2C_0 M$. Hence, there is no need to apply Proposition 9.8 when $N=0$, and we may interpret $(2N)^N|_{N=0} = 1$ in (5.14).

It remains to bound the second summand in the RHS of (5.8). Note that for $z \in D$ and $r \in [R_1, R_2]$, we have that $|z/r| \leq 1$. Thus, we can again apply Proposition 9.2 in Appendix A with $R=1$. Using this result together with (4.2) and the Cauchy Schwarz inequality, we find that for $z \in D$,

$$\begin{aligned} \left| 2 \int_{R_1}^{R_2} f_k(r) \frac{T_k\left(\frac{z}{r}\right)}{\sqrt{1-\frac{z^2}{r^2}}} dr \right| &= 2 \left| \int_{R_1}^{R_2} (f_k(r) \sqrt{r}) \left(\frac{T_k\left(\frac{z}{r}\right)}{\sqrt{r} \sqrt{1-\frac{z^2}{r^2}}} \right) dr \right| \\ &\leq 2 \left(\int_{R_1}^{R_2} |f_k(r)|^2 r dr \right)^{1/2} \left(\int_{R_1}^{R_2} \frac{|T_k\left(\frac{z}{r}\right)|^2}{r |1-\frac{z^2}{r^2}|} dr \right)^{1/2} \\ &\leq 2 \sqrt{\frac{2}{\pi}} M \left((1+\sqrt{2})^{2k} \int_{R_1}^{R_2} \frac{r}{r^2-|z|^2} dr \right)^{1/2} \\ &= 2 \sqrt{\frac{2}{\pi}} M (1+\sqrt{2})^k \left(\frac{1}{2} \ln(r^2-|z|^2) \Big|_{R_1}^{R_2} \right)^{1/2} \\ &\leq 2 \sqrt{\frac{2}{\pi}} M (1+\sqrt{2})^k \sqrt{\ln R_2} \end{aligned} \quad (5.15)$$

Finally, we combine estimates (5.14) and (5.15) with (5.8) to find that for $z \in D$,

$$\begin{aligned} |g_k(z)| &\leq \sum_{j=0}^N \sup_{z \in \overline{D}} |I_{j,k}\left(\frac{z}{R_1}\right)| + \sup_{z \in \overline{D}} \left| 2 \int_{R_1}^{R_2} f_k(r) \frac{T_k\left(\frac{z}{r}\right)}{\sqrt{1-\frac{z^2}{r^2}}} dr \right| \\ &\leq \sum_{j=0}^N \sup_{w \in \overline{D_1}} |I_{j,k}(w)| + 2 \sqrt{\frac{2}{\pi}} M (1+\sqrt{2})^k \sqrt{\ln R_2} \\ &\leq 4R_1 M C_N (N+1)^2 (2N)^N \left(\frac{\pi(1+\sqrt{2})^k}{2\sqrt{\delta_1}} + \frac{1}{\sqrt{\delta_1}} + \frac{\pi}{2(1-\delta_1)} \right) + 2 \sqrt{\frac{2}{\pi}} M (1+\sqrt{2})^k \sqrt{\ln R_2} \\ &:= M_{1,k} \end{aligned} \quad (5.16)$$

Now, we are ready to apply Nevanlinna's principle to $g_k(z)$ on D^+ . Let $\varphi_1(z)$ be the harmonic function in D^+ that satisfies $\varphi_1(z) = 1$ for $z \in (\cdot, R_1 - \cdot)$ and $\varphi_1(z) = 0$ for $z \in D^+ \setminus (\cdot, R_1 - \cdot)$. By Nevanlinna's principle and the bounds (4.3) and (5.16), we have

$$|g_k(z)| \leq M_{1,k} \left(\frac{2\varepsilon}{M_{1,k}} \right)^{\omega_1(z)} \quad \text{for } z \in D^+. \quad (5.17)$$

Note that if we extend $\varphi_1(z)$ to D^- by writing $\varphi_1(z) := \varphi_1(\bar{z})$, then (5.17) holds for all $z \in D$ by symmetry. Hence we arrive at the main result of this section:

$$|g_k(z)| \leq M_{1,k} \left(\frac{2\varepsilon}{M_{1,k}} \right)^{\omega_1(z)} \quad \text{for all } z \in D. \quad (5.18)$$

Step 3: Application of Cauchy's Integral Formula—Now that we have a bound on $|g_k(z)|$ in D , we will use Cauchy's integral formula to obtain an expression for $g'_k(p)$ in terms of $g_k(z)$. Let C be the circle with diameter $(2\delta, R_1 - 2\delta)$, as in Figure 1. Then, for any $p \in [3\delta, R_1 - 3\delta]$, Cauchy's integral formula gives us

$$g_k(p) = \frac{1}{2\pi i} \int_{C_\delta} \frac{g_k(z)}{z-p} dz. \quad (5.19)$$

Taking the derivative with respect to p in (5.19), we find that

$$g'_k(p) = \frac{1}{2\pi i} \int_{C_\delta} \frac{g_k(z)}{(z-p)^2} dz. \quad (5.20)$$

Thus, from (5.18) and (5.20), we get that for $p \in [3\delta, R_1 - 3\delta]$,

$$\begin{aligned} |g'_k(p)| &\leq \frac{1}{2\pi} \pi R_1 \sup_{z \in C_\delta} |g_k(z)| \frac{1}{\delta^2} \\ &\leq \frac{R_1}{2\delta^2} \sup_{z \in C_\delta} M_{1,k} \left(\frac{2\varepsilon}{M_{1,k}} \right)^{\omega_1(z)} \leq \frac{R_1 M_{1,k}}{2\delta^2} \left(\frac{2\varepsilon}{M_{1,k}} \right)^{\inf_{C_\delta} \omega_1(z)}. \end{aligned} \quad (5.21)$$

In Proposition 9.6 of Appendix B, we prove that $\inf_{C_\delta} \omega_1(z) \geq \frac{4\delta}{\pi R_1}$. From this fact, (5.21), and (5.2), we obtain the following result:

$$|g'_k(p)| \leq \frac{R_1 M_{1,k}}{\delta^2} \left(\frac{\varepsilon}{M_{1,k}} \right)^{\frac{4\delta}{\pi R_1}} \quad \text{for } p \in [3\delta, R_1 - 3\delta]. \quad (5.22)$$

6. Obtaining a bound for $P_k(r)$

Our stability result will consist of a bound for $P(r)$ on B_{R_1} . From (4.5), we see that this can be done by bounding $|P_k(r)|$ for each k . We will achieve this in the present section by starting from (4.12) and using our bound for $g'_k(p)$ in conjunction with Nevanlinna's principle. As before, fix $k \geq 0$. Let $p_0 := R_1 - 3\delta$. Note that this is the upper limit of the interval on which we have bounded $g'_k(p)$ in (5.22). In view of this, we split up the integral in (4.12) as

$$f_k(r) = -\frac{1}{\pi} \int_r^{p_0} g'_k(p) \frac{T_k\left(\frac{p}{r}\right)}{\sqrt{p^2 - r^2}} dp + \frac{-1}{\pi} \int_{p_0}^{R_2} g'_k(p) \frac{T_k\left(\frac{p}{r}\right)}{\sqrt{p^2 - r^2}} dp. \quad (6.1)$$

Define the interval $I := (3\delta, p_0 - \delta)$. For $r \in I$, we can use (4.7) to rearrange (6.1) as

$$P_k(r) + \frac{1}{\pi} \int_{p_0}^{R_2} g'_k(p) \frac{T_k\left(\frac{p}{r}\right)}{\sqrt{p^2 - r^2}} dp = -\frac{1}{\pi} \int_r^{p_0} g'_k(p) \frac{T_k\left(\frac{p}{r}\right)}{\sqrt{p^2 - r^2}} dp =: \varphi_k(r). \quad (6.2)$$

Note our definition of the function $\kappa(r)$. From the left-most expression in (6.2), it is clear that $\kappa(r)$ is real-analytic on I .

Define the region $\mathbb{C}$ as the annulus centered at $(R_1 + R_2)/2$, which intersects $\mathbb{R}$ in the intervals I and J , as pictured in Figure 3. As shown, let $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ be regions such that $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 = \mathbb{C}$. Also, define the upper and lower 3/4 annuli as follows: $\Omega_U := \Omega_1 \cup \Omega_2$ and let $\Omega_L := \Omega_3 \cup \Omega_4$. We obtain a bound on $P_k(r)$ in the following three steps (see Figure 3):

1. First, we analytically continue $\kappa(r)$ into the regions Ω_U and Ω_L to obtain the functions $\varphi_k^U(z)$ and $\varphi_k^L(z)$, respectively.
2. Then, we apply Nevanlinna's principle to $\varphi_k^U(z)$ and $\varphi_k^L(z)$ in Ω_U and Ω_L , respectively.
3. Finally, by adding $\varphi_k^U(z)$ and $\varphi_k^L(z)$ and using the cancelation of the integral terms, we recover $P_k(r)$ (see (6.16)). Thus, combining the bounds on $\varphi_k^U(z)$ and $\varphi_k^L(z)$ obtained in (2), we bound $P_k(r)$ on J . We then use Proposition 9.9 to translate this into a bound for $P_k(r)$ on the interval $[0, R_1]$, which proves Theorem 3.2.

Step 1: Analytic Continuation of $\varphi_k(r)$

First, we will analytically continue $\kappa(r)$ from I to Ω_U . From the left-most expression of (6.2), we see that if we are able to define the square root appropriately, then we will be able to find the analytic continuation of $\kappa(r)$ by replacing r with z . To do this, we will verify that when $p \in (p_0, R_2)$ and $z \in \Omega_U$, the argument $p^2 - z^2$ of the square root in (6.2) lies in a domain not containing a loop around the origin. Define the set

$$D_U := \{p^2 - z^2 \mid p \in (p_0, R_2), z \in \Omega_U\}. \quad (6.3)$$

Also, for $c > 0$, define the half strip $H_c := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0, 0 < \operatorname{Im} z < c\}$. It will suffice to prove that we can choose a small enough c such that $D_U \subset H_c$. Now, define $z =: x + iy$. Then, we have that $\operatorname{Im}(p^2 - z^2) = -2xy$. As a consequence, $0 < \operatorname{Im}(p^2 - z^2) < c$ implies that $-c/2 < xy < 0$. The subset H_c of Ω_U that has this property is the shaded region in Figure 4. Note that as c tends to 0, the region H_c approaches the interval J . Also note that the set $\{p^2 - z^2 \mid p \in (p_0, R_2), z \in J\}$ is an interval of the negative real axis. Thus, by the continuity of the mapping $z \mapsto p^2 - z^2$, it is clear that we can take a sufficiently small c so that $\{p^2 - z^2 \mid p \in (p_0, R_2), z \in H_c\}$ is a subset of the left half-plane $\operatorname{Re} z < 0$. It follows that $D_U \cap H_c = \{p^2 - z^2 \mid p \in (p_0, R_2), z \in H_c\} \subset \{z \mid \operatorname{Re} z < 0\} \subset \{z \mid \operatorname{Re} z < 0, \operatorname{Im} z > 0\}$, as needed.

It follows that for $z \in \Omega_U$ and $p \in (p_0, R_2)$, we can define an analytic continuation $(p^2 - z^2)_U^{1/2}$ of the square root function. Hence, the analytic continuation of $\kappa(r)$ from I into Ω_U is

$$\varphi_k^U(z) := P_k(z) - \frac{1}{\pi} \int_{p_0}^{R_2} g'_k(p) \frac{T_k\left(\frac{p}{z}\right)}{(p^2 - z^2)_U^{1/2}} dp. \quad (6.4)$$

Defining $(p^2 - z^2)_L^{1/2}$ analogously, we find that the analytic continuation of $\kappa(r)$ from I into Ω_L is

$$\varphi_k^L(z) := P_k(z) - \frac{1}{\pi} \int_{p_0}^{R_2} g_k'(p) \frac{T_k\left(\frac{p}{z}\right)}{(p^2 - z^2)^{1/2}} dp. \quad (6.5)$$

Step 2: Applying Nevanlina's Principle to $\varphi_k^U(z)$ and $\varphi_k^L(z)$

Now that we have defined analytic functions $\varphi_k^U(z)$ and $\varphi_k^L(z)$ in the regions U and L , respectively, we can apply Nevanlina's principle to both. Let us focus on the case of $\varphi_k^U(z)$. Since we have the bound (5.22), we will use the interval I as our B (in the notation of Theorem 5.1). It is now our task to find bounds for $\varphi_k^U(z)$ on I and on the remainder of the boundary. To do the former, we consider the right equality of (6.2). From that equation, (5.22), and Proposition 9.3 in Appendix A, we find

$$\begin{aligned} |\varphi_k^U(r)| &\leq \frac{1}{\pi} \int_r^{p_0} |g_k'(p)| \frac{|T_k\left(\frac{p}{r}\right)|}{\sqrt{p^2 - r^2}} dp \leq \frac{1}{\pi} \left(\frac{R_1 M_{1,k}}{\delta^2} \left(\frac{\varepsilon}{M_{1,k}} \right)^{\frac{4\delta}{\pi R_1}} \right) \left(2 \frac{p_0}{3\delta} \right)^k \int_r^{p_0} \frac{1}{\sqrt{p^2 - r^2}} dp \\ &= \frac{1}{\pi} \left(\frac{R_1 M_{1,k}^{1 - \frac{4\delta}{\pi R_1}}}{\delta^2} \varepsilon^{\frac{4\delta}{\pi R_1}} \right) \left(2 \frac{p_0}{3\delta} \right)^k \ln \left(\frac{p_0}{r} + \sqrt{\left(\frac{p_0}{r} \right)^2 - 1} \right) \\ &\leq \frac{1}{\pi} \left(\frac{R_1 M_{1,k}^{1 - \frac{4\delta}{\pi R_1}}}{\delta^2} \right) \left(\frac{2R_1}{3\delta} \right)^k \ln \left(\frac{2R_1}{3\delta} \right) \varepsilon^{\frac{4\delta}{\pi R_1}} =: \varepsilon_{2,k}, \quad r \in I_\delta. \end{aligned} \quad (6.6)$$

Next, we will find a bound for $\varphi_k^U(z)$ on the closure $\overline{\Omega_U}$. In a computation very similar to (5.9), we find that for $p \in [p_0, R_2]$ and $z \in \overline{\Omega_U}$,

$$|p^2 - z^2|^{1/2} \geq \sqrt{R_1} \delta. \quad (6.7)$$

Considering the integral in (6.4), we note that it involves $g_k'(p)$ on the interval (p_0, R_2) where we have no bounds for this function. To overcome this difficulty, we integrate by parts:

$$\begin{aligned} \int_{p_0}^{R_2} g_k'(p) \frac{T_k\left(\frac{p}{z}\right)}{(p^2 - z^2)^{1/2}} dp &= \int_{p_0}^{R_2} \frac{T_k\left(\frac{p}{z}\right)}{(p^2 - z^2)^{1/2}} d(g_k(p)) \\ &= \left[\frac{T_k\left(\frac{p}{z}\right)}{(p^2 - z^2)^{1/2}} g_k(p) \right]_{p_0}^{R_2} - \int_{p_0}^{R_2} g_k(p) d \left(\frac{T_k\left(\frac{p}{z}\right)}{(p^2 - z^2)^{1/2}} \right) \\ &= \frac{-T_k\left(\frac{p_0}{z}\right)}{(p_0^2 - z^2)^{1/2}} g_k(p_0) - \int_{p_0}^{R_2} g_k(p) \left(\frac{\frac{1}{2} T_k'\left(\frac{p}{z}\right)}{(p^2 - z^2)^{1/2}} - \frac{p T_k\left(\frac{p}{z}\right)}{(p^2 - z^2)^{3/2}} \right) dp. \end{aligned} \quad (6.8)$$

Note that in the last step above, we used that $g_k(R_2) = 0$. For $z \in \overline{\Omega_U}$,

$$3\delta \leq |z| \leq R_1 + R_2. \quad (6.9)$$

By the first inequality of (6.9), if $p \in [p_0, R_2]$ and $z \in \overline{\Omega_U}$, then

$$\left| \frac{p}{z} \right| \leq \frac{R_2}{3\delta}. \quad (6.10)$$

Also note that for $R \geq 1$, we have $R + \sqrt{R^2 + 1} \leq 3R$. It follows by (6.10) and Propositions 9.2 and 9.4 in Appendix A that for $p \in [p_0, R_2]$ and $z \in \overline{\Omega_U}$,

$$\left| T_k \left(\frac{p}{z} \right) \right| \leq \left(\frac{R_2}{\delta} \right)^k, \quad (6.11)$$

and

$$\left| T'_k \left(\frac{p}{z} \right) \right| \leq \frac{k \left(\frac{R_2}{\delta} \right)^k}{\frac{R_2}{3\delta}} = 3k \left(\frac{R_2}{\delta} \right)^{k-1}. \quad (6.12)$$

Now, we can use the bounds (4.4), (4.10), Proposition 9.9 (with $R = R_1 + R_2$ and $[a, b] = [0, R_1]$), (6.7), (6.9), (6.11), and (6.12), along with equations (6.4) and (6.8) to obtain that for $z \in \overline{\Omega_U}$,

$$\begin{aligned} |\varphi_k^U(z)| &\leq 2C_N M(N+1) \left(\frac{(R_1+R_2)N}{R_1} \right)^N + \frac{1}{\pi} \left| -\frac{T_k \left(\frac{p_0}{z} \right)}{(p_0^2 - z^2)^{1/2}} g_k(p_0) - \int_{p_0}^{R_2} g_k(p) \left(\frac{\frac{1}{z} T'_k \left(\frac{p}{z} \right)}{(p^2 - z^2)^{1/2}} - \frac{p T_k \left(\frac{p}{z} \right)}{(p^2 - z^2)^{3/2}} \right) dp \right| \\ &\leq 2C_N M(N+1) \left(\frac{(R_1+R_2)N}{R_1} \right)^N + \frac{1}{\pi} \frac{|T_k \left(\frac{p_0}{z} \right)|}{|p_0^2 - z^2|^{1/2}} |g_k(p_0)| + \frac{1}{\pi} \int_{p_0}^{R_2} |g_k(p)| \left(\frac{\frac{1}{|z|} |T'_k \left(\frac{p}{z} \right)|}{|p^2 - z^2|^{1/2}} + \frac{p |T_k \left(\frac{p}{z} \right)|}{|p^2 - z^2|^{3/2}} \right) dp \\ &\leq 2C_N M(N+1) \left(\frac{(R_1+R_2)N}{R_1} \right)^N + \frac{1}{\pi} \left(\frac{R_2}{\delta} \right)^k K_s M + \frac{1}{\pi} (R_2 - p_0) K_s M \left(\frac{\frac{1}{3\delta} 3k \left(\frac{R_2}{\delta} \right)^{k-1}}{\sqrt{R_1 \delta}} + \frac{R_2 \left(\frac{R_2}{\delta} \right)^k}{(R_1 \delta)^{3/2}} \right) \\ &= 2C_N M(N+1) \left(\frac{(R_1+R_2)N}{R_1} \right)^N + \frac{K_s M}{\pi \sqrt{R_1 \delta}} \left(\frac{R_2}{\delta} \right)^k \left(1 + (R_2 - p_0) \left(\frac{k}{R_2} + \frac{R_2}{R_1 \delta} \right) \right) \\ &=: M_{2,k} \end{aligned} \quad (6.13)$$

Note that the above expression is undefined for $N=0$ because of the term with N as an exponent. However, as in (5.14), we can interpret that exponential as 1 and the result will hold. From (6.6) and (6.13) we have the necessary components to apply Nevanlinna's principle to $\varphi_k(z)$ in Ω_U . Let ω_2^U be the harmonic function on Ω_U such that $\omega_2^U(p)=1$ for $p \in I$ and $\omega_2^U(z)=0$ for $z \in \Omega_U \setminus I$. Then,

$$|\varphi_k^U(z)| \leq M_{2,k} \left(\frac{\mathcal{E}_{2,k}}{M_{2,k}} \right)^{\omega_2^U(z)} \quad \text{for } z \in \Omega_U. \quad (6.14)$$

By symmetry, if we define ω_2^L on Ω_L as $\omega_2^L(z) := \omega_2^U(\bar{z})$, then

$$|\varphi_k^L(z)| \leq M_{2,k} \left(\frac{\mathcal{E}_{2,k}}{M_{2,k}} \right)^{\omega_2^L(z)} \quad \text{for } z \in \Omega_L. \quad (6.15)$$

Step 3: Obtaining a bound for $|P_k(r)|$

Consider the functions φ_k^U and φ_k^L on the interval J . From (6.4) and (6.5), we see that the crucial difference between the two is the definition of the square root. It is easy to see that for $r \in J$, $(p^2 - r^2)_U^{1/2} = -i\sqrt{r^2 - p^2}$, while $(p^2 - r^2)_L^{1/2} = i\sqrt{r^2 - p^2}$. Thus, the two square roots differ by a sign. Hence, for $r \in J$, we get the following formula:

$$P_k(r) = \frac{1}{2} \left(P_k(r) + \frac{1}{\pi} \int_{p_0}^{R_2} g'_k(p) \frac{T_k\left(\frac{p}{z}\right)}{i\sqrt{r^2-p^2}} dp + P_k(r) - \frac{1}{\pi} \int_{p_0}^{R_2} g'_k(p) \frac{T_k\left(\frac{p}{z}\right)}{i\sqrt{r^2-p^2}} dp \right) \\ = \frac{1}{2} (\varphi_k^U(r) + \varphi_k^L(r)). \quad (6.16)$$

This formula is key because it expresses $P_k(r)$ in terms of φ_k^U and φ_k^L , for which we have the bounds (6.14) and (6.15). Since $\omega_2^U(r) = \omega_2^L(r)$ for $r \in J$, (6.16) implies that

$$|P_k(r)| \leq \frac{1}{2} (|\varphi_k^U| + |\varphi_k^L|) \leq M_{2,k} \left(\frac{\varepsilon_{2,k}}{M_{2,k}} \right)^{\omega_2^U(r)}. \quad (6.17)$$

We seek a uniform bound on $P_k(r)$, and we see from (6.17) that this can be obtained if we uniformly bound $\omega_2^U(r)$ from below. This is not possible on the whole segment J , because $\omega_2^U(r) \rightarrow 0$ as r tends to the endpoints of J . To overcome this problem, we consider a closed interval $J'_\delta \subset J_\delta$, which is centered on J , but shorter on both sides by δ . Then, $\omega_2^U(r)$ attains a minimum on J'_δ because the latter interval is a compact set. Since $0 \leq \omega_2^U(r) \leq 1$ on $\overline{\Omega_U}$, and since the nonconstant harmonic function $\omega_2^U(z)$ cannot attain a minimum at an interior point of $\overline{\Omega_U}$, we find that

$$0 < \omega_2 := \inf_{r \in J'_\delta} \omega_2^U(r). \quad (6.18)$$

Though a lower bound for ω_2 is not readily apparent, please see Appendix C for a discussion of ω_2 . (6.17) and (6.18) imply that

$$|P_k(r)| \leq M_{2,k} \left(\frac{\varepsilon_{2,k}}{M_{2,k}} \right)^{\omega_2}, \quad r \in J'_\delta. \quad (6.19)$$

We have bounded $|P_k(r)|$ on the interval J'_δ , but we seek to find a bound of that quantity on $[0, R_1]$. For this, we may once again apply Proposition 9.9, choosing $[a, b] := J'_\delta$, and letting D be a disk of diameter $R_1 + R_2$ that contains both J'_δ and $[0, R_1]$. By Proposition 9.9 and by (6.19), we find that for $r \in [0, R_1]$,

$$|P_k(r)| \leq M_{2,k} \left(\frac{\varepsilon_{2,k}}{M_{2,k}} \right)^{\omega_2} (N+1) \left(\frac{(R_1+R_2)N}{R_1-9\delta} \right)^N. \quad (6.20)$$

As with (5.14) and (6.13), we can interpret the expression with N in the exponent as 1 when $N=0$. Note that the $R_1-9\delta$ appearing in (6.20) is the length of J'_δ . The requirement that this length be positive led us to assume that $\delta = R_1/18$ ($0, R_1/9$) following Theorem 5.1. Combining (6.20) with (6.6), we find that for $r \in [0, R_1]$,

$$\begin{aligned}
 |P_k(r)| &\leq M_{2,k}^{1-\omega_2} \left(\frac{1}{\pi} \left(\frac{R_1 M_{1,k}^{1-\frac{4\delta}{\pi R_1}}}{\delta^2} \right) \left(\frac{2R_1}{3\delta} \right)^k \ln \left(\frac{2R_1}{3\delta} \right) \varepsilon^{\frac{4\delta}{\pi R_1}} \right)^{\omega_2} (N+1) \left(\frac{(R_1+R_2)N}{R_1-9\delta} \right)^N \\
 &= M_{1,k}^{\left(1-\frac{4\delta}{\pi R_1}\right)\omega_2} M_{2,k}^{1-\omega_2} M_{3,k} \varepsilon^{\frac{4\delta}{\pi R_1}\omega_2},
 \end{aligned} \tag{6.21}$$

where

$$M_{3,k} := \left(\frac{1}{\pi} \left(\frac{R_1}{\delta^2} \right) \left(\frac{2R_1}{3\delta} \right)^k \ln \left(\frac{2R_1}{3\delta} \right) \right)^{\omega_2} (N+1) \left(\frac{(R_1+R_2)N}{R_1-9\delta} \right)^N. \tag{6.22}$$

We have obtained the above estimate of $|P_k(r)|$ for $k \geq 0$. As noted before, this estimate holds for negative k as well, once we replace k by $|k|$. Combining (4.5) and (6.21), we obtain the bound

$$|P(x)| \leq \left(\sum_{k=-N}^N M_{1,|k|}^{\left(1-\frac{4\delta}{\pi R_1}\right)\omega_2} M_{2,|k|}^{1-\omega_2} M_{3,|k|} \right) \varepsilon^{\frac{4\delta}{\pi R_1}\omega_2}, \quad x \in B_{R_1}. \tag{6.23}$$

As our last step, we determine the dependency on M of the above expression. From (5.16), (6.13), and (6.22), we can see that $M_{1,k}$ and $M_{2,k}$ depend linearly on M , while $M_{3,k}$ has no dependency on M . Thus, it follows that the quantity in (6.23) is proportional to $M^{1-\frac{4\delta}{\pi R_1}\omega_2}$. Consequently, we can restate (6.23) as

$$|P(x)| \leq AM \left(\frac{\varepsilon}{M} \right)^{\omega}, \quad x \in B_{R_1}, \tag{6.24}$$

where A depends only on R_1, R_2, s, N and where $\omega := \frac{4\delta}{\pi R_1}\omega_2$. This completes the proof of Theorem 3.2.

7. Extension

As in [CNDK08], we will extend our stability result in Theorem 3.1 to functions that are almost polynomial on their ROI. In other words, we would like to prove Theorem 3.1, but modified in the following way. Instead of f_a being exactly polynomial on the ROI, suppose there is an error function $E \in H_0^s(B_{R_2})$ so that $f_a(x) - E(x) = P_a(x)$ on the ROI and $\|E\|_s = \varepsilon_0$. The idea is to apply Theorem 3.1 to $f_a - E$ instead of f_a . In effect, the introduction of E increases the error in measurement, as quantified by

$$\|h - R(f_a - E)\|_{R_1}^* \leq \|h - Rf_a\|_{R_1}^* + \|RE\|_{R_1}^* \leq \varepsilon + c_s C_s \|E\|_s \leq \varepsilon + c_s C_s \varepsilon_0, \tag{7.1}$$

where we have used (2.3) and (2.7). Moreover, the error function E contributes to a slight increase in the bound on f_a :

$$\|f_a - E\|_s \leq \|f_a\|_s + \|E\|_s \leq M + \varepsilon_0. \tag{7.2}$$

Thus, we may apply Theorem 3.1 with f_a replaced by $f_a - E$, M replaced by $M + \varepsilon_0$ and replaced by $\varepsilon + c_s C_s \varepsilon_0$ to obtain the following extension:

Corollary 7.1

Let $s > 0$ and let $f_a \in H_0^s(B_{R_2})$ with $\|f_a\|_s \leq M$. Suppose there is a function $E \in H_0^s(B_{R_2})$ such that $f_a(x) - E(x)|_{B_{R_1}} = P_a(x)$, where P_a is a polynomial of degree at most N and $\|E\|_s = 0$.

Suppose that the Radon data h measured on Z_{R_1} satisfy $\|h\|_{R_1}^* < \infty$, and that the error in measurement is small: $\|h - Rf_a\|_{R_1}^* < \varepsilon$, for some $\varepsilon > 0$. Let f_r be a reconstruction satisfying the conditions imposed by $f_r \in S_{+c_s C_s 0, M+\varepsilon 0}$. Then there is a constant A , depending only on R_1, R_2, s, N , and an exponent $\omega > 0$, depending only on R_1, R_2 , such that

$$|P_a(x) - P_r(x)| \leq A(M + \varepsilon_0) \left(\frac{\varepsilon + c_s C_s \varepsilon_0}{M + \varepsilon_0} \right)^\omega, \quad x \in B_{R_1}, \quad (7.3)$$

where c_s and C_s are defined in (2.7) and (2.3), respectively.

Corollary 7.1 provides an important step towards proving stability in the case when f_a is piecewise polynomial on the ROI. Suppose that the ROI can be broken up into finitely many regions where f_a is polynomial, and the boundaries of these regions are, for example, smooth curves with non-zero curvature. In theory, local tomography allows us to find the locations of the boundaries and values of jumps of f_a . Using this information, we can generate a piecewise polynomial function $Q(x)$, which has exactly the same jumps as f_a inside the ROI. Subtracting the Radon transform of Q from the data converts our problem to the one with the polynomial attenuation function on the ROI $f_a(x) - Q(x)$. If the jumps (both locations and values) are recovered with some error, the difference between f_a and the estimated Q is an almost polynomial function on the ROI. The latter function can be written in the form: polynomial + error term $E(x)$. As is easily checked, the assumption of non-zero curvature implies that the Fourier transform of $E(x)$ satisfies $\tilde{E}(\xi) = O(|\xi|^{-3/2})$. Thus, if E is suitably extended to all of B_{R_2} , we get $E \in H_0^s(B_{R_2})$ for any $s < 1/2$, and Corollary 7.1 applies for any $s \in (0, 1/2)$. Hence, obtaining an estimate of $E(x)$ in an appropriate Sobolev norm would lead to a proof of stability in the piecewise polynomial case. This is a separate problem which is not addressed in this paper.

8. A Uniqueness Result

On the one hand, [YYJW10] prove the uniqueness of the solution to the interior problem in the case of a piecewise polynomial attenuation function on the ROI. On the other hand, [CNDK08] prove uniqueness in the case of a known subregion. In Theorem 8.1 below, we will prove a more general uniqueness result, which assumes only that the attenuation function is, in a certain sense, analytic on a subregion. From this result, we will be able to recover the uniqueness results of [YYJW10] and [CNDK08], which is discussed in the remarks after the theorem. The reader is referred to [Hor83, KP02] for a nice presentation of the theory of distributions and the theory of real analytic functions.

Theorem 8.1

Let $f \in \mathcal{E}'(\mathbb{R}^2)$ be a compactly supported distribution, and let S be a connected component of the support of f . Suppose there is an open disk $U \subset S$ such that $f|_U = F|_U$, where F is a real analytic function in a neighborhood V of S . If $Rf = 0$ for all lines passing through U , then $f = 0$.

Proof—Let r be the radius of the disk U . We may assume without loss of generality that V is a bounded set, and thus that $\bar{V}$ is compact. Then, S and $\bar{V}$ are nonintersecting compact sets, so let $d > 0$ denote the distance between them. We may also assume without loss of

generality that S is the only connected component of $\text{supp} f$ lying inside V (by shrinking V if necessary). Take $\{K\}_{>0}$ as in (3.2), and define $f := f^* K$. This way, $f_\eta \in C_0^\infty$. Using the fact that the support of K is a ball of radius η , it is easy to check that f inherits the following properties from f . First, we define U to be the slightly smaller disk concentric to U with radius $r - \eta$, and define the slightly contracted set $V := \{x \in V : \text{dist}(x, \partial V) > \eta\}$. It follows that $f|_U = F|_U$, where $F := F^* K$ is real-analytic on V . Also,

$$Rf_\eta = 0 \text{ on } U_\eta, \quad (8.1)$$

because $R(f^* K) = Rf \otimes RK$ ([RK96], eq. (2.1.18)), where $\otimes$ denotes the one-dimensional convolution with respect to p .

Let us now examine the support of f . The support expands by η , and in particular, the connected component S expands to become $S := \{x \in \mathbb{R}^2 : \text{dist}(x, S) \leq \eta\}$. We choose η small enough that U is still a disk of positive radius ($\eta < r$) and the inclusion $S \subset V$ still holds ($2\eta < d$). We assumed at the beginning of the proof that the other connected components of $\text{supp} f$ lay outside of V . After our convolution with K , these connected components expanded by η , while V contracted by η , so they still lie outside of V . More precisely,

$$(\text{supp } f_\eta) \cap V_\eta = S_\eta. \quad (8.2)$$

Since V is V contracted by η , and S is S expanded by η , we have

$$\text{dist}(S_\eta, \partial V_\eta) = d - 2\eta > 0. \quad (8.3)$$

Let L be an arbitrary line through the center of U (see Figure 5). Consider the restriction f_η^1 of f to L . Let a_2 and a_3 be the coordinates on L of the endpoints of $L \cap U$. Let $(a_0, a_5) \subset L$ be the connected component (i.e., interval) of $L \cap V$ that contains $L \cap U$. Also, let $a_1 := a_0 + (d - 2\eta)$, and $a_4 := a_5 - (d - 2\eta)$ (see Figure 5). By (8.2) and (8.3),

$$f_\eta^1 = 0 \text{ on } (a_0, a_1) \text{ and on } (a_4, a_5). \quad (8.4)$$

Choose a system of coordinates for $\mathbb{R}^2$ such that the x -axis coincides with L . Then, by (8.1) and the differentiated backprojection formula [GG91, ZP04, PNC05, YYW11],

$$Hf_\eta^1(y) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \left[\frac{\partial}{\partial s} Rf_\eta(s, \alpha) \right]_{s=y \cos \alpha} d\alpha = 0, \quad y \in (a_2, a_3), \quad (8.5)$$

where Hf_η^1 is the Hilbert transform of f_η^1 . Let $(a_0, a_5) \subset L$ be the connected component (i.e., interval) of $L \cap V$ that contains $L \cap U$ (see Figure 5). Let $a_1 := a_0 + (d - 2\eta)$, and $a_4 := a_5 - (d - 2\eta)$. By (8.2) and (8.3),

$$f_\eta^1 = 0 \text{ on } (a_0, a_1) \text{ and on } (a_4, a_5). \quad (8.6)$$

Moreover, because the restriction of a real-analytic function to a line is also real-analytic, we find that

$$f_\eta^1|_{(a_2, a_3)} = F_\eta^1|_{(a_2, a_3)}, \quad (8.7)$$

where $F_\eta^1 = F_\eta^1|_L$ is real-analytic on (a_0, a_5) . By this real-analyticity, we can take a disk in the complex plane about each point of (a_0, a_5) in which the power series of F_η^1 converges. If we let $D \subset \mathbb{C}$ be the union of these disks, then it is an open set on which F_η^1 is complex-analytic, with $(a_0, a_5) \subset D$. Moreover, we can construct regions $U, L \subset D$ very similar to those discussed in Section 6 (see Figure 6).

By (8.5), (8.6), and the definition of the Hilbert transform, we have

$$0 = H f_\eta^1(y) = \frac{1}{\pi} \left(\int_{-\infty}^{a_0} + \int_{a_1}^{a_2} + \int_{a_2}^{a_3} + \int_{a_3}^{a_4} + \int_{a_5}^{\infty} \right) \frac{f_\eta^1(x)}{x-y} dx, \quad y \in (a_2, a_3). \quad (8.8)$$

Note that in (8.8), each of the integrals except for the middle one may be analytically continued into $U \cup L$, simply replacing y by z . Thus, we may group those four integrals into one function $A(z)$, which is analytic in $U \cup L$:

$$0 = A(y) + \int_{a_2}^{a_3} \frac{F_\eta^1(x)}{x-y} dx, \quad y \in (a_2, a_3). \quad (8.9)$$

Note that we have used (8.7) to replace f_η^1 by F_η^1 in the above equation. By Cauchy's integral formula and partial fractions, if we choose a closed contour D containing (a_2, a_3) , then we can rewrite the integral in (8.9) as

$$\begin{aligned} \int_{a_2}^{a_3} \frac{F_\eta^1(x)}{x-y} dx &= \int_{a_2}^{a_3} \frac{F_\eta^1(x) - F_\eta^1(y)}{x-y} dx + \int_{a_2}^{a_3} \frac{F_\eta^1(y)}{x-y} dx \\ &= \left(\int_{a_2}^{a_3} \frac{1}{2\pi i} \int_{\Gamma} \frac{F_\eta^1(\zeta)}{(\zeta-x)(\zeta-y)} d\zeta dx \right) + F_\eta^1(y) (\ln(a_3-y) - \ln(y-a_2)). \end{aligned} \quad (8.10)$$

It is clear that in (8.10), the double integral can be analytically continued into $U \cup L$, again replacing y by z . Also, D does not contain a loop around the branch point a_2 of $\ln(y-a_2)$, and so this logarithm can be analytically continued into $U \cup L$ as well. From these facts, we may absorb the two terms just discussed into $A(y)$ in (8.9):

$$0 = A(y) + F_\eta^1(y) \ln(a_3-y) \quad (8.11)$$

for $y \in (a_2, a_3)$, where $A(z)$ is analytic in $U \cup L$. Now, U and L do not contain closed loops around a_3 , so the logarithm $\ln(a_3-y)$ may be analytically continued into these two regions. The analytic continuation of $\ln(a_3-y)$ into U is $\ln|a_3-z| + i \arg_U(a_3-z)$, where $\arg_U \in (-2\pi, 0]$, while the analytic continuation of $\ln(z-a_3)$ into L is $\ln|a_3-z| + i \arg_L(a_3-z)$, where $\arg_L \in [0, 2\pi)$. Consider these two analytic continuations on the segment (a_4, a_5) . From (8.11), we find that

$$0 = A(y) + F_\eta^1(y) (\ln(y-a_3) + i \arg_U(a_3-y)) = A(y) + F_\eta^1(y) (\ln(y-a_3) - i\pi) \quad (8.12)$$

and

$$0 = A(y) + F_{\eta}^1(y)(\ln(y - a_3) + i \arg_L(a_3 - y)) = A(y) + F_{\eta}^1(y)(\ln(y - a_3) - i\pi) \quad (8.13)$$

for $y \in (a_4, a_5)$. If we subtract (8.12) from (8.13), then we obtain that

$$F_{\eta}^1(y) = 0 \text{ for } y \in \Omega \cap (a_4, a_5). \quad (8.14)$$

Because F_{η}^1 is analytic in D , we may conclude from (8.14) that $F_{\eta}^1 \equiv 0$ on all of D . In particular, it follows that $f_{\eta}^1(x) = F_{\eta}^1(x) = 0$ for $x \in (a_2, a_3)$. Now, since $f_{\eta}^1(x) = H f_{\eta}^1(x) = 0$ on (a_2, a_3) , it follows by the Plemelj-Sokhotski formula that

$$f_{\eta}^1 \equiv 0 \quad (8.15)$$

(see Lemma 2.1 of [CNDK08] for a proof). Since f_{η}^1 is the restriction of f to an arbitrary line L through the center of U , we may conclude from (8.15) that $f \equiv 0$. Finally, since f as a distribution is 0 in the sense of distributions, and $f \equiv 0$ for all sufficiently small $\epsilon > 0$, we conclude that $f \equiv 0$.

Remark 8.1—Let us compare Theorem 8.1 to the piecewise polynomial uniqueness result in [YYJW10]. This result states that a function, piecewise polynomial on its ROI, is determined by its Radon transform on an ROI. First, we should note that at the core of both results is the property of logarithms that allows one to isolate a function from its Hilbert transform. We used this property in arriving at (8.12) and (8.13), which led to the crucial equation (8.14). The same principle is used in Lemma 1 of [YYJW10]. However, in our paper, we apply this property of logarithms in a simpler, and consequently more general framework (see the manipulation in (8.10)), which allowed us to replace the piecewise polynomial assumption with the weaker analytic subregion assumption. In fact, it is easy to see that by taking a disk in any one of the polynomial components of the ROI, the uniqueness result of [YYJW10] follows immediately from Theorem 8.1, because polynomials are entire. Moreover, whereas [YYJW10] assumes that the attenuation function is piecewise smooth outside the ROI, our uniqueness result has only minimal restrictions on the function outside the ROI.

Remark 8.2—Let us also compare Theorem 8.1 to the known subregion uniqueness result in [CNDK08]. This result states that a function is uniquely determined by its Radon transform on a subregion, where the function's value is also exactly known. This result is also an immediate corollary of our theorem. Indeed, let f be the unknown attenuation function. Suppose there is a subregion U on which we know Rf , and that we also know a distribution $F \in \mathcal{E}'(\mathbb{R}^2)$ such that $f|_U = F|_U$. Consider the function $f - F$. We know its Radon transform on U , and have $f - F = 0$ on U . By Theorem 8.1, since 0 is an entire function, $f - F$ is uniquely determined, and thus so is the original function f , thereby proving the uniqueness result in [CNDK08].

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9. Appendices

Here, we obtain some useful auxiliary results and bounds. Some of these are well-known, but they are stated here for convenience. We will adopt the convention of writing $\log z$ for complex arguments z and $\ln x$ for real arguments x .

Appendix A. Bounds on Chebyshev Polynomials

The following are well-known facts about the Chebyshev polynomials T_k :

$$T_k(\cos\theta) = \cos(k\theta), \quad (9.1)$$

([AS70], eq. 22.3.15). Note that the classical formula (9.1) holds for all complex z by analytic continuation. It follows immediately from (9.1) that

$$|T_k(x)| \leq 1 \text{ for } -1 \leq x \leq 1. \quad (9.2)$$

Also,

$$T_k(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m c_{k,m} x^{k-2m}, \quad (9.3)$$

where the $c_{k,m}$ are positive integers ([AS70], eq. 22.3.6).

Lemma 9.1

Let $z \in \mathbb{C}$. Then, $|\cos z| \leq e^{|\operatorname{Im} z|}$.

Proof

This statement is very easy to check.

Throughout this Appendix, it will suffice for our purposes to assume $k \geq 0$.

Proposition 9.2

Let $R > 0$. For $z \in \mathbb{C}$ such that $|z| \leq R$, we have $|T_k(z)| \leq (R + \sqrt{R^2 + 1})^k$.

Proof

We see from (9.3) that the coefficients of $T_k(x)$ are real and alternate in sign. Moreover, each $T_k(x)$ has either only odd or only even powers of x . From this, it follows that if we let

$$T_k(x) = \sum_{m=0}^k c_m x^m, \text{ then for } |z| \leq R,$$

$$|T_k(z)| \leq \sum_{m=0}^k |c_m| |z|^m \leq \sum_{m=0}^k |c_m| R^m = |T_k(Ri)|. \quad (9.4)$$

Now, one easily checks that $\cos(i \log(i(R + \sqrt{R^2 + 1}))) = Ri$. It follows by (9.1) and Lemma 9.1 that

$$|T_k(Ri)| = \left| \cos(ki \log(i(R + \sqrt{R^2 + 1}))) \right| \leq e^{k \ln(R + \sqrt{R^2 + 1})} = (R + \sqrt{R^2 + 1})^k, \quad (9.5)$$

as needed.

For real x , we can obtain a slightly better bound on $T_k(x)$:

Proposition 9.3

Let $R \geq 1$. Then, for $|x| \leq R$, we have the estimate $|T_k(x)| \leq (2R)^k$.

Proof

Consider first the case when $x \geq 1$. As is well known,

$$\cos^{-1}(z) = -i \log(z + \sqrt{z^2 - 1}), \quad (9.6)$$

where the square root is two-valued. Now, (9.1) implies that $T_k(x) = \cos(k \cos^{-1} x)$. We have $\cos^{-1}(x) = \pm i \ln(x + \sqrt{x^2 - 1}) =: \pm ic$, where c is real. Note that $|c| \leq \ln(2x)$. We use this and Lemma 9.1 to obtain

$$|T_k(x)| = |\cos(k \cdot ic)| \leq e^{|kc|} \leq e^{k \ln(2x)} = (2x)^k \leq (2R)^k. \quad (9.7)$$

The above result also holds for $|x| \leq 1$ because in that case, $|T_k(x)| \leq 1$, and it also holds for $x \leq -1$ because each $T_k(x)$ is either even or odd.

Proposition 9.4

Let $R > 0$. For $z \in \mathbb{C}$ such that $|z| \leq R$, we have

$$|T'_k(z)| \leq \frac{k(R + \sqrt{R^2 + 1})^k}{R}. \quad (9.8)$$

Proof

Note that the coefficients of $T'_k(x)$ also have the property used in the beginning of the proof of Proposition 9.2. Thus, in a calculation similar to (9.4), we obtain

$$|T'_k(z)| \leq |T'_k(Ri)| \text{ for } |z| \leq R. \quad (9.9)$$

To calculate $T'_k(Ri)$, we use (9.1) to obtain $T_k(z) = \cos(k \cos^{-1}(z))$. $\cos^{-1}(z)$ is an analytic function in $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$, and thus so is $T_k(z)$. Hence, in this region, we may take the derivative to obtain

$$T'_k(z) = \frac{k \sin(k \cos^{-1}(z))}{\sqrt{1 - z^2}}. \quad (9.10)$$

In the proof of Proposition 9.2, we saw that $\cos^{-1}(Ri) = i \log(i(R + \sqrt{R^2 + 1}))$. Noting that Lemma 9.1 holds for $\sin z$ instead of $\cos z$, it follows that

$$|\sin(k \cos^{-1}(Ri))| \leq (R + \sqrt{R^2 + 1})^k. \quad (9.11)$$

Combining (9.9), (9.10), and (9.11), we get that for $|z| \leq R$,

$$|T'_k(z)| \leq |T'_k(Ri)| = \frac{|k \sin(k \cos^{-1}(Ri))|}{\sqrt{1 - (Ri)^2}} \leq \frac{k(R + \sqrt{R^2 + 1})^k}{\sqrt{1 + R^2}} \leq \frac{k(R + \sqrt{R^2 + 1})^k}{R}, \quad (9.12)$$

as desired.

Appendix B: Nevanlinna's Exponent for D^+

Proposition 9.5

Let $D^+ \subset \mathbb{C}$ be the upper semi-disk with diameter $(-R, R)$. Then, Nevanlinna's exponent $\omega_1(z)$ that satisfies $\omega_1(z) = 1$ for $z \in (-R, R)$ and $\omega_1(z) = 0$ for $z \in D^+ \setminus (-R, R)$ has the form

$$\omega_1(z) = \omega_H(z_3(z_2(z_1(z))))), \quad (9.13)$$

where $z_3(z) := z^2$, $z_2(z) := \frac{1+z}{1-z}$, $z_1(z) := \frac{2z-R}{R_1-2\delta}$, and $\omega_H(z) := 1 - \frac{1}{\pi} \text{Arg } z$.

Proof

In a calculation very similar to that in [CNDK08], we map our region to the upper half-plane by a series $z_3 \circ z_2 \circ z_1$ of conformal maps. We will see that the problem reduces to finding a harmonic function $\omega_H(z)$ in the upper half plane that is 1 on $(0, \infty)$ and 0 on $(-\infty, 0)$. It is easy to check that $\omega_H(z) := 1 - \frac{1}{\pi} \text{Arg } z$ satisfies these requirements. Now, consider Figure 7 below. We see from the definition of z_1 that it is a shift composed with a dilation that sends D^+ to the upper half of the unit disk. Now, we proceed as do [CNDK08]. We compose z_1 with the linear fractional transformation z_2 , which maps the upper semi-disk to the first quadrant. Finally, z_3 sends the quarter-plane to a half-plane. It is easy to verify that the images of $(-R, R)$ and $D^+ \setminus (-R, R)$ are as drawn in the diagram. Hence, $z_3 \circ z_2 \circ z_1$ indeed maps $(-R, R)$ to $(0, \infty)$ and $D^+ \setminus (-R, R)$ to $(-\infty, 0)$. Thus we see that ω_H satisfies the correct boundary conditions, and so (9.13) is the correct expression for Nevanlinna's exponent.

Ultimately, the purpose of finding $\omega_1(z)$ was to calculate $\inf_{z \in C} \omega_1(z)$. This is what we turn to in the following proposition.

Proposition 9.6

Nevanlinna's exponent in (9.13) satisfies the following inequality:

$$\inf_{z \in C_\delta} \omega_1(z) \geq \frac{4\delta}{\pi R_1} \quad (9.14)$$

Proof

Let S' be the upper half of C (See Figure 1). It is clear that z_1 maps S' to a semicircle S'_δ in the upper half-plane centered at 0 with radius $\frac{R_1-4\delta}{R_1-2\delta}$. Then, from writing the map z_2 as $z_2(z) = -1 + \frac{2}{1-z}$, we can see that it sends S'_δ to another semicircle S''_δ in the upper half-plane (see Figure 8). To find the location of S''_δ , we calculate endpoints of its diameter, which are

$$z_2\left(\frac{R_1-4\delta}{R_1-2\delta}\right) = \frac{R_1-3\delta}{\delta} \text{ and } z_2\left(-\frac{R_1-4\delta}{R_1-2\delta}\right) = \frac{\delta}{R_1-3\delta}. \quad (9.15)$$

Thus, we find that the radius of S''_δ is

$$r_\delta := \frac{1}{2} \left(\frac{R_1-3\delta}{\delta} - \frac{\delta}{R_1-3\delta} \right) = \frac{R_1^2-6R_1\delta+8\delta^2}{2\delta(R_1-3\delta)} \quad (9.16)$$

and that the x -coordinate of the center is

$$x_\delta := \frac{1}{2} \left(\frac{R_1-3\delta}{\delta} + \frac{\delta}{R_1-3\delta} \right) = \frac{R_1^2-6R_1\delta+10\delta^2}{2\delta(R_1-3\delta)} \quad (9.17)$$

Now, using (9.13), (9.16), (9.17), Figure 8, and the symmetry of γ_1 with respect to the x -axis, we have

$$\begin{aligned} \inf_{z \in C_\delta} \omega_1(z) &= \inf_{z \in S_\delta} \omega_1(z) = \inf_{z \in S_\delta} \omega_H(z_3(z_2(z_1(z)))) = \inf_{z \in S''_\delta} \omega_H(z_3(z)) = \inf_{z \in S''_\delta} \left[1 - \frac{1}{\pi} \text{Arg}(z^2) \right] \\ &= \inf_{z \in S''_\delta} \left[1 - \frac{2}{\pi} \text{Arg}(z) \right] = 1 - \frac{2}{\pi} \sup_{z \in S''_\delta} \text{Arg}(z) = 1 - \frac{2}{\pi} \sin^{-1} \left(\frac{R_1^2-6R_1\delta+8\delta^2}{R_1^2-6R_1\delta+10\delta^2} \right) \end{aligned} \quad (9.18)$$

To better understand the above expression, we expand it in a power series. By ([AS70], eq. 4.4.41), we have

$$1 - \frac{2}{\pi} \sin^{-1}(1-x) = 1 - \frac{2}{\pi} \left[\frac{\pi}{2} - \sqrt{2x} \left(1 + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^{2k}(2k+1)k!} x^k \right) \right] \geq \frac{2}{\pi} \sqrt{2x}, \quad (9.19)$$

for $x \geq 0$. Thus, we have

$$\inf_{z \in C_\delta} \omega_1(z) = 1 - \frac{2}{\pi} \sin^{-1} \left(1 - \frac{2\delta^2}{R_1^2-6R_1\delta+10\delta^2} \right) \geq \frac{2}{\pi} \sqrt{\frac{4\delta^2}{R_1^2-6R_1\delta+10\delta^2}} \geq \frac{2}{\pi} \sqrt{\frac{4\delta^2}{R_1^2}} = \frac{4\delta}{\pi R_1}, \quad (9.20)$$

as desired.

Appendix C: Nevanlinna's Exponent for ΩU

In (6.18), we defined $\omega_2 := \inf_{t \in J_\delta^U} \omega_2^U(t)$. Here, we will discuss possible lower bounds on ω_2 . Recall that ω_2^U was defined in the region Ω^U (see Figure 3). It will be convenient to change

to polar coordinates, and to place the origin at the center of the annular region U . For this, let A be the annular region $\{(r, \theta)/r_1 \leq r \leq r_2, 0 \leq \theta < 2\pi\}$, with $r_1 < r_2$. Let (r, θ) be the harmonic function in A that satisfies $(r, 0) = 1$, $(r, 3\pi/2) = 0$ for $r_1 \leq r \leq r_2$, and $(r_1, \theta) = (r_2, \theta) = 0$ for $0 \leq \theta < 3\pi/2$. Thus, A corresponds to U , ∂U corresponds to ω_2^U , and the interval J'_δ corresponds to the interval with endpoints $(r_1 + \delta, 0)$ and $(r_2 - \delta, 3\pi/2)$.

