An ecological perspective on marine reserves in prey–predator dynamics

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Abstract This paper describes a prey–predator type fishery model with prey dispersal in a two-patch environment, one of which is a free fishing zone and other is a protected zone. The existence of possible steady states, along with their local stability, is discussed. A geometric approach is used to derive the sufficient conditions for global stability of the system at the positive equilibrium. Relative size of the reserve is considered as control in order to study optimal sustainable yield policy. Subsequently, the optimal system is derived and then solved numerically using an iterative method with Runge–Kutta fourth-order scheme. Numerical simulations are carried out to illustrate the importance of marine reserve in fisheries management. It is noted that the marine protected area enables us to protect and restore multi-species ecosystem. The results illustrate that dynamics of the system is extremely interesting if simultaneous effects of a regulatory mechanism like marine reserve is coupled with harvesting effort. It is observed that the migration of the resource, from protected area to unprotected area and vice versa, is playing an important role towards the standing stock assessment in both the areas which ultimately control the harvesting efficiency and enhance the fishing stock up to some extent.

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1 Introduction

Mathematical biology has been applied in recent years to various problems in ecology [1, 2]. Biologists and policy-makers often advocate marine protected areas (MPAs) on the basis that they provide economic benefits, though there is little evidence that such benefits actually occur [3–6]. A marine protected area is defined as a spatially defined area in which all populations are free of exploitation. Marine reserves are established as a fisheries management tool for various reasons. It may be introduced as a protective measure because it is hoped that adult or juvenile migration will replenish depleted fishing grounds beyond the borders of the marine reserve. The advantages of creating marine reserves can go beyond the protection of a specific overfished fish population as they can protect multiple stocks from being overfished. In certain predominantly tropical fisheries where the existence of numerous species prevents managers from conducting single-stock assessment techniques; marine reserves may be the only viable way to protect stocks. MPAs can protect the marine landscape from degradation caused by destructive fishing practices, provide an important opportunity to learn about marine ecosystems and species dynamics, and protect all components of a marine community. MPAs have provided many general benefits as a tool for conservation and marine environmental management [7–9].

Takashina et al. [10] analyzed the potential impact of establishing MPAs on marine ecosystems using mathematical models and demonstrated that establishment of an MPA can sometimes result in a considerable decline, or even extinction, of a species. Their analyses revealed that the establishment of the MPA can cause a reduction in prey abundance, and even extinction of the prey. Wang and Takeuchi [11] proposed mathematical models to simulate migrations of prey and predators between patches. They concluded that the adaptation of prey and predators increases the survival probability of predators from extinction in both patches to the persistence in one patch. They have also demonstrated that there exists a pattern that prey and predators cooperate well through adaptations such that predators are permanent in every patch in the case that predators become extinct in each patch in the absence of adaptations. Mougi and Iwasa [12] studied the predator–prey coevolutionary dynamics when a prey’s defense and a predator’s offense change in an adaptive manner, either by genetic evolution or phenotypic plasticity, or by behavioral choice and obtained several important results regarding prey–predator dynamics.

However, Luck et al. [13] asserted that MPAs can be viewed as a kind of insurance against scientific uncertainty, stock assessments, or regulation errors. Hartmann et al. [14] investigated the economic optimality of implementing an MPA to get more informative data about fish population, thereby allowing a better management strategy. Dubey [15] proposed a mathematical model to study the role of a reserved zone on the dynamics of predator–prey system. Many of the benefits associated with MPAs have been widely investigated and the field is an active area of research in theoretical ecology [16–19]. However, scientists and researchers consider the increasing scope of closed areas for the conservation of marine biodiversity [20–22]. Where ecosystems are fragile, the main aim of an MPA is to support

ecosystem functioning. MPAs also have the potential to provide a margin of safety and perhaps even enhance the productivity of some fisheries.

Despite the growing interest in marine reserves both the economic benefits and the conservation impact of marine reserves have recently been questioned [23]. Dixon [24] observed that the objectives of creation of a marine reserve area can lead to conflicts between the various economic agents involved with the area. Conard [25] showed that, in the absence of ecological uncertainty and in the context of optimal harvesting, reserve generates no economic benefits to fishermen. Such a result coincides with the perspectives of many fishermen and also some economists. It has also been noted that the reserves often become highly beneficial by becoming a prime source of recruitment in times of local overfishing in unprotected areas outside reserves. Srinivasu and Gayetri [26] proposed the dynamics of a prey predator model incorporating reserve area to protect certain number of prey population from predator population. They have also discussed the conservation of the prey population and analyzed the behavioral dependency of the predator population on the reserve capacity. Dubey et al. [27] analyzed a mathematical model to study the dynamics of a fishery resource system in an aquatic environment that consists of two zones, one was free fishing area and another was reserve area. It may also be noted that prey–predator interactions do matter when the implementation of a reserve is considered in the system and reserves will be most effective when coupled with fishing effort controls in adjacent fisheries [28–30].

