Radial differential interior tomography and its image reconstruction with differentiated backprojection and projection onto convex sets

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Purpose: Interior tomography has been recognized as one of the most effective approaches in computed tomography (CT) to reduce radiation dose rendered to patients. In this work, the authors propose and evaluate an imaging method of radial differential interior tomography.

Methods: In interior tomography, an x-ray beam is collimated to only irradiate the region of interest (ROI) with suspected lesions while the surrounding area/volume of normal tissues/organs is spared. In the proposed imaging method of radial differential interior tomography, the outcome is a ROI image that has gone through a radial differential filtering. The image reconstruction algorithm for the radial differential interior tomography is kept in the fashion of differentiated backprojection and projection onto convex sets, but the required *a priori* knowledge in a small round area becomes zero and may be more readily available in practice.

Results: Using the projection data simulated by computer and acquired by CT scanner, the authors evaluate and verify the performance of the proposed radial differential interior tomography method and its associated image reconstruction algorithm. The preliminary results show that the proposed imaging method can generate an image that is the radial differentiation of a conventional tomographic image and is robust over noise that inevitably exist in practice.

Conclusions: It is believed that the proposed imaging method may find its utility in advanced clinical applications wherein a ROI-based image processing and analysis is required for lesion visualization, characterization, and diagnosis. © 2013 American Association of Physicists in Medicine. [<http://dx.doi.org/10.1118/1.4812676>]

Key words: CT, tomography, interior tomography, reconstruction, derivative backprojection, projection onto convex sets

1. INTRODUCTION

X-ray computed tomography (CT) is one of the most widely used imaging modalities for diagnostic tasks in the clinic. However, the x-ray radiation rendered to a patient might potentially induce clinical consequences, such as cancer.^{1–7} Though the medical benefits provided by diagnostic CT substantially outweigh the potential risks, it is still mandatory to reduce the radiation dose to a patient as much as possible while maintaining the image quality for diagnosis.

In the conventional CT imaging, a tomographic image is reconstructed from the projection data without lateral truncation. Among the methods to reduce CT radiation dose that have been proposed in the literature,^{8–24} the so-called method of interior tomography^{16–23} is of relevance, wherein only the region of interest (ROI) with suspected lesions is irradiated via dynamic x-ray beam collimation. As a result, the normal tissues/organs surrounding the ROI are spared of x-ray radiation dose, which is definitely of clinical relevance, especially in advanced clinical applications, such as cardiovascular imaging. This means that in the interior

tomography an image is reconstructed from the projection data with lateral truncation. Thus far, under the framework of total-variation (TV) minimization based compressed sensing,^{16,17} or the scheme of differentiated backprojection (DBP) and projection onto convex sets (POCS),^{18–23} the development of image reconstruction algorithms for interior tomography has become a subject of active research.

In the clinical applications of conventional tomography, certain filtering for edge detection or enhancement, which is mathematically local,^{24–27} may facilitate the tasks in image analysis and characterization to improve the accuracy and precision in computer aided diagnosis (CAD) and treatment planning in image-guided radiation therapy.^{28,29} For example, the Laplacian for edge detection/enhancement has played an important role in contour outlining for area/volume measurement, feature extraction for image registration/fusion, and segmentation for image understanding.^{26–29} It is straightforward for us to understand that such edge detection/enhancement filtering may also be desirable in the clinical applications of interior tomography. Moreover, it is our opinion that a radial differential filtering, which is a local

and linear operator for edge detection/enhancement, may provide the morphological information that is relevant for image analysis and characterization in clinical applications. Hence, we propose the imaging method of radial differential interior tomography in this paper and derive its associated image reconstruction algorithm.

In conventional tomography, a transform or filtering may be carried out after a tomographic image has been reconstructed by a filtered backprojection (FBP) algorithm, i.e., via a two-step process. Similarly, in radial differential interior tomography, the radial differential filtering may also be carried out once the tomographic image has been reconstructed with the DBP-POCS algorithm.^{18–23} However, it is not hard to understand that the computation efficiency of such a two-step approach can be significantly improved if the radial differential filtering is integrated into image reconstruction and thus the postreconstruction operation can be avoided. It also should be noted that any artifact existing in the reconstructed interior tomographic image may be picked up or even amplified by the postreconstruction radial differential filtering. In addition, by avoiding the postreconstruction filtering, certain properties of the reconstructed images, e.g., the spatial resolution, may not be degraded. All these points clearly show that, in the radial differential interior tomography, the DBP-POCS based one-step image reconstruction algorithm (namely, radial differential DBP-POCS algorithm henceforth) is more desirable than the DBP-POCS based two-step approach.

With the projection data of a phantom simulated by computer and a large animal (sheep) acquired by diagnostic CT scanner, we evaluate the performance of the proposed radial differential interior tomography and the derived DBP-POCS based one-step image reconstruction algorithm. Moreover, we compare the performance of the DBP-POCS based one-step algorithm with that of the DBP-POCS based two-step algorithm. In such a way, the proposed imaging method of radial differential interior tomography and its reconstruction algorithm can be thoroughly investigated and characterized.

2. MATERIALS AND METHODS

2.A. Reconstruction of derivative object function with DBP

In the Cartesian coordinate system, an object function can be denoted as a real function,

$$f(\mathbf{x}) = f(x_1, x_2), \quad \mathbf{x} = (x_1, x_2) \in \mathfrak{R}^2, \quad (1)$$

while its parallel-beam projection, i.e., Radon transform, is defined as³⁰

$$\begin{aligned} p(\phi, r) &= \int_{\mathfrak{R}} f(r\boldsymbol{\phi} + t\boldsymbol{\phi}^\perp) dt, \quad \boldsymbol{\phi} = (\cos \phi, \sin \phi), \\ \boldsymbol{\phi}^\perp &= (-\sin \phi, \cos \phi), \quad r \in \mathfrak{R}, \quad \phi \in [0, \pi). \end{aligned} \quad (2)$$

The Hilbert transform of a one-dimensional (1D) function $g(r)$ is defined as

$$Hg(r) = \int_{\mathfrak{R}} \frac{g(r')}{\pi(r-r')} dr' = \int_{\mathfrak{R}} -i \operatorname{sgn}(\rho) G(\rho) e^{i2\pi r \rho} d\rho, \quad (3)$$

where $\operatorname{sgn}(\rho)$ is the sign function of variable ρ , and $G(\rho)$ is the Fourier transform of $g(r)$, i.e.,

$$G(\rho) = \int_{\mathfrak{R}} g(r) e^{-i2\pi r \rho} dr.$$

In the local polar coordinate system with respect to reference point $\mathbf{x}_0$, (therein the origin of local polar coordinate system and the reference point are coincident), the object function and its derivative along the radial direction can be expressed as

$$\begin{aligned} \tilde{f}_{\mathbf{x}_0}(s, \theta) &= f(s\boldsymbol{\theta} + \mathbf{x}_0), \quad \boldsymbol{\theta} = (\cos \theta, \sin \theta), \\ s &\in \mathfrak{R}, \quad \theta \in [0, \pi), \end{aligned} \quad (4)$$

$$\begin{aligned} \partial_1 \tilde{f}_{\mathbf{x}_0}(s, \theta) &= \frac{\partial}{\partial s} \tilde{f}_{\mathbf{x}_0}(s, \theta) = \frac{\partial}{\partial s} f(s\boldsymbol{\theta} + \mathbf{x}_0) \\ &= \frac{\partial}{\partial \mathbf{x}} f(s\boldsymbol{\theta} + \mathbf{x}_0) \cdot \boldsymbol{\theta}, \quad s \in \mathfrak{R}, \quad \theta \in [0, \pi), \end{aligned} \quad (5)$$

and the radial Hilbert filtering of the derivative object function is defined as

$$\begin{aligned} H_\theta^1 \tilde{f}_{\mathbf{x}_0}(s) &= \int_{\mathfrak{R}} \frac{1}{\pi(s-s')} \partial_1 \tilde{f}_{\mathbf{x}_0}(s', \theta) ds' \\ &= \int_{\mathfrak{R}} \frac{1}{\pi(s-s')} \frac{\partial}{\partial \mathbf{x}} f(s'\boldsymbol{\theta} + \mathbf{x}_0) \cdot \boldsymbol{\theta} ds'. \end{aligned} \quad (6)$$

Starting from the Fourier series of object function $f(\mathbf{x})$

$$f(\mathbf{x}) = \int_{\mathfrak{R}^2} F(\mathbf{v}) e^{i2\pi \mathbf{v} \cdot \mathbf{x}} d\mathbf{v}, \quad (7)$$

where $F(\mathbf{v}) = F(v_1, v_2) = \int_{\mathfrak{R}^2} f(\mathbf{x}) e^{-i2\pi \mathbf{v} \cdot \mathbf{x}} d\mathbf{x}$, we have

$$\frac{\partial}{\partial \mathbf{x}} f(\mathbf{x}) = i2\pi \int_{\mathfrak{R}^2} F(\mathbf{v}) e^{i2\pi \mathbf{v} \cdot \mathbf{x}} \mathbf{v} d\mathbf{v}, \quad (8)$$

and

$$\begin{aligned} H_\theta^1 \tilde{f}_{\mathbf{x}_0}(s) &= i2\pi \int_{\mathfrak{R}^2} F(\mathbf{v}) e^{i2\pi \mathbf{v} \cdot \mathbf{x}_0} (\mathbf{v} \cdot \boldsymbol{\theta}) \\ &\quad \times \int_{\mathfrak{R}} \frac{1}{\pi(s-s')} e^{i2\pi s' \mathbf{v} \cdot \boldsymbol{\theta}} ds' d\mathbf{v}. \end{aligned} \quad (9)$$

Changing the integral variable as

$$\int_{\mathfrak{R}} \frac{1}{\pi(s-s')} e^{i2\pi s' \mathbf{v} \cdot \boldsymbol{\theta}} ds' = \int_{\mathfrak{R}} \frac{1}{\pi s'} e^{i2\pi (s-s') \mathbf{v} \cdot \boldsymbol{\theta}} ds', \quad (10)$$

we can get

$$\begin{aligned}
 H_\theta^1 \tilde{f}_{x_0}(s) &= i2\pi \int_{\mathbb{R}^2} F(v) e^{i2\pi v \cdot x_0} (v \cdot \theta) \int_{\mathbb{R}} \frac{1}{\pi s'} e^{i2\pi(s-s')v \cdot \theta} ds' dv \\
 &= i2\pi \int_0^\pi \int_{\mathbb{R}} F(\rho\phi) e^{i2\pi\rho x_0 \cdot \phi} \rho(\phi \cdot \theta) \\
 &\quad \times \int_{\mathbb{R}} \frac{1}{\pi s'} e^{i2\pi(s-s')\rho\phi \cdot \theta} ds' |\rho| d\rho d\phi \\
 &= i2\pi \int_0^\pi \int_{\mathbb{R}} F(\rho\phi) e^{i2\pi\rho x_0 \cdot \phi} \rho(\phi \cdot \theta) \\
 &\quad \times \int_{\mathbb{R}} \frac{1}{\pi s'} e^{i2\pi(s-s')\rho \cos(\phi-\theta)} ds' |\rho| d\rho d\phi \\
 &= i2\pi \int_0^\pi \int_{\mathbb{R}} \text{sgn}(\rho) \rho^2 (\phi \cdot \theta) P(\phi, \rho) e^{i2\pi\rho x_0 \cdot \phi} \\
 &\quad \times \int_{\mathbb{R}} \frac{1}{\pi s'} e^{i2\pi(s-s')\rho \cos(\phi-\theta)} ds' d\rho d\phi. \tag{11}
 \end{aligned}$$

Defining

$$K_p(\theta, s, \phi, \rho) = \int_{\mathbb{R}} \frac{1}{\pi s'} e^{i2\pi(s-s')\rho \cos(\phi-\theta)} ds', \tag{12}$$

we can have

$$\begin{aligned}
 H_\theta^1 \tilde{f}_{x_0}(s) &= i2\pi \int_0^\pi \int_{\mathbb{R}} \text{sgn}(\rho) \rho^2 (\phi \cdot \theta) P(\phi, \rho) \\
 &\quad \times K_p(\theta, s, \phi, \rho) e^{i2\pi\rho x_0 \cdot \phi} d\rho d\phi. \tag{13}
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 K_p(\theta, s, \phi, \rho) &= e^{i2\pi s \rho \cos(\phi-\theta)} \int_{\mathbb{R}} \frac{1}{\pi s'} e^{-i2\pi s' \rho \cos(\phi-\theta)} ds' \\
 &= -i2e^{i2\pi s \rho \cos(\phi-\theta)} \int_{\mathbb{R}^+} \frac{\sin(2\pi s' \rho \cos(\phi-\theta))}{\pi s'} ds' \\
 &= -i\frac{2}{\pi} e^{i2\pi s \rho \cos(\phi-\theta)} \text{sgn}(\rho \cos(\phi-\theta)) \int_{\mathbb{R}^+} \frac{1}{\lambda} \sin \lambda d\lambda \\
 &= -ie^{i2\pi s \rho \cos(\phi-\theta)} \text{sgn}(\rho \cos(\phi-\theta)). \tag{14}
 \end{aligned}$$

So, we arrive at

$$\begin{aligned}
 H_\theta^1 \tilde{f}_{x_0}(s) &= -2\pi \int_0^\pi \int_{\mathbb{R}} \text{sgn}(\rho) (i\rho)^2 (\phi \cdot \theta) P(\phi, \rho) \\
 &\quad \times e^{i2\pi\rho(s \cos(\phi-\theta) + x_0 \cdot \phi)} \text{sgn}(\rho \cos(\phi-\theta)) d\rho d\phi
 \end{aligned}$$

$$\begin{aligned}
 &= -2\pi \int_0^\pi \int_{\mathbb{R}} \text{sgn}(\rho) (i\rho)^2 P(\phi, \rho) \\
 &\quad \times e^{i2\pi\rho(s \cos(\phi-\theta) + x_0 \cdot \phi)} \text{sgn}(\rho) d\rho |\cos(\phi-\theta)| d\phi \\
 &= -2\pi \int_0^\pi \int_{\mathbb{R}} (i\rho)^2 P(\phi, \rho) \\
 &\quad \times e^{i2\pi\rho(s \cos(\phi-\theta) + x_0 \cdot \phi)} d\rho |\cos(\phi-\theta)| d\phi, \tag{15}
 \end{aligned}$$

where

$$\begin{aligned}
 &\int_{\mathbb{R}} (i\rho)^2 P(\phi, \rho) e^{i2\pi\rho(s \cos(\phi-\theta) + x_0 \cdot \phi)} d\rho \\
 &= \frac{1}{4\pi^2} p''(\phi, s \cos(\phi-\theta) + x_0 \cdot \phi). \tag{16}
 \end{aligned}$$

Then, the DBP formula of the derivative object function along the radial direction in the parallel-beam geometry becomes

$$\begin{aligned}
 H_\theta^1 \tilde{f}_{x_0}(s) &= -\frac{1}{2\pi} \int_0^\pi p''(\phi, (s \cos(\phi-\theta) \\
 &\quad + x_0 \cdot \phi)) |\cos(\phi-\theta)| d\phi. \tag{17}
 \end{aligned}$$

In practice, the DBP algorithm to reconstruct the derivative object function along the radial direction in the fan-beam geometry is needed. Since $p''(\phi + \pi, r) = p''(\phi, -r)$ holds in general, we can rewrite Eq. (17) in the case of a full (2π) scan as

$$\begin{aligned}
 H_\theta^1 \tilde{f}_{x_0}(s) &= -\frac{1}{4\pi} \int_0^{2\pi} p''(\phi, (s \cos(\phi-\theta) \\
 &\quad + x_0 \cdot \phi)) |\cos(\phi-\theta)| d\phi, \tag{18}
 \end{aligned}$$

and then

$$\begin{aligned}
 H_\theta^1 \tilde{f}_{x_0}(s) &= -\frac{1}{4\pi} \int_0^{2\pi} \int_{\mathbb{R}} p(\phi, r') \delta''(s \cos(\phi-\theta) \\
 &\quad + x_0 \cdot \phi - r') |\cos(\phi-\theta)| dr' d\phi. \tag{19}
 \end{aligned}$$

Letting $\bar{p}(\xi, u)$ denote projection in the equidistant fan-beam geometry (ξ, u) , there exist the relationships

$$\phi = \xi + \arctan(u/D), \quad r = R_0 u / \sqrt{D^2 + u^2}, \tag{20}$$

$$\xi = \phi - \arctan\left(r / \sqrt{R_0^2 - r^2}\right), \quad u = Dr / \sqrt{R_0^2 - r^2}, \tag{21}$$

$$p(\phi, r) = \bar{p}(\xi, u), \tag{22}$$

and the Jacobean

$$\begin{aligned} drd\phi &= \left\| \frac{\partial r}{\partial \phi} \frac{\partial u}{\partial \xi} \right\| \left\| \frac{\partial r}{\partial \phi} \frac{\partial u}{\partial \xi} \right\| dud\xi = \left\| \frac{\partial r}{\partial \phi} \frac{\partial u}{\partial \xi} \right\| \left\| \frac{\partial r}{\partial \phi} \frac{\partial u}{\partial \xi} \right\| dud\xi \\ &= |\partial r / \partial u| dud\xi = \frac{R_0 D^2}{(\sqrt{D^2 + u^2})^3} dud\xi. \end{aligned} \quad (23)$$

Here D and R_0 are the distances from x-ray source to the detector and the rotation axis, respectively. By denoting $s\theta + x_0$

with $\mathbf{x}$, the integral kernel in Eq. (19) becomes $\delta''(s\cos(\phi - \theta) + \mathbf{x}_0 \cdot \boldsymbol{\phi} - r') = \delta''(\mathbf{x} \cdot \boldsymbol{\phi} - r')$. Furthermore, according to the Appendix, we have

$$\delta''(\mathbf{x} \cdot \boldsymbol{\phi} - r') = \delta''(u^* - u)(\sqrt{D^2 + u^2})^3 / T^3, \quad (24)$$

where $T = R_0 - \mathbf{x} \cdot \boldsymbol{\xi}^\perp$, $u^* = D\mathbf{x} \cdot \boldsymbol{\xi} / T$.

In a way similar to that in Ref. 31 the DBP algorithm to reconstruct the derivative object function along the radial direction in the equidistant fan-beam geometry can be obtained

$$\begin{aligned} H_\theta^1 \tilde{f}_{x_0}(s) &= -\frac{1}{4\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \int_{\Re} \bar{p}(\xi, u) \cos(\phi - \theta) \delta''(u^* - u) \text{sgn}(\cos(\phi - \theta)) dud\xi \\ &= -\frac{1}{4\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \frac{\partial^2}{\partial u^2} (\bar{p}(\xi, u) \cos(\phi - \theta) \text{sgn}(\cos(\phi - \theta))) \Big|_{u=u^*} d\xi \\ &= -\frac{1}{4\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \frac{\partial^2}{\partial u^2} (\bar{p}(\xi, u) \cos(\phi - \theta) \text{sgn}(\cos(\phi - \theta))) \Big|_{u=u^*} d\xi \\ &\quad - \frac{1}{2\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \frac{\partial}{\partial u} (\bar{p}(\xi, u) \cos(\phi - \theta)) \frac{\partial}{\partial u} (\text{sgn}(\cos(\phi - \theta))) \Big|_{u=u^*} d\xi \\ &\quad - \frac{1}{4\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \bar{p}(\xi, u) \cos(\phi - \theta) \frac{\partial^2}{\partial u^2} (\text{sgn}(\cos(\phi - \theta))) \Big|_{u=u^*} d\xi \\ &= -\frac{1}{4\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \frac{\partial^2}{\partial u^2} (\bar{p}(\xi, u) \cos(\phi - \theta) \text{sgn}(\cos(\phi - \theta))) \Big|_{u=u^*} d\xi + r_1 + r_2. \end{aligned} \quad (25)$$

In fact, we have

$$\begin{aligned} r_1 &= -\frac{1}{2\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \frac{\partial}{\partial u} (\bar{p}(\xi, u) \cos(\phi - \theta)) \frac{\partial}{\partial u} (\text{sgn}(\cos(\phi - \theta))) \Big|_{u=u^*} d\xi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \frac{\partial}{\partial u} (\bar{p}(\xi, u) \cos(\phi - \theta)) \times 2\delta(\cos(\phi - \theta)) \sin(\phi - \theta) \frac{D}{D^2 + u^2} \Big|_{u=u^*} d\xi \\ &= \frac{1}{\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \frac{\partial}{\partial u} \bar{p}(\xi, u) \Big|_{u=u^*} \cos(\phi^* - \theta) \delta(\cos(\phi^* - \theta)) \sin(\phi^* - \theta) \frac{D}{D^2 + u^{*2}} d\xi \\ &\quad - \frac{1}{\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \bar{p}(\xi, u^*) \delta(\cos(\phi^* - \theta)) \sin^2(\phi^* - \theta) \frac{D^2}{(D^2 + u^{*2})^2} d\xi \\ &= -\frac{1}{\pi} \int_0^{2\pi} \frac{R_0 D^4}{T^3 (D^2 + u^{*2})^2} \bar{p}(\xi, u^*) \delta(\cos(\phi^* - \theta)) \sin^2(\phi^* - \theta) d\xi, \end{aligned} \quad (26)$$

where $\phi^* = \xi + \arctan \frac{u^*}{D}$. By referring to Eq. (A5), we can further have

$$r_1 = -\frac{1}{\pi} \sum_{i=1}^2 \frac{D^2}{T_i^2(D^2 + u_i^{*2})} \bar{p}(\xi_i, u_i^*) = -\frac{1}{\pi} \sum_{i=1}^2 \frac{\bar{p}(\xi_i, u_i^*)}{\|x - v_i\|^2}, \quad (27)$$

where $v_i = R_0 \xi_i^\perp$, $i = 1, 2$. Meanwhile, we have

$$\begin{aligned} r_2 &= -\frac{1}{4\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \bar{p}(\xi, u) \cos(\phi - \theta) \frac{\partial^2}{\partial u^2} (\text{sgn}(\cos(\phi - \theta))) \Big|_{u=u^*} d\xi \\ &= -\frac{1}{4\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \bar{p}(\xi, u) \cos(\phi - \theta) \frac{\partial}{\partial u} \left(-2\delta(\cos(\phi - \theta)) \sin(\phi - \theta) \frac{D}{D^2 + u^2} \right) \Big|_{u=u^*} d\xi \\ &= -\frac{1}{2\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \bar{p}(\xi, u^*) \cos(\phi^* - \theta) \left(\delta'(\cos(\phi^* - \theta)) \sin^2(\phi^* - \theta) \frac{D^2}{(D^2 + u^{*2})^2} \right. \\ &\quad \left. - \delta(\cos(\phi^* - \theta)) \cos(\phi^* - \theta) \frac{D^2}{(D^2 + u^{*2})^2} + \delta(\cos(\phi^* - \theta)) \sin(\phi^* - \theta) \frac{2Du^*}{(D^2 + u^{*2})^2} \right) d\xi \\ &= -\frac{1}{2\pi} \int_0^{2\pi} \frac{R_0 D^4}{T^3(D^2 + u^{*2})^2} \bar{p}(\xi, u^*) \cos(\phi^* - \theta) \delta'(\cos(\phi^* - \theta)) \sin^2(\phi^* - \theta) d\xi. \end{aligned} \quad (28)$$

By referring to Eq. (A6), we obtain

$$r_2 = -\frac{1}{2\pi} \sum_{i=1}^2 \frac{1}{R_0 T_i} \bar{p}(\xi_i, u_i^*) \left(1 + \frac{D}{D^2 + u^{*2}} \frac{du^*}{d\xi} \right) \Big|_{\xi=\xi_i}, \quad (29)$$

where

$$\frac{du^*}{d\xi} = \frac{DR_0}{T} - \frac{D^2 + u^{*2}}{D}. \quad (30)$$

Subsequently, we have

$$\begin{aligned} r_2 &= -\frac{1}{2\pi} \sum_{i=1}^2 \frac{1}{R_0 T_i} \bar{p}(\xi_i, u_i^*) \left(1 + \frac{D}{D^2 + u^{*2}} \left(\frac{DR_0}{T} - \frac{D^2 + u^{*2}}{D} \right) \right) \Big|_{\xi=\xi_i} \\ &= -\frac{1}{2\pi} \sum_{i=1}^2 \frac{D^2}{T_i^2(D^2 + u_i^{*2})} \bar{p}(\xi_i, u_i^*) = -\frac{1}{2\pi} \sum_{i=1}^2 \frac{\bar{p}(\xi_i, u_i^*)}{\|x - v_i\|^2}. \end{aligned} \quad (31)$$

By integrating Eqs. (27) and (31) into Eq. (25), we obtain the following DBP formula of the derivative object function along the radial direction in the equidistant fan-beam geometry,

$$\begin{aligned} H_\theta^1 \tilde{f}_{x_0}(s) &= -\frac{1}{4\pi} \int_0^{2\pi} \frac{R_0 D^2}{T^3} \text{sgn}(\cos(\phi - \theta)) \frac{\partial^2}{\partial u^2} (\bar{p}(\xi, u) \cos(\phi - \theta)) \Big|_{u=u^*} d\xi \\ &\quad - \frac{3}{2\pi} \sum_{i=1}^2 \frac{\bar{p}(\xi_i, u_i^*)}{\|x - v_i\|^2}. \end{aligned} \quad (32)$$

Letting $\hat{p}(\xi, \gamma)$ denote projection in the equiangular fan-beam geometry (ξ, γ) , there exist the relationships

$$\phi = \xi + \gamma, \quad r = R_0 \sin \gamma, \quad (33)$$

$$\xi = \phi - \gamma, \quad \gamma = \arcsin(r / R_0), \quad (34)$$

$$p(\phi, r) = \hat{p}(\xi, \gamma), \quad (35)$$

where R_0 is the distances from x-ray source to the rotation axis. Meanwhile, between the equiangular and equidistant fan-beam geometries, there exist

$$u = D \tan \gamma, \quad \bar{p}(\xi, u) = \bar{p}(\xi, \gamma), \quad (36)$$

and

$$\begin{aligned} \partial/\partial u &= (\cos^2 \gamma / D) \partial/\partial \gamma, \\ \partial^2/\partial u^2 &= (\cos^4 \gamma / D^2) (\partial^2/\partial \gamma^2 - 2 \tan \gamma \partial/\partial \gamma). \end{aligned} \quad (37)$$

In a similar way, the DBP algorithm to reconstruct the derivative object function along the radial direction in the equiangular fan-beam geometry can be obtained,

$$\begin{aligned} H_\theta^1 \tilde{f}_{x_0}(s) &= -\frac{1}{4\pi} \int_0^{2\pi} \frac{R_0 \cos^4 \gamma^*}{T^3} \text{sgn}(\cos(\phi - \theta)) \\ &\times \left(\frac{\partial^2}{\partial \gamma^2} - 2 \tan \gamma \frac{\partial}{\partial \gamma} \right) (\bar{p}(\xi, \gamma)) \\ &\times \cos(\phi - \theta)|_{\gamma=\gamma^*} d\xi \\ &- \frac{3}{2\pi} \sum_{i=1}^2 \frac{\bar{p}(\xi_i, \gamma_i^*)}{\|x - v_i\|^2}, \end{aligned} \quad (38)$$

where $\gamma_i^* = \arctan(x_i \cdot \xi_i / T_i)$, $T_i = R_0 - x \cdot \xi_i^\perp$, $i = 1, 2$, and $\xi_1 = \theta - \pi/2 + \arcsin(x_0 \cdot \theta^\perp / R_0)$, $\xi_2 = 2\theta - \xi_1$. The above DBP formula will be used to demonstrate the feasibility of the proposed imaging method of radial differential interior tomography.

2.B. Projection of differentiated backprojection onto convex sets (DBP-POCS)

It should be noted that the right side of both Eqs. (32) and (38) represent the radial Hilbert transform of the derivative object function $\partial_1 \tilde{f}_{x_0}(s, \theta)$. Moreover, it is important to note that the Hilbert transform is not mathematically local.^{31,32} Hence, the reconstruction of the derivative object function $\partial_1 \tilde{f}_{x_0}(s, \theta)$ inside an ROI using either Eq. (32) or Eq. (38) becomes an interior tomography problem that can be dealt with using the DBP-POCS based methods.¹⁸⁻²³ Hereafter, the dependence on x_0 and θ will be omitted for brevity. The implementation of a DBP-POCS based method is carried out along each Hilbert line associated with a local coordinate system,¹⁸⁻²³ and thus the derivative object function

$\partial_1 \tilde{f}(s)$ can be viewed as a 1D function of variable s with a definition domain of normalized interval $(-1, 1)$. On each Hilbert line, Λ_K , Λ_H , Λ , and X represent the intervals with the *a priori* knowledge $\partial_1 \tilde{f}^{(p)}(s) = 0$, the DBP values that can be calculated from Eq. (38), the compact support of the object function $\partial_1 \tilde{f}(s)$, and the entire interval wherein the DBP-POCS based method is implemented.¹⁸⁻²³ For the case considered in this paper, there is a relationship among these intervals: $\Lambda_H \subseteq X$, $\Lambda_K \subseteq \Lambda \subseteq X = (-1, 1)$. $L_W^2(X)$ represents a weighted L^2 space with $W(s) = (1 - s^2)^{0.5}$, and the associated inner product and norm are defined by

$$\begin{aligned} (\partial_1 \tilde{f}_1, \partial_1 \tilde{f}_2)_W &= \int_{-1}^1 ds \partial_1 \tilde{f}_1(s) \partial_1 \tilde{f}_2(s) W(s), \\ \|\partial_1 \tilde{f}\|_W^2 &= (\partial_1 \tilde{f}, \partial_1 \tilde{f})_W. \end{aligned} \quad (39)$$

To implement the DBP-POCS based method, four convex sets are defined¹⁸⁻²³

$$\begin{aligned} C_1 &= \{\partial_1 \tilde{f} \in L_W^2(X) | (H_X \partial_1 \tilde{f})(s) = g(s) \text{ for } s \in \Lambda_H\}, \\ (H_X \partial_1 \tilde{f})(s) &= \frac{1}{\pi} p.v. \int_{-1}^1 ds \frac{1}{s - s'} \partial_1 \tilde{f}(s'); \\ C_2 &= \{\partial_1 \tilde{f} \in L_W^2(X) | \partial_1 \tilde{f}(s) = 0 \text{ for } s \notin \Lambda\}; \\ C_3 &= \{\partial_1 \tilde{f} \in L_W^2(X) | \partial_1 \tilde{f}(s) = \partial_1 \tilde{f}^{(p)}(s) = 0 \text{ for } s \in \Lambda_K\}; \\ C_4 &= \left\{ \partial_1 \tilde{f} \in L_W^2(X) \left| \int_{\Lambda \setminus \Lambda_K} ds \partial_1 \tilde{f}(s) = C_\Lambda^{(p)} \right. \right\}, \\ C_\Lambda^{(p)} &= \pi C_\Lambda - \int_{\Lambda_K} ds \partial_1 \tilde{f}^{(p)}(s) = \pi C_\Lambda; \end{aligned} \quad (40)$$

where πC_Λ is equal to the ideal projection data of the derivative object function acquired along the Hilbert line. In the equiangular fan-beam geometry, we have

$$\begin{aligned} C_\Lambda &= \frac{1}{\pi} \int_{\mathfrak{H}} \partial_1 \tilde{f}_{x_0}(s, \theta) ds = \frac{1}{\pi} \int_{\mathfrak{H}} \frac{\partial}{\partial s} f(s\theta + x_0) ds \\ &= \frac{1}{\pi} f(s\theta + x_0)|_{-\infty}^{+\infty} \equiv 0, \end{aligned} \quad (41)$$

and then $C_\Lambda^{(p)} = 0$.

Projection operator P_i corresponding to each convex set C_i is required to implement the DBP-POCS based algorithm.^{18,19} Projectors P_2 and P_3 are trivial, and P_1 and P_4 are specified as^{18,19}

$$\begin{aligned} (P_1 \partial_1 \tilde{f})(s) &= \left(\frac{1}{\pi} \frac{1}{W(s)} \int_{-1}^1 ds' \partial_1 \tilde{f}(s') \right) + (H_X^+ M H_X \partial_1 \tilde{f})(s), \\ (H_X^+ h)(s) &= \frac{1}{\pi} \frac{1}{W(s)} p.v. \int_{-1}^1 ds' \frac{1}{s' - s} h(s') W(s'), \\ (Mh)(s) &= \begin{cases} g(s) & (s \in \Lambda_H), \\ h(s) & (s \notin \Lambda_H); \end{cases} \end{aligned}$$

$$\begin{aligned}
(P_4 \partial_1 \tilde{f})(s) &= \begin{cases} \partial_1 \tilde{f}(s) + \frac{\left(C_\Lambda^{(p)} - \int_{\Lambda \setminus \Lambda_K} ds' \partial_1 \tilde{f}(s')\right) / W(s)}{\int_{\Lambda \setminus \Lambda_K} ds' / W(s')} & (s \in \Lambda \setminus \Lambda_K) \\ \partial_1 \tilde{f}(s) & (s \notin \Lambda \setminus \Lambda_K). \end{cases} \\
&= \begin{cases} \partial_1 \tilde{f}(s) - \frac{\int_{\Lambda \setminus \Lambda_K} ds' \partial_1 \tilde{f}(s') / W(s)}{\int_{\Lambda \setminus \Lambda_K} ds' / W(s')} & (s \in \Lambda \setminus \Lambda_K), \\ \partial_1 \tilde{f}(s) & (s \notin \Lambda \setminus \Lambda_K). \end{cases} \quad (42)
\end{aligned}$$

To determine the derivative object function $\partial_1 \tilde{f}(s)$ that falls into the intersection of the four convex sets defined in Eq. (40), the following iterative algorithm is utilized,

$$\partial_1 \tilde{f}^{(k+1)} = P_4 P_3 P_2 P_1 \partial_1 \tilde{f}^{(k)}, \quad k = 0, 1, \dots, K-1, K, \quad (43)$$

where $\partial_1 \tilde{f}^{(0)}$ is the initial image with each pixel set to zero. In this work, the maximal iteration number K is empirically set as 100, since no improvement is noticeable after the 100th iteration.

3. PERFORMANCE EVALUATION

The proposed imaging method of radial differential interior tomography and its reconstruction algorithm are evaluated using the projection data simulated by computer and acquired by CT scanner.

3.A. Evaluation with projection data simulated by computer

The modified Shepp-Logan mathematical phantom³³ is used to simulate the projection data in the equiangular fan-beam geometry,³⁴ in which the distance from the x-ray source to the rotation axis (R_0) is 541 mm. The projection data are uniformly acquired in a full scan (360°) at 1160 angular positions, with a detector array consisting of 888 cells that span a 52.1° full fan-angle. The interior tomography is simulated by setting the radius of the ROI to be scanned as 40.5 mm and truncating the projection data appropriately with lateral collimation. The origin x_0 of the local polar coordinate system is fixed at (0, 0) mm. In such a way, the *a priori* knowledge of the Shepp-Logan phantom can be determined as 0.3. Transaxial tomographic images are initially reconstructed in the polar coordinate system at dimension 2048×2048 , i.e., 2048 sampling points along the azimuthal direction (360°) and radial direction (a distance of 128 mm from the origin x_0), and then converted into a 512×512 matrix with pixel size $0.5 \times 0.5 \text{ mm}^2$ in the Cartesian coordinate system, in which a bilinear interpolation is utilized. A region with a constant value (centered at the origin x_0 , and with a radius of 10^2 sampling points along the radial direction) is chosen as the small round area where the *a priori* knowledge is required.

3.B. Evaluation with projection data acquired by CT scanner

The performance of the proposed imaging method and its reconstruction algorithm is further evaluated and verified using the projection data of a large animal (sheep) acquired by a CT scanner. In data acquisition, the distance from x-ray source to the rotation axis is 570 mm. The projection data are also uniformly acquired at 1160 angular positions over a full scan (360°). The detector array consists of 672 cells spanning a 52.1° full fan-angle. Truncated projection data are acquired by setting the radius of the ROI to be scanned as 50 mm. Again, transaxial tomographic images are initially reconstructed in the polar coordinate system at dimension $2,048 \times 2,048$, i.e., 2048 sampling points along the azimuthal direction (360°) and radial direction (a distance of 128 mm from the origin x_0), and then converted into a 512×512 matrix with pixel size $0.5664 \times 0.5664 \text{ mm}^2$ in the Cartesian coordinate system. An air-filled cavity in the sheep's body (the bronchus with its attenuation approximately equal to zero, centered at origin x_0 , and with a radius of ten sampling points along the radial direction) is chosen as the small round area wherein the *a priori* knowledge is required.

4. RESULTS

The simulation study focuses on the evaluation of reconstruction accuracy, while the animal study verifies the performance in the presence of noise in a practical situation.

4.A. Performance evaluation—phantom study

The modified Shepp-Logan phantom³³ is utilized as the basis for performance evaluation. Shown in Fig. 1(a) is the sinogram of the modified Shepp-Logan phantom simulated in the equiangular fan-beam geometry, in which no lateral truncation occurs. The transaxial image reconstructed from the sinogram by the FBP algorithm is presented in Fig. 1(b).

The transaxial images displayed in the upper row of Fig. 2 are obtained from the projection data with lateral truncation, while those from the data without truncation are presented in the lower row as a reference. The images shown in Figs. 2(a) and 2(a') are reconstructed using the DBP-POCS algorithm²³ and those in Figs. 2(c) and 2(c') are reconstructed using the radial differential DBP-POCS

FIG. 1. (a) The sinogram of the modified Shepp-Logan phantom; and (b) a transaxial image reconstructed from the sinogram [display window $(0\ 0.5)\ \text{mm}^{-1}$].

algorithm proposed in this work. For comparison, the images obtained by applying a postreconstruction radial differentiation on the image reconstructed by the DBP-POCS algorithm, i.e., the two-step radial differential tomography, are presented in Figs. 2(b) and 2(b'). Meanwhile, the numerical radial derivative of the Shepp-Logan phantom obtained via one-dimensional radial differential filtering with adequate windowing in the frequency domain is included in Figs. 2(d) and 2(d') as the ground truth for comparison. As demonstrated in the areas indicated by the dotted and solid arrows in Figs. 2(a) and 2(c), the proposed radial differential DBP-POCS algorithm outperforms the DBP-POCS algorithm in reconstruction accuracy. We speculate that such an advantage may be attributed to the fact that the *a priori* knowledge required by the proposed radial differential DBP-POCS reconstruction is always zero, rather than a constant as required in

the DBP-POCS reconstruction. Moreover, it is observed that the artifact existing in the image reconstructed by the DBP-POCS algorithm is picked up by the postreconstruction radial differentiation [see Fig. 2(b)].

For quantitative evaluation, the profiles along the central horizontal and vertical lines of the images obtained by the postreconstruction radial differentiation [Figs. 2(b) and 2(b')] and the proposed radial differential DBP-POCS algorithm [Figs. 2(c) and 2(c')] are plotted in Fig. 3. The RMSE (root mean square error) between Figs. 2(b) and 2(d) and Figs. 2(c) and 2(d) are 0.0242 and 0.0173, respectively, showing that the proposed radial differential interior tomography method indeed outperforms the two-step method significantly (almost 40%) in artifact avoidance.

4.B. Performance evaluation—animal study

The transaxial images presented in the left column of Fig. 4 are obtained from the projection data without lateral truncation, while those from the data with truncation are presented in the right column. The images shown in the top row [display window $(0\ 0.5)\ \text{mm}^{-1}$] of Fig. 4 are reconstructed using the DBP-POCS algorithm²³ and those in the bottom row [display window $(-0.02\ 0.02)\ \text{mm}^{-2}$] are reconstructed using the radial differential DBP-POCS algorithm proposed in this work. For comparison, the images generated by the two-path method are presented in the middle row. A close inspection of Fig. 4(b') shows that edges of the sheep's tissues/organs in the ROI are enhanced significantly, verifying that the proposed radial differential interior tomography

FIG. 2. Transaxial images of the Shepp-Logan phantom reconstructed from the projection data with (top) and without (bottom) lateral truncation using: (a-a') the DBP-POCS algorithm, (b-b') a postreconstruction radial differentiation algorithm, (c-c') the radial differential DBP-POCS algorithm, and (d-d') numerical derivative of the Shepp-Logan phantom [display window: $(-0.1\ 0.1)\ \text{mm}^{-1}$ in (a-a'), and $(-0.1\ 0.1)\ \text{mm}^{-2}$ in (b-b'), (c-c'), and (d-d')].

FIG. 3. Profiles along the horizontal (left) and vertical (right) lines of the central areas in the images displayed in Figs. 2(b)–2(b'), 2(c)–2(c'), and 2(d)–2(d').

FIG. 4. Transaxial images of the sheep reconstructed from the projection data without (left) and with (right lateral truncation: (a-a') using the DBP-POCS algorithm, (a'-b') the postreconstruction radial differentiation algorithm, and (a''-b'') the proposed radial differential DBP-POCS algorithm [display window: (0 0.5) mm⁻¹ in (a-a') and (-0.02 0.02) mm⁻² in (a'-b') and (a''-b'')].

and its associated reconstruction algorithm indeed can provide information complementary to the original reconstruction. Meanwhile, it also shows that the proposed radial differential interior tomography and its image reconstruction perform well in the presence of noise. For quantitative evaluation, the profiles along the central horizontal and vertical lines of the images obtained by the proposed radial differential DBP-POCS algorithm [Figs. 4(a')–4(b')] are plotted in Fig. 5.

5. DISCUSSIONS AND CONCLUSIONS

In this work, we propose the imaging method of radial differential interior tomography and derive the associated image reconstruction algorithm. Using the projection data of the phantom simulated by computer and a large animal (sheep) acquired by a CT scanner, we evaluate and verify its imaging performance. The results of the preliminary phantom study demonstrate the reconstruction accuracy of the proposed radial differential interior tomography, while the animal study data show its robustness in the presence of noise. What follows is a summary of the points we would like to make based on the preliminary results.

It is important for us to point out that the radial differential interior tomography proposed in this work is essentially different from the so-called Lambda tomography.^{35–40} In general, the Lambda tomography is under the framework of local tomography, in which the algorithm for image reconstruction is in the fashion of FBP. Specifically, in the Lambda tomography, the projection data undergo a 2nd-order differentiation that is mathematically local and then are backprojected to form a tomographic image. As such, the two-dimensional Fourier transform of the Lambda tomographic image is equal to that of the conventional tomographic image with a radial ramp filtering.^{35–38} It should be noted that a ramp filtered conventional tomographic image is actually equivalent

FIG. 5. Profiles along the central horizontal (left) and vertical (right) lines of Figs. 4(a')–4(b') and 4(a'')–4(b'').

to an Hilbert transform of the differentiated conventional tomographic image.³² It should also be noted that another local tomography image reconstruction algorithm called local BPF algorithm³⁹ has been proposed in the literature, in which the projection data go through a 1st-order differentiation, back-projection onto chords and then another 1st-order differentiation along the chords. It should be pointed out that such a tomographic image is actually equivalent to a Hilbert transform of the differentiated conventional tomographic image^{31,32} too.

On the other hand, the image generated by the radial differential interior tomography proposed in this work is a version of the conventional tomographic image with a radial differential filtering carried out in the image domain, rather than a version with a Hilbert transform applied. Hence, the proposed imaging method is under the framework of interior tomography. This is the underlying reason why either the POCS method,^{18–23} which is adopted in this work, or the TV-minimization based compressed sensing method,^{16,17} has to be employed. It is our opinion that, from the practical point of view of image processing, visualization, fusion, and understanding, a differentiated conventional tomographic image should be more meaningful than its counterpart with a Hilbert transform applied.

In the DBP-POCS based image reconstruction solution for the interior tomography, *a priori* knowledge in a small area inside the ROI to be reconstructed is required. The *a priori* knowledge can be available in a small area that is relatively uniform in attenuation, such as air-filled cavities (sinus) or fluid-filled compartments (ventricles). In the proposed radial differential interior tomography, the reconstructed image is a version of the object function with a radial differential operation. Consequently, the required *a priori* knowledge becomes zero if the air-filled cavity or fluid-filled compartment, regardless of its attenuation, is still chosen as the small area to provide the *a priori* knowledge. We would like to indicate that, even though the *a priori* knowledge of zero does not differ from that of a nonzero value in principle, a zero *a*

priori knowledge may be more readily available in practical situations.

It is observed in our preliminary investigation that the proposed radial differential DBP-POCS algorithm is indeed more efficient than the DBP-POCS reconstruction with postreconstruction radial filtering. This is because the radial differential filtering in the former is actually carried out in the projection domain, whereas that in the latter is carried out postreconstruction, which results in additional computational overhead. It is also observed [see Figs. 2(a)–2(c)] that the radial differential DPB-POCS algorithm outperforms the DBP-POCS two-step reconstruction algorithm in terms of artifact avoidance. We think this may be attributed to the fact that the required *a priori* knowledge in the former is zero, whereas that in the latter is nonzero and thus is harder to obtain in practical situations.

Interior tomography has recently been recognized as one of the promising methods to reduce CT's radiation dose in diagnostic imaging without compromising image quality.^{16,40} It is straightforward to understand that, as demonstrated in Figs. 4(a') and 4(b'), a radial differential interior tomographic image may stand alone to reveal more details about the ROI in which pathophysiological lesions are suspected. Meanwhile, the radial differential interior tomographic image may provide extra information complementary to the interior tomographic image (see Fig. 4). It should be noted that such extra information may be of relevance in image analysis, e.g., contour outlining for area/volume measurement, edge detection for image segmentation and morphologic feature extraction for image registration. All these operations are of significance in the clinical applications wherein the CAD based on image processing, visualization, fusion, and understanding is desirable. Hence, it is hoped that the imaging method of radial differential interior tomography and its associated image reconstruction algorithm (i.e., the radial differential DPB-POCS algorithm) may find their applications in the future.

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APPENDIX: PROPERTIES OF THE DIRAC FUNCTION USED IN ALGORITHM DERIVATION

Regarding the Dirac function,²⁴ one has

$$\delta^{(n)}(\lambda s) = \delta^{(n)}(s) / \lambda^{n+1}, \quad \lambda > 0, \quad n = 0, 1, 2, \dots, \quad (\text{A1})$$

where the n th order derivative of Dirac function, $\delta^{(n)}$, is homogeneous of degree $-(n+1)$.

In order to derive the DBP formula in the equidistant fan-beam geometry, we need to convert $\delta''(\mathbf{x} \cdot \boldsymbol{\phi} - r')$ into $\delta''(u^* - u)$, as shown below

$$\begin{aligned} \mathbf{x} \cdot \boldsymbol{\phi} - r' &= x_1 \cos(\xi + \arctan(u/D)) + x_2 \sin(\xi + \arctan(u/D)) - r' \\ &= x_1 \cos \xi \cos(\arctan(u/D)) - x_1 \sin \xi \sin(\arctan(u/D)) \\ &\quad + x_2 \sin \xi \cos(\arctan(u/D)) + x_2 \cos \xi \sin(\arctan(u/D)) - r' \\ &= x_1 \cos \xi \frac{D}{\sqrt{D^2 + u^2}} - x_1 \sin \xi \frac{u}{\sqrt{D^2 + u^2}} + x_2 \sin \xi \frac{D}{\sqrt{D^2 + u^2}} + x_2 \cos \xi \frac{u}{\sqrt{D^2 + u^2}} - \frac{R_0 u}{\sqrt{D^2 + u^2}} \\ &= (x_1 \cos \xi + x_2 \sin \xi) \frac{D}{\sqrt{D^2 + u^2}} - (x_1 \sin \xi - x_2 \cos \xi + R_0) \frac{u}{\sqrt{D^2 + u^2}} \\ &= (u^* - u) \frac{T}{\sqrt{D^2 + u^2}}, \end{aligned}$$

where $T = R_0 - \mathbf{x} \cdot \boldsymbol{\xi}^\perp$, $u^* = D\mathbf{x} \cdot \boldsymbol{\xi}/T$. According to Eq. (A1), we have

$$\delta''(\mathbf{x} \cdot \boldsymbol{\phi} - r') = \delta''(u^* - u)(\sqrt{D^2 + u^2})^3 / T^3. \quad (\text{A2})$$