Having defined ω , we can apply a standard separation of variables approach to arrive at the following series expansion:

$$\omega(r, \theta) = \frac{4}{\pi} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\sinh\left(\frac{\frac{3\pi}{2} - \theta}{\ln \frac{r_2}{r_1}} \pi n\right)}{n \sinh\left(\frac{3\pi}{2 \ln \frac{r_2}{r_1}} \pi n\right)} \sin\left(\frac{\ln \frac{r}{r_1}}{\ln \frac{r_2}{r_1}} \pi n\right) \quad (9.21)$$

We are interested in the value of the above expression when $\theta = 3\pi/2$. It will be convenient to denote $c := \frac{\pi^2}{2 \ln(r_2/r_1)}$ and $x := \pi \frac{\ln(r/r_1)}{\ln(r_2/r_1)}$. With these notations, we introduce the one-dimensional function $\varphi_c(x)$:

$$\begin{aligned} \varphi_c(x) &:= \omega(r, \pi) = \frac{4}{\pi} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\sinh(cn)}{n \sinh(3cn)} \sin(xn) \\ &= \frac{4}{\pi} \sum_{\substack{n \geq 1 \\ n \text{ odd}}} \frac{\sin(xn)}{n(2 \cosh(2cn) + 1)}. \end{aligned} \quad (9.22)$$

We claimed right before (3.10) that ω depends continuously on r ; we are now ready to substantiate that claim. Up to a constant that depends continuously on r , ω equals to the infimum of (9.22) over an interval that also depends continuously on r . Moreover, c and x depend continuously on r . The series in (9.22) is convergent absolutely and uniformly in $x \in \mathbb{R}$ and $c \in [a, b]$ for any $0 < a < b$. Hence it is obvious that ω indeed depends continuously on r .

As r ranges through $[r_1, r_2]$, x ranges through $[0, \pi]$. As we expect, we can see from (9.22) that $\varphi_c(0) = \varphi_c(\pi) = 0$. Now, the interval in the x -variable over which we seek to minimize $\varphi_c(x)$ is the one corresponding to $r \in [r_1 + \delta, r_2 - \delta]$, which we will denote by $[a, b] \subset [0, \pi]$.

The task of finding an explicit lower bound for $\varphi_c(x)$, as written in (9.22), is a difficult one. However, we will discuss next an indirect expression for such a lower bound. Although this fact is difficult to deduce analytically for small c , it is clear from Figure 9 that for each value of c , the function $\varphi_c(x)$ is concave down for $x \in [0, \pi]$. (For large c , this fact can be shown by taking two derivatives in x in (9.22), and then summing the series by parts). It follows that $\varphi_c(x)$ attains its minimum on $[a, b]$ on the boundary; i.e., at a or b . Thus, we can obtain an expression, though not a straightforward one, for ω_2 :

$$\omega_2 := \inf_{t \in J'_\delta} \omega_2^U(t) = \inf_{x \in [a, b]} \varphi_c(x) = \min(\varphi_c(a), \varphi_c(b)). \quad (9.23)$$

Note that the constant c depends on the size of the ROI relative to that of the whole object. A larger ROI corresponds to a smaller c , which as we see from Figure 9, leads to a better

(larger) ω_2 . In the limiting case, as we can see from Figure 9, we have $\omega_2(x) \rightarrow 1/3$ as $c \rightarrow 0^+$ for all $x \in (0, 1)$. Hence, for any fixed $[a, b] \subset (0, 1)$, ω_2 gets close to $1/3$ for small c .

It is also useful to consider the case $N=0$, which was discussed in the remark following the proof of Theorem 3.2. As discussed in that remark, the case $N=0$ allows us to define ω_2 as the *supremum* of ω_2^U instead of the infimum. This gives us much flexibility, and allows us to take $x = \pi/2$ to find a lower bound for the supremum. Then, the series becomes alternating, and the sum of the first two terms gives a lower bound. After a bit of simplification, the following bound can be attained:

$$\omega_2 \geq \varphi_c(\pi/2) \geq \frac{4}{\pi} \left(e^{-2c} - e^{-4c} - \frac{1}{3}e^{-6c} + \frac{1}{3}e^{-12c} \right). \quad (9.24)$$

However, as stated, this result is valid only for the case $N=0$.

Appendix D: Bounds on polynomials

Here, we will prove two bounds on polynomials. In both cases, we will use the Lagrange interpolation formula, which we state in the following theorem:

Theorem 9.7

Let $P(x)$ be a polynomial of degree at most N , and let $c_0, c_1, \dots, c_n$ be distinct real numbers. Then, we have

$$P(x) = \sum_{i=0}^N P(c_i) \frac{(x-c_0) \dots (\widehat{x-c_i}) \dots (x-c_N)}{(c_i-c_0) \dots (\widehat{c_i-c_i}) \dots (c_i-c_N)}, \quad (9.25)$$

where the hats denote omitted terms.

Proposition 9.8

Let $P(x) = a_N x^N + \dots + a_1 x + a_0$ be a polynomial such that $|P(x)| \leq C$ for $x \in [0, R_1]$. Then, we have the following bound on each a_j :

$$|a_j| R_1^j \leq C(N+1)(2N)^N \quad (9.26)$$

Proof

Let $0 = c_0 < \dots < c_N = R_1$, where the c_i are equally spaced. From Theorem 9.7, we find that

$$a_j = \sum_{i=0}^N P(c_i) \frac{(-1)^{N-j} q_{i,j}(c_0, \dots, c_N)}{(c_i-c_0) \dots (\widehat{c_i-c_i}) \dots (c_i-c_N)}, \quad (9.27)$$

where $q_{i,j}$ is the elementary symmetric polynomial in $c_0, \dots, \widehat{c_i}, \dots, c_N$ of degree $N-j$. Note

that this polynomial has $\binom{N}{j}$ terms. From (9.27), we find that

$$|a_j| \leq \sum_{i=0}^N C \frac{\binom{N}{j} R_1^{N-j}}{(R_1/N)^N} \leq (N+1)C \frac{2^N N^N}{R_1^j}, \quad (9.28)$$

from which the conclusion follows.

Proposition 9.9

Let $P(x) = a_N x^N + \dots + a_1 x + a_0$ be a polynomial such that $|P(x)| \leq C$ for $x \in [a, b]$. Also, let $D \subset \mathbb{C}$ be a disk such that its diameter is an interval of the real line, has length R , and contains $[a, b]$. Then, we have

$$|P(z)| \leq C(N+1) \left(\frac{RN}{b-a} \right)^N, \quad z \in \overline{D}. \quad (9.29)$$

Proof

Taking $a = c_0 < \dots < c_N = b$, where the c_i are equally spaced, we use the Lagrange interpolation formula to obtain that for $z \in D$,

$$|P(z)| \leq \sum_{i=0}^N |P(c_i)| \frac{|z-c_0| \dots |z-c_{i-1}| |z-c_{i+1}| \dots |z-c_N|}{|c_i-c_0| \dots |c_i-c_{i-1}| |c_i-c_{i+1}| \dots |c_i-c_N|} \leq \sum_{i=0}^N C \frac{R^N}{((b-a)/N)^N} = C(N+1) \left(\frac{RN}{b-a} \right)^N, \quad (9.30)$$

as needed.

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Figure 1.

The setup for finding a bound for $g'_k(p)$.

Figure 2.
Contour of integration .

Figure 3.
Analytic continuations of $k(r)$ from the interval I .

Figure 4.

Figure 5.

Figure 6.

Figure 7.

Conformal mappings taking initial semi-disk D^+ to upper half-plane. The boldface lines indicate the images of $(\delta, R_1 - \delta)$, while the dotted lines indicate the images of $D^+ \setminus (\delta, R_1 - \delta)$.

Figure 8.
 S''_{δ} and the maximum argument on the semicircle.

Figure 9.
A plot of $\phi_c(x)$ for three different values of c