The present paper deals with a prey predator system where a fractional part of the prey population is considered as protected (where harvesting will not take place) and the rest part of the prey stock is considered for harvesting of the resource. Relative size of the reserve ($0 < \alpha < 1$) is considered as a control instrument to regulate the system. We first outline the basic theoretical model describing the biological dynamics of two homogeneous stocks and the importance of the dynamics in the reserves and fishing zones, and then we add exploitation to the system. The dynamical analysis of the system is carried out. The optimal control problem is formulated and solved numerically using an iterative method with Runge–Kutta fourth-order scheme. We compute some numerical simulations to determine the optimal conditions under which the biological steady state can be attained and to draw some important conclusions regarding the design of reserve area.

2 Model and its qualitative properties

We consider a prey–predator type system with Holling type II functional response. The ecological set up of the system is based on the following major assumptions:

- The entire system is divided in two-patch environment: a patch where harvesting is not allowed (patch1) i.e. reserve area of prey and the other one being an unreserved area (patch2) where harvesting is allowed.
- Assuming, total region under consideration is unit and α ($0 < \alpha < 1$) is the reserved area, consequently $(1 - \alpha)$ is the unreserved area.
- The growth of prey population in both patches is assumed to be logistic type.
- The prey population migrates from reserved area to unreserved area. This is happened due to the difference in density of the population in the concerned patches.

- The predator population consumes prey population from both the patches by Holling type II functional response.
- The predator grows logistically with intrinsic growth rate ω and carrying capacity proportional to the prey population size [18, 29]. It is also assumed that the carrying capacity for the predator population could not be zero due to the existence of reserve area for the prey population.

Let us assume x , y and z are respectively the size of the prey population, in reserved and unreserved area, and predator population at time t . Then basic model governed by the following ordinary differential equations:

$$\begin{aligned}\frac{dx}{dt} &= rx \left(1 - \frac{x}{K\alpha}\right) - M - \frac{mxz}{a+x}, \\ \frac{dy}{dt} &= sy \left(1 - \frac{y}{K(1-\alpha)}\right) + M - \frac{nyz}{b+y} - h(t), \\ \frac{dz}{dt} &= \omega z \left(1 - \frac{\gamma z}{x + \beta y}\right) - dz.\end{aligned}\quad (2.1)$$

where $r(K)$ and $s(K)$ are the respective intrinsic growth rates (carrying capacities) of prey species inside the protected and unprotected areas, m and n are respectively the maximal predator per capita consumption rate, a and b are the half capturing saturation constant [31], d is the natural death rate of predator population, β is the prey predators' conversion factor and γ is the amount of prey required to support one predator at equilibrium. $h(t)$ is the total amount of harvest at time t from unreserved area.

The functional form of harvest is generally considered using the term catch-per-unit-effort (CPUE) hypothesis [32] to describe an assumption that catch per unit effort is proportional to the stock level, that is $h = qEy$. But this functional form gives us some unrealistic features like, unbounded linear increase of h with y for fixed E etc. These hypothetical features are removed in the functional form of $h = qEy/(uE + vy)$. Here, we note that $h \rightarrow (q/u)y$ as $E \rightarrow \infty$ for a fixed value of y and $h \rightarrow (q/v)E$ as $y \rightarrow \infty$ for a fixed value of E . Thus, the catch rate exhibits solution effects with respect to both the stock and effort levels.

Therefore, we take functional form of harvest as follows [33]:

$$h = \frac{qEy}{uE + vy} \quad (2.2)$$

where E is the harvesting effort used to harvest prey population from unprotected area, q is the catchability coefficient of prey population and u , v are the positive constants.

The migration of prey population from reserved area to unreserved area is considered in the following form [29]:

$$M = \sigma \left(\frac{x}{\alpha K} - \frac{y}{(1-\alpha)K} \right) \quad (2.3)$$

σ the net transfer rate or migration.

Thus, using (2.2) and (2.3), system (2.1) becomes:

$$\begin{aligned}\frac{dