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Adaptive Kaczmarz Method for Image Reconstruction in Electrical Impedance Tomography

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Abstract

We present an adaptive Kaczmarz method for solving the inverse problem in electrical impedance tomography and determining the conductivity distribution inside an object from electrical measurements made on the surface. To best characterize an unknown conductivity distribution and avoid inverting the Jacobian-related term $J^T J$ which could be expensive in terms of computation cost and memory in large scale problems, we propose solving the inverse problem by applying the optimal current patterns for distinguishing the actual conductivity from the conductivity estimate between each iteration of the block Kaczmarz algorithm. With a novel subset scheme, the memory-efficient reconstruction algorithm which appropriately combines the optimal current pattern generation with the Kaczmarz method can produce more accurate and stable solutions adaptively as compared to traditional Kaczmarz and Gauss-Newton type methods. Choices of initial current pattern estimates are discussed in the paper. Several reconstruction image metrics are used to quantitatively evaluate the performance of the simulation results.

Keywords

electrical impedance tomography; image reconstruction; optimal current pattern

1. Introduction

Electrical impedance tomography (EIT) is an imaging modality that determines the internal conductivity distribution based on the measurements made on an object's surface. It is a non-invasive imaging technique with non-ionizing radiation, low cost instrumentation and easy portability (Saulnier 2001). These advantages over other medical imaging techniques make it favorable for clinical applications such as monitoring lung and heart functions in the thorax, and detecting breast cancer (Holder 2005). In EIT, a set of current patterns is applied to electrodes placed on the surface of the object, and the resulting electrode voltages are measured. The reconstruction problem is an ill-posed inverse problem, therefore, in order to obtain an accurate internal conductivity distribution, appropriate reconstruction methods and current patterns have to be chosen.

Different reconstruction methods proposed to solve the non-linear least-squares problem in EIT inversion can be categorized as direct methods and iterative methods (Holder 2005, Mueller and Siltanen 2012). A direct method, the NOSER algorithm (Cheney *et al* 1990), uses only one step of the minimization to solve the least-squares problem in EIT, and it is considered to be a fast but less accurate algorithm. In contrast, iterative methods tend to

search for more accurate solutions by iteration. However, these methods, such as Gauss Newton (Yorkey *et al* 1987), can be expensive in terms of computation and memory storage in large scale problems (Vauhkonen 2004, Horesh 2005).

The choice of current patterns will have a great impact on a system's distinguishability. Distinguishability, and the optimal current patterns needed to achieve a maximum of this distinguishability, was introduced by Isaacson in 1986 to provide a quantitative measure of the ability of a system to distinguish between two conditions based on the measurements. A system with limited measurement precision that uses current patterns with high distinguishability, called optimal current patterns in the previous and current papers, is more likely to detect small conductivity changes in an object. A procedure to generate a complete set of optimal current patterns using the Adaptive Current Tomography (ACT) system is given in Isaacson *et al* in 1996. Note that different notions of optimal current patterns have been introduced such as a statistical criterion to minimize the total variance of the estimation for the resistance or conductance matrix (Demidenko *et al* 2005). We use the definitions and notions of optimal current patterns designed to maximize the distinguishability as introduced in the 1986 work of Isaacson in what follows.

The objective of the present work is to design an efficient reconstruction algorithm which can adaptively improve the accuracy of the reconstructed images using optimal current patterns. The Kaczmarz method, also known as the Algebraic Reconstruction Technique (ART), is an efficient iterative algorithm to solve the non-linear least-squares problem by recursively solving each row of the system (Kaczmarz 1937, Gordon 1970), and it is widely used in applications ranging from signal processing to medical imaging (Herman 2009, Byrne 2009). Similarly, the generalized block Kaczmarz method partitions the system into subsets and recursively solves the subsets during iterations. In order to best characterize an unknown conductivity in a cost-efficient manner, we introduce an adaptive Kaczmarz method, which solves the inverse problem by applying the optimal current patterns for distinguishing the actual conductivity from the conductivity estimate between each iteration of the block Kaczmarz algorithm. With an ordered subset scheme, the novel reconstruction method appropriately combines the optimal current pattern generation with the block Kaczmarz method and produces accurate solutions without inverting large dimensional matrices during the computation.

The idea of the ordered subset is also applied to the field of transmission tomography's maximum likelihood problem and is shown to be an improvement over the original algorithm (Erdogan *et al* 1999).

The paper is organized as follows: first we review both the Kaczmarz method and the procedure to generate the optimal current patterns in EIT. Next, an ordered subset scheme is proposed and the adaptive Kaczmarz method is introduced. Simulations are conducted for a two-dimensional (2D) unit radius circular tank with different targets, including lung and heart phantoms. Reconstructed images are evaluated quantitatively using several image quality metrics. The adaptive Kaczmarz method is then tested on experimental data collected on a 2D phantom. We show that the adaptive Kaczmarz method is able to produce stable and accurate solutions cost-efficiently as compared to other methods.

2. Method

In this paper the inverse problem of EIT is viewed as a least-squares problem. Specifically, the goal is to minimize the objective functional in the least-square sense:

$$E(\rho) = \|V - U(\rho)\|_2^2 \quad (1)$$

where $\|\cdot\|_2$ denotes the L_2 norm, ρ is the piecewise constant conductivity distribution, $U(\rho)$ are the voltages that are predicted by a forward model given that the conductivity distribution in the body is ρ , and V are the measured voltages on the electrodes.

2.1. Kaczmarz Method

The Kaczmarz method is an iterative algorithm to solve the least-squares problem (1) by recursively solving each row (a single current excitation pattern in the EIT case) of the system (Natterer *et al* 2001, Vauhkonen 2004). In the EIT application, an iteration is defined to be a single execution to pass through all applied current patterns and the corresponding voltage measurements. Further iterations are followed with the initial solution taken to be the one provided by the previous iteration. At the k th iteration of the i th current excitation, the algorithm projects the solution $\rho_{k,i-1}$ obtained from the $(i-1)$ th excitation onto the solution space of $\langle j_{k,i}, \rho_{k,i-1} \rangle = V_i$ that is

$$\rho_{k,i} = \rho_{k,i-1} + a_{k,i} j_{k,i}^T (j_{k,i} j_{k,i}^T + \lambda_{k,i} I)^{-1} [V_i - U_i(\rho_{k,i-1})], \quad (2)$$

where $i = 1, 2, \dots, L-1$ is the current index, L is the total number of electrodes, $a_{k,i}$ is the step length at iteration k for current i , $j_{k,i}$ is the i th row of the Jacobian matrix J obtained at iteration k , $\lambda_{k,i}$ is the regularization parameter, and V_i and $U_i(\rho_{k,i-1})$ are the measured and predicted voltages corresponding to the i th current pattern given the conductivity distribution is $\rho_{k,i-1}$. The approach to select the rows of J at each iteration is either to go through the rows in order or to pick out the index i uniformly at random (Strohmer *et al* 2009) if we do not have prior information about the current patterns.

Although the Kaczmarz method is well suited for many medical imaging reconstruction problems, its original form might not be a good choice in EIT, where the reconstruction problem is very ill-posed and underdetermined. It is not adequate to obtain an accurate solution update from only a single set of measurements and its corresponding current pattern. In order to overcome the problem, two or more sets of measurements and current patterns can be grouped into blocks to provide an acceptable solution. Hence the block version of the Kaczmarz method should be used.

The block Kaczmarz method (Elfving 1980, Eggermont *et al* 1981, Byrne 1996) also starts with an initial estimate and a proper partition scheme to divide the Jacobian matrix J into n subsets. At the k th iteration of the i th subset, the algorithm projects the solution $\rho_{k,i-1}$ onto the solution space of $\langle J_{k,i}, \rho_{k,i-1} \rangle = V_i$ that is

$$\rho_{k,i} = \rho_{k,i-1} + a_{k,i} J_{k,i}^T (J_{k,i} J_{k,i}^T + \lambda_{k,i} I)^{-1} [V_i - U_i(\rho_{k,i-1})], \quad (3)$$

The update step of the block Kaczmarz method looks similar to equation (2) but with a different interpretation. The index $i = 1, \dots, n$ is the subset index, $a_{k,i}$ is the step length at iteration k for subset i , $J_{k,i}$ is the row submatrix of the Jacobian obtained at iteration k corresponding to the i th subset, V_i and $U_i(\rho_{k,i-1})$ are the measured and predicted voltages corresponding to the i th subset given the conductivity distribution is $\rho_{k,i-1}$.

2.2. Ordered Subset Scheme and Optimal Current Generation

The block Kaczmarz method requires that the complete set of both current patterns and measurements should be appropriately divided into subsets to improve the performance of the traditional Kaczmarz method. One subset scheme is to partition the complete set in the order that the current patterns are applied and sequentially pick the block at each iteration. We call this the sequential subset scheme. Another subset scheme, which randomly picks the block from a pre-partitioned complete set at each iteration, is called the randomized subset scheme (Needell *et al* 2012).

We propose a novel subset scheme so that we can partition and order the subsets of current patterns and measurements by their ability to characterize the conductivity distribution.

Specifically, we partition the current patterns $I = \{I_{sub}^1, I_{sub}^2, \dots, I_{sub}^n\}$ and measurements $V = \{V_{sub}^1, V_{sub}^2, \dots, V_{sub}^n\}$ so that the pair (I_{sub}^1, V_{sub}^1) is the best subset to distinguish the actual conductivity distribution from our conductivity estimate, the pair (I_{sub}^2, V_{sub}^2) is the second best subset and so on. By ordering the subsets of current patterns and measurements in this way, the best “blockwise” conductivity estimate from the current subset will be obtained and passed down as the starting point to the next subset. Compared to the randomized or sequential subset scheme, the ordered subset scheme can accelerate the convergence by avoiding the situation where the conductivity estimates from the first few iterations are suboptimal.

To obtain the ordered subset of the current patterns, the complete set of the “best” current patterns should be generated and partitioned according to the size of each subset I_i . The “best” is in the sense that the current patterns can best distinguish the actual conductivity distribution from a conductivity distribution estimate, meaning we are looking for the best I

that satisfies $\max \frac{\|V - U(\rho^{(0)})I\|}{\|I\|}$, where V and $U^{(0)}$ are the measured voltages and the predicted voltages given a conductivity estimate $\rho^{(0)}$. The procedure of finding the optimal current patterns is described in Algorithm 1.

Algorithm 1: Producing the optimal current patterns

Input: orthonormal current matrix: T , conductivity estimate $\rho^{(0)}$

Initialize $T = T$;

repeat

- 1 Apply T and measure voltages V ;
 - 2 Apply T and compute the predicted voltages $U^{(0)}$;
 - 3 Form the voltage difference matrix $D = V - U^{(0)}$ and $P = D^*D$;
 - 4 Perform eigenvalue decomposition of P and form the eigenvector matrix E so that their corresponding eigenvalues are sorted in descending order;
 - 5 $T = TE$;
 - 6 **until**
 - 7 *the current pattern converges*;
 - 8 **Output:** Optimal current matrix: T
-

The algorithm usually converges within 3 iterations. The optimal current patterns represent the modes of the system and the larger the eigenvalue associated with a given pattern, the stronger the mode and the better that pattern's ability to distinguish the actual conductivity distribution from the estimate. By sorting the eigenvalues of P in descending order, it is

guaranteed that the new current patterns T are ordered by each excitation's ability to distinguish the conductivity distribution. Denote the desired subset size to be S and the number of all subsets to be $n = (L - 1) \bmod S + 1$, the i th current subset can be obtained $I_{sub}^i = I_{1+(i-1)S:iS}$, and the i th measured voltage subset can be obtained $V_{sub}^i = V_{1+(i-1)S:iS}$. With the ordered subset scheme, the image can be reconstructed by the block Kaczmarz method described in equation (3).

2.3. Adaptive Kaczmarz method with Ordered Subsets

An adaptive Kaczmarz method with ordered subsets is proposed based on the block Kaczmarz method and optimal current pattern generation. It begins with an initial guess for the current patterns $I^{(0)}$ and an initial guess for the conductivity distribution $\rho^{(0)}$ and proceeds to generate the optimal current patterns $I^{(1)}$ to best distinguish the actual conductivity distribution from the guess $\rho^{(0)}$. Then $I^{(1)}$ with the ordered subset scheme is used to solve for the conductivity distribution with one-iteration of the Kaczmarz method to obtain $\rho^{(1)}$. The more accurate conductivity estimate $\rho^{(1)}$ is used for the next iteration to find the optimal current patterns $I^{(2)}$ to best distinguish the actual conductivity distribution from the more accurate estimate $\rho^{(1)}$, which are then used to solve for $\rho^{(2)}$. Therefore, a new conductivity estimate and a set of optimal current patterns are obtained adaptively at each iteration. The iteration continues until a reasonable conductivity estimate is obtained, or in the best case, the conductivity estimate and the actual conductivity are not distinguishable from the voltage difference. The algorithm is described in Algorithm 2.

In the algorithm, regularization with the identity matrix is needed due to the rank-deficiency of $J_i J_i^T$. When prior information, e.g. smoothness, of the solution is known, other regularization matrices, which are widely used in the Tikhonov regularization, can also be applied in the adaptive Kaczmarz method. The generalized update step is

$$\rho_i = \rho_{i-1} + a_i (Q^{-1} J_i^T (J_i Q^{-1} J_i^T + \lambda_i W^{-1})^{-1} W^{-1} [V_i - U_i(\rho_{i-1})]), \quad (4)$$

where Q is a regularization matrix and W is a weighting matrix. Regularization parameter λ_i is fixed during iterations. The derivation is based on the analogous Gauss-Newton update step and the matrix inverse lemma and is shown in Appendix A.

Algorithm 2: Adaptive Kaczmarz method

Input: orthonormal current matrix T , initial conductivity guess $\rho^{(0)}$, number of iterations m

Initialize $T = T$, $\rho = \rho^{(0)}$;

for $k = 1:m$ **do**

a. repeat

- 1 Apply T and measure voltages V ;
- 2 Apply T and compute the simulated voltages U ;
- 3 Form the voltage difference matrix $D = V - U$ and $P = D^* D$;
- 4 Perform eigenvalue decomposition of P and form the eigenvector
- 5 matrix E so that their corresponding eigenvalues are sorted in descending order;
- 6 $T = TE$;
- 7 **until**

 the current pattern converges;

b. Obtained the ordered subsets $I=\{I_{sub}^1, I_{sub}^2, \dots, I_{sub}^n\}$, $V=\{V_{sub}^1, V_{sub}^2, \dots, V_{sub}^n\}$, the row submatrix $J=\{J_1, J_2, \dots, J_n\}$, $\rho_0 = \rho$;

c. **for** $i=1:n$ **do** $\rho_i = \rho_{i-1} + a_i J_i^T (J_{k,i} J_i^T + \lambda_i I)^{-1} (V_i - U_i(\rho_{i-1}))$;

end

d. $\rho = \rho_i$;

end

Output: conductivity distribution

Instead of iterating using one current pattern in the block Kaczmarz method, the adaptive Kaczmarz method updates adaptively both the optimal current patterns and a more accurate conductivity distribution estimate at each iteration. Hence, we are able to use the best conductivity estimate when we solve for the current patterns, and the best current patterns when we solve for the conductivity. It is empirically observed that an accurate and stable solution can be obtained within 5 iterations.

Note that the Gauss Newton method with the update step at iteration k has the form of

$$\rho_k = \rho_{k-1} + a_k (J_k^T J_k + \lambda_k I)^{-1} J_k^T [V - U(\rho_{k-1})]. \quad (5)$$

The relationship between the Gauss Newton update step and the Kaczmarz update step is shown in Appendix A. This relationship can be expressed as

$$(J_k^T J_k + \lambda_k I)^{-1} J_k^T [V - U(\rho_{k-1})] = J_k^T J_k^k (J_k^T J_k^k + \lambda_k I)^{-1} [V - U(\rho_{k-1})],$$

when both the weighting matrix and the regularization matrix are identity matrices and there is no row partition in the Kaczmarz method. The $J_k^T J_k$ term has the size of $M \times M$ and the $J_k J_k^T$ term has the size of $N \times N$, where M is the number of the elements in the finite element inverse mesh (the number of columns of J_k) and N is the number of measurements (the number of rows of J_k). Since the cost to invert a matrix is $O(n^3)$, when $M > N$ the Kaczmarz method is preferable because it inverts a smaller matrix $J_k J_k^T$ at each step and when $N > M$, Gauss Newton is preferable because $J_k^T J_k$ has a smaller dimension. In many realistic EIT problems, the dimension of $J_k J_k^T$ is considerably smaller than the dimension of $J_k^T J_k$ since thousands of conductivity voxels have to be determined. This limits the use of the Gauss Newton algorithm to relatively small scale problems. In the 2D two-target simulation shown in the next section, Gauss Newton method inverts $J^T J$ with a dimension of 2992×2992 at each update step, while the adaptive Kaczmarz method inverts $J_i J_i^T$ with a dimension of 192×192 assuming the inverse mesh has 2992 elements and the subset size used in the Kaczmarz is 6. In terms of memory requirement, the Gauss Newton method with 10 iterations requires 231.4 Megabytes, while the adaptive Kaczmarz method with 10 iterations requires only 31.2 Megabytes, which is about 6 times less than Gauss Newton. Moreover, the ordered subset scheme in the adaptive Kaczmarz splits the complete set into blocks and allows separated and parallel computation, which will further reduce the memory storage and the computational time. The advantage of the adaptive Kaczmarz in terms of computation cost and memory requirement will be reflected even more in large scale realistic problems, especially in 3D cases.

3. Results

In this section, reconstructed images using different methods are shown and evaluated. Simulations are conducted in EIDORS (Adler and Lionheart 2006) with a forward finite-element model created by Netgen (Schoberl 1997). A normalized 2D circular tank with unit radius and 32 equally-spaced square electrodes is modeled. The complete electrode model is

assumed with each electrode's length to be $\frac{1}{6}$ and the contact impedance to be 0.0001. Meshes are refined around the square electrodes to improve the accuracy of the model. The inverse mesh (2992 elements) is designed to be coarser than the forward mesh (70394 elements) to avoid an *inverse crime*. The subset size used in the block Kaczmarz and the adaptive Kaczmarz method is 6. The iteration number used in all methods is 10.

3.1. Comparison of different subset schemes

In this simulation, two circular targets with radius $r_{target} = 0.15$ are placed inside the unit radius tank. The conductor is placed at $(0.4, -0.35)$ with conductivity 2 S/m, and the insulator is placed at the location $(-0.3, 0.2)$ with the conductivity 0 S/m. Trigonometric current patterns, which are the optimal current patterns for the homogeneous case, are used with the maximum amplitude of 1 mA. The default order of the applied current patterns is defined by specifying the i th current pattern at electrode l as

$$I_i^l = \begin{cases} M \cos(i\theta_l) & \text{for } i=1, \dots, \frac{L}{2} - 1 \\ M \cos(\pi l) & \text{for } i=\frac{L}{2} \\ M \sin((i - L/2)\theta_l) & \text{for } i=\frac{L}{2}+1, \dots, L-1 \end{cases} \quad (6)$$

where M is the maximum current amplitude, $L = 32$ is the total number of the electrodes, and $\theta = \frac{2\pi l}{L}$ is the angle of the center of the l th electrode.

Images are reconstructed using the non-adaptive block Kaczmarz method with the sequential subset scheme, the randomized subset scheme, and the proposed ordered subset scheme. This simulation serves as a guidance for choosing an appropriate partition scheme for the block Kaczmarz method. The subset size $|I_{sub}^i| = |V_{sub}^i| = 6$, meaning 6 current excitations and the corresponding measurements are grouped in a single subset. The sequential subset scheme and the ordered subset scheme are described in the previous section. For the randomized subset scheme, we test both the case when the subsets are formed using sequential patterns and the case when the subsets are formed using ordered patterns. At each update step of the Kaczmarz method, a subset is then drawn uniformly at random from the partition. The results are shown in Figure 1.

It is observed in Figure 1 that the reconstructed image with the ordered subset scheme is more accurate than the images with other schemes. Reconstructed images with the randomized scheme and the sequential scheme have target deformation and artifacts on different levels, while the image with the ordered subset more accurately reflects the actual target's location, size, and shape.

3.2. One target reconstruction with measurement truncation

This simulation is conducted to demonstrate the distinguishability improvement of the adaptive Kaczmarz over its non-adaptive version and to compare different choices for the initial current estimate for the adaptive Kaczmarz method.

A circular target with conductivity 1 S/m with radius of 0.015 is placed at the center of the tank (0, 0). Voltages measured on the electrodes are truncated after 4 digits to model a limited measurement precision. This approach to modeling the limited precision of an EIT measurement system is considered to be more appropriate than adding the Gaussian noise to the measurement channels. Adjacent pair, opposite pair, and trigonometric current patterns are applied using the block Kaczmarz and the adaptive Kaczmarz methods. The maximum current is set to be 1 mA for adjacent pair and opposite pair. Two cases are tested for the trigonometric patterns. In one case, the amplitude is set to be 1 mA (same max amplitude as the adjacent pattern) and, in the other case, the amplitude is set to be 0.25 mA (same applied power as the adjacent pattern). Truncation is scaled according to the measured voltages' dynamic range, modeling the case where the full scale of the analog-to-digital converter is adjusted to equal the maximum measured voltage.

The reconstructed images from the non-adaptive block Kaczmarz method and adaptive Kaczmarz method are shown in Figure 2. Due to their smaller distinguishability, adjacent pair and opposite pair patterns are unable to detect the small target in the center of the tank, while the trigonometric pattern can detect the target even with the same applied power as the adjacent pair. The adaptive Kaczmarz method improves the reconstructions over the non-adaptive version, especially for the cases starting with adjacent pair and opposite pair current pattern. The algorithm generates the optimal patterns at each iteration, improving the ability to detect the small conductivity changes in the center. Figure 3 illustrates this point using the first adjacent pair pattern. Before the optimal current pattern generation, the voltage difference between the inhomogeneous case and the homogeneous case is zero due to the truncation after 4 digits, meaning that it is not possible to distinguish the actual conductivity distribution from the homogeneous background with 4 digits of precision. In contrast, after the optimal current pattern generation step, the current converges to the trigonometric shape pattern and the distinguishability is improved, as shown by the fact that the voltage difference has a trigonometric shape and is detectable. It is worth noting that the voltage difference is detectable (nonzero) without optimal current generation if 6 or more digits of precision are allowed.

The adaptive algorithm can start with any orthogonal current pattern set and will converge to the optimal patterns. The convergence will be accelerated by starting with a near optimal current pattern set. For the 2D circular tank, the trigonometric patterns are optimal for the homogeneous case and are a good starting point for the algorithm.

3.3. Two-target reconstruction

In this section, the non-adaptive block Kaczmarz method, the Gauss Newton method, the adaptive Kaczmarz method, the adaptive Gauss Newton method, and the GREIT method (Adler A *et al* 2009) are compared for their reconstruction images. The adaptive Gauss Newton method combines the Gauss Newton method and the optimal current pattern generation in a similar way as the adaptive Kaczmarz method does, where the optimal current pattern are adaptively determined between each iteration. The GREIT method reconstructs the image by forming a linear image reconstruction matrix encoded with a set of performance requirements. The two-target model used earlier is once again used and the current patterns (initial for the adaptive cases) are the trigonometric patterns with a maximum amplitude of 1 mA. The reconstructed images using different methods are shown in Figure 4. Note that the GREIT method reconstructs the image onto a 64×64 grid, which is different from the inverse mesh used in other methods. The improvement of the adaptive Kaczmarz method and the adaptive Gauss Newton method over their non-adaptive versions can be easily observed.

These images are further compared using several consensus figures of merit described in (Adler *et al* 2009):

Position Error is defined to be $PE = r_t - r_q$ where $r_t = \sqrt{x_t^2 + y_t^2}$ is the target position and $r_q = \sqrt{x_q^2 + y_q^2}$ is the center of gravity (CoG) of the reconstructed target.

Blurring Error is defined to be $BE = \frac{AreaRecon}{AreaSim}$, where $AreaSim$ is the target area and $AreaRecon$ is defined to be the area that has a reconstructed value 50% of the peak amplitude.

Peak Amplitude is defined to be the biggest pixel values of the reconstructed target.

Amplitude Response is defined to be the sum of the pixels value in $AreaRecon$.

The measures are applied to the reconstructed conductor target produced by different methods. The simulated conductor target position is at (0.4, -0.35), its area is $(0.15)^2$, and its peak value is 2. The comparison of different methods is shown in Table 1.

From the error table, it is observed that the adaptive Kaczmarz has the smallest position error and blurring error among all methods. Moreover, the peak value of the adaptive Kaczmarz is the closest to the true peak value 2. GREIT method has the best amplitude response performance since it uses training data to make sure the amplitude response of the same target will be constant regardless of its location.

3.4. 2D lung and heart simulated phantom

The third simulation is to test the adaptive Kaczmarz method with more complicated targets in the unit 2D circular region. A lung and heart phantom is created by using ellipses and a circle. To make the phantom more realistic, the right ellipse is slight bigger than the left ellipse, and the circle is located closer to the left ellipse. The background conductivity is 0.424 S/m, the conductivity of the lungs is 0.240 S/m, and the heart conductivity is 0.750 S/m. Trigonometric current pattern is applied with a maximum amplitude of 1mA. Reconstructed images of Gauss Newton, Kaczmarz method, adaptive Gauss Newton, and the adaptive Kaczmarz methods are compared in Figure 5. It is observed that the adaptive Kaczmarz method more accurately recovers the shapes and relative locations of the lung-heart phantom. The reconstructed image shows fewer artifacts, especially near the center of the tank.

3.5. 2D lung and heart experimental phantom

The adaptive Kaczmarz method is then tested on experimental data collected on a phantom which consisted of agar heart and lungs in a saline bath in a 2D circular tank of radius 15 cm with 32 electrodes of size 1.6 cm high and 2.5 cm wide. The conductivity of the saline is 0.424 S/m, the conductivity of the agar lungs is 0.240 S/m, and the conductivity of the agar heart is 0.750 S/m. A photo of the configuration is shown in Figure 6. The data was collected using the ACT3 system (Edic *et al* 1995) at Rensselaer Polytechnic Institute. Trigonometric current patterns are applied with a maximum amplitude of 0.2 mA. Reconstructed images of the adaptive Kaczmarz method and the adaptive Gauss Newton are shown in Figure 6. It is observed that both methods have similar performance, accurately recovering the relative size and position of heart and lungs with small artifacts from the experimental data.

4. Discussion and Conclusion

We introduce the adaptive Kaczmarz method to solve the inversion problem in electrical impedance tomography. This cost-efficient method appropriately combines the block Kaczmarz method and optimal current pattern generation to produce accurate and stable solutions. Unlike the Gauss Newton method, the adaptive Kaczmarz method avoids the expense of inverting some large scale matrices and is quite memory-efficient. The simulation results show that the adaptive Kaczmarz algorithm is able to provide more accurate 2D reconstruction images within a few iterations than the Gauss Newton method.

The ordered subset scheme, which partitions and orders the subsets by the ability to distinguish the actual conductivity distribution from the conductivity estimate, is coupled with optimal pattern generation to improve the Kaczmarz method. Besides its improvement in reconstructing images, the ordered subset scheme provides two alternative directions to further increase the efficiency of the algorithm: 1) in the case when we know that some subsets or the corresponding current patterns contribute little information to refining the solution, it is reasonable to discard them in order to accelerate the algorithm. In some of our simulations, it is observed that with proper partitioning, using only the first half of the subsets or less, we were still able to obtain reconstruction as good as with the complete subsets. 2) The subset scheme splits the complete set into blocks and allows separated and parallel computation, which will further reduce the memory requirement and speed up the algorithm.

Future efforts will involve further improvement in the efficiency of the algorithm and the generalization to the 3D case.

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Appendix A: Derivation of the update step equation of the Kaczmarz method with regularization

Define the search direction to be d_i at i th subset, the measurements corresponding to i th subset to be V_i , the predicted voltage corresponding to i th subset given the conductivity estimate from $(i-1)$ th subset to be $U_i(\rho_{i-1})$, the row submatrix of the Jacobian corresponding to i th subset to be J_i , λ to be the regularization parameter, W to be the weighting matrix, and Q to be the regularization matrix. The Gauss Newton update step would be

$$d_i = (J_i^T W J_i + \lambda Q)^{-1} J_i^T [V_i - U_i(\rho_{i-1})]$$

$$\rho_i = \rho_{i-1} + \alpha_i d_i$$

By using the Matrix Inverse Lemma:

$$(A+UCV)^{-1}=A^{-1}-A^{-1}U(C^{-1}+VA^{-1}U)^{-1}VA^{-1}$$

We can rewrite d_i

$$\begin{aligned} d_i &= (J_i^T W J_i + \lambda_i Q)^{-1} J_i^T [V_i - U_i(\rho_{i-1})] \\ &= (\lambda_i^{-1} Q^{-1} - \lambda_i^{-1} Q^{-1} J_i^T (W^{-1} + J_i \lambda_i^{-1} Q^{-1} J_i^T)^{-1} J_i \lambda_i^{-1} Q^{-1}) J_i^T [V_i - U_i(\rho_{i-1})] \\ &= (\lambda_i^{-1} Q^{-1} - \lambda_i^{-1} Q^{-1} J_i^T (\lambda_i^{-1} (\lambda_i W^{-1} + J_i Q^{-1} J_i^T))^{-1} J_i \lambda_i^{-1} Q^{-1}) J_i^T [V_i - U_i(\rho_{i-1})] \\ &= (\lambda_i^{-1} Q^{-1} - \lambda_i^{-1} Q^{-1} J_i^T (\lambda_i W^{-1} + J_i Q^{-1} J_i^T)^{-1} J_i Q^{-1}) J_i^T [V_i - U_i(\rho_{i-1})] \\ &= (\lambda_i^{-1} Q^{-1} J_i^T - \lambda_i^{-1} Q^{-1} J_i^T (\lambda_i W^{-1} + J_i Q^{-1} J_i^T)^{-1} J_i Q^{-1} J_i^T) [V_i - U_i(\rho_{i-1})] \\ &= (\lambda_i^{-1} Q^{-1} J_i^T (\lambda_i W^{-1} + J_i Q^{-1} J_i^T)^{-1} (\lambda_i W^{-1} + J_i Q^{-1} J_i^T) - \lambda_i^{-1} Q^{-1} J_i^T (\lambda_i W^{-1} + J_i Q^{-1} J_i^T)^{-1} J_i Q^{-1} J_i^T) [V_i - U_i(\rho_{i-1})] \\ &= (\lambda_i^{-1} Q^{-1} J_i^T (\lambda_i W^{-1} + J_i Q^{-1} J_i^T)^{-1} (\lambda_i W^{-1} + J_i Q^{-1} J_i^T) - J_i Q^{-1} J_i^T) [V_i - U_i(\rho_{i-1})] \\ &= (\lambda_i^{-1} Q^{-1} J_i^T (\lambda_i W^{-1} + J_i Q^{-1} J_i^T)^{-1} (\lambda_i W^{-1}) [V_i - U_i(\rho_{i-1})] \\ &= (Q^{-1} J_i^T (\lambda_i W^{-1} + J_i Q^{-1} J_i^T)^{-1} W^{-1}) [V_i - U_i(\rho_{i-1})] \end{aligned}$$

When $W = I$, $d_i = (Q^{-1} J_i^T (\lambda_i I + J_i Q^{-1} J_i^T)^{-1} (V_i - U_i(\rho_{i-1})))$. We can use this search direction in our Kaczmarz method update step. When computing the inverse of the regularization matrix Q , sometimes $(Q + rI)^{-1}$ is used instead.

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Figure 1.

Reconstructed images for the case of two targets. Difference images are reconstructed using the non-adaptive block Kaczmarz method with different subset schemes.

Figure 2.

Reconstructed difference images for the case of one centered target with voltage measurements truncated after 4 digits. The images in the top row are reconstructed using the non-adaptive block Kaczmarz method with 5 iterations. The images in the bottom row are reconstructed using the adaptive Kaczmarz method with 5 iterations. The current patterns applied in the top row and used to initialize the adaptive method in the bottom row are labeled for each image.

Figure 3.

Comparison of the first adjacent current pattern before the optimal current generation (upper) and after the optimal current generation (lower). Plots of the current pattern, the measured voltages (truncated after 4 digits), and the voltage difference (truncated after 4 digits) are shown.

Figure 4.

Reconstructed images for the case of two targets. Difference images are reconstructed using Kaczmarz, Gauss Newton, adaptive Kaczmarz, adaptive Gauss Newton, and GREIT method.

Figure 5. Reconstructed images of the 2D Lung and Heart Phantom from simulated data

Figure 6. Reconstructed images of the 2D Lung and Heart Phantom from experimental data

Table 1

Comparison of different methods. AR stands for amplitude response, PE stands position error, BE stands for blurring error, and PV stands for peak value.

	AR	PE	BE	PV
Gauss Newton	0.0222	0.0202	3.0261	0.4499
Kaczmarz	0.0199	−0.0166	2.6690	0.7336
Adaptive Gauss Newton	0.0233	−0.0166	1.8037	0.7974
Adaptive Kaczmarz	0.0233	0.0032	1.5262	0.9351
GREIT	0.2026	0.2806	2.2420	5.6727