In the integral calculations, we have made use of the following properties of the Dirac function,

$$\int \delta(k(t)) g(t) dt = \int \delta(k(t)) \frac{g(t)}{k'(t)} dk(t) = \sum_{\substack{k(t)=0 \\ k'(t) \neq 0}} \frac{g(t)}{|k'(t)|}, \quad (\text{A3})$$

$$\begin{aligned} \int \delta'(k(t)) g(t) dt &= \int \delta'(k(t)) \frac{g(t)}{k'(t)} dk(t) = \int \delta'(\zeta) \frac{g(t(\zeta))}{|k'(t(\zeta))|} d\zeta \\ &= \sum_{\substack{k(t)=0 \\ k'(t) \neq 0}} \frac{d}{d\zeta} \left(\frac{g(t(\zeta))}{|k'(t(\zeta))|} \right) = \sum_{\substack{k(t)=0 \\ k'(t) \neq 0}} \frac{d}{dt} \left(\frac{g(t)}{|k'(t)|} \right) \frac{dt}{d\zeta} = \sum_{\substack{k(t)=0 \\ k'(t) \neq 0}} \frac{d}{dt} \left(\frac{g(t)}{|k'(t)|} \right) \frac{1}{k'(t)}. \end{aligned} \quad (\text{A4})$$

Especially, there exist

$$\begin{aligned} \int_0^{2\pi} \Psi(\xi) \delta \left(\cos \left(\xi + \arctan \frac{u^*}{D} - \theta \right) \right) \sin \left(\xi + \arctan \frac{u^*}{D} - \theta \right) d\xi \\ = \sum_{i=1}^2 \frac{\Psi(\xi_i) \sin \left(\xi_i + \arctan \frac{u_i^*}{D} - \theta \right)}{\left| \frac{d}{d\xi} \cos \left(\xi + \arctan \frac{u^*}{D} - \theta \right) \right|_{\xi=\xi_i}} = \sum_{i=1}^2 \frac{\Psi(\xi_i) \sin \left(\xi_i + \arctan \frac{u_i^*}{D} - \theta \right)}{\left| \sin \left(\xi_i + \arctan \frac{u_i^*}{D} - \theta \right) \right| \left| 1 + \frac{D}{D^2 + u^{*2}} \left(\frac{du^*}{d\xi} \right) \right|_{\xi=\xi_i}} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^2 \frac{\Psi(\xi_i) \operatorname{sgn} \left(\sin \left(\xi_i + \arctan \frac{u_i^*}{D} - \theta \right) \right)}{\left| 1 + \frac{D}{D^2 + u^{*2}} \left(\frac{du^*}{d\xi} \right) \right|_{\xi=\xi_i}} = \sum_{i=1}^2 \frac{(-1)^i \Psi(\xi_i)}{\left| 1 + \frac{D}{D^2 + u^{*2}} \left(\frac{DR_0}{T} - \frac{D^2 + u^{*2}}{D} \right) \right|_{\xi=\xi_i}} \\
&= \sum_{i=1}^2 \frac{(-1)^i T_i (D^2 + u_i^{*2}) \Psi(\xi_i)}{D^2 R_0}, \tag{A5}
\end{aligned}$$

$$\begin{aligned}
&\int_0^{2\pi} \Psi(\xi) \delta' \left(\cos \left(\xi + \arctan \frac{u^*}{D} - \theta \right) \right) \sin \left(\xi + \arctan \frac{u^*}{D} - \theta \right) d\xi \\
&= \sum_{i=1}^2 \frac{d}{d\xi} \left(\frac{\Psi(\xi) \sin \left(\xi + \arctan \frac{u^*}{D} - \theta \right)}{\left| \frac{d}{d\xi} \cos \left(\xi + \arctan \frac{u^*}{D} - \theta \right) \right|} \right) \bigg|_{\xi=\xi_i} \frac{1}{\frac{d}{d\xi} \cos \left(\xi + \arctan \frac{u^*}{D} - \theta \right)} \bigg|_{\xi=\xi_i} \\
&= \sum_{i=1}^2 \frac{d}{d\xi} \left(\frac{\Psi(\xi) \sin \left(\xi + \arctan \frac{u^*}{D} - \theta \right)}{\left| \sin \left(\xi + \arctan \frac{u^*}{D} - \theta \right) \right| \left| 1 + \frac{D}{D^2 + u^{*2}} \left(\frac{du^*}{d\xi} \right) \right|} \right) \bigg|_{\xi=\xi_i} \frac{1}{-\sin \left(\xi + \arctan \frac{u^*}{D} - \theta \right) \left(1 + \frac{D}{D^2 + u^{*2}} \left(\frac{du^*}{d\xi} \right) \right)} \bigg|_{\xi=\xi_i} \\
&= \sum_{i=1}^2 \frac{d}{d\xi} \left(\frac{\Psi(\xi) \operatorname{sgn} \left(\sin \left(\xi + \arctan \frac{u^*}{D} - \theta \right) \right)}{\left| 1 + \frac{D}{D^2 + u^{*2}} \left(\frac{du^*}{d\xi} \right) \right|} \right) \bigg|_{\xi=\xi_i} \frac{1}{-\sin \left(\xi + \arctan \frac{u^*}{D} - \theta \right) \left(1 + \frac{D}{D^2 + u^{*2}} \left(\frac{du^*}{d\xi} \right) \right)} \bigg|_{\xi=\xi_i} \\
&= \sum_{i=1}^2 (-1)^i \frac{d}{d\xi} \left(\frac{\Psi(\xi)}{\left| 1 + \frac{D}{D^2 + u^{*2}} \left(\frac{DR_0}{T} - \frac{D^2 + u^{*2}}{D} \right) \right|} \right) \bigg|_{\xi=\xi_i} \frac{1}{-\sin \left(\xi + \arctan \frac{u^*}{D} - \theta \right) \left(1 + \frac{D}{D^2 + u^{*2}} \left(\frac{DR_0}{T} - \frac{D^2 + u^{*2}}{D} \right) \right)} \bigg|_{\xi=\xi_i} \\
&= \sum_{i=1}^2 \frac{(-1)^i}{D^4 R_0^2} \frac{d}{d\xi} \left(T (D^2 + u^{*2}) \Psi(\xi) \right) \bigg|_{\xi=\xi_i} \frac{T_i (D^2 + u_i^{*2})}{-\sin \left(\xi_i + \tan^{-1} \frac{u_i^*}{D} - \theta \right)} \\
&= - \sum_{i=1}^2 \frac{T_i (D^2 + u_i^{*2})}{D^4 R_0^2} \frac{d}{d\xi} \left(T (D^2 + u^{*2}) \Psi(\xi) \right) \bigg|_{\xi=\xi_i}, \tag{A6}
\end{aligned}$$

where

$$\begin{aligned}
\xi_1 &= \theta - \frac{\pi}{2} - \arctan(u_1^*/D) = \theta - \frac{\pi}{2} + \arcsin(\mathbf{x}_0 \cdot \boldsymbol{\theta}^\perp / R_0), \\
\xi_2 &= \theta + \frac{\pi}{2} - \arctan(u_2^*/D) = \theta + \frac{\pi}{2} - \arcsin(\mathbf{x}_0 \cdot \boldsymbol{\theta}^\perp / R_0), \\
u_i^* &= D\mathbf{x} \cdot \boldsymbol{\xi}_i / T_i, \quad T_i = R_0 - \mathbf{x} \cdot \boldsymbol{\xi}_i^\perp, \\
\sin(\xi_i + \arctan(u_i^*/D) - \theta) &= (-1)^i, \\
\cos(\xi_i + \arctan(u_i^*/D) - \theta) &= 0, \quad i = 1, 2. \tag{A7}
\end{aligned}$$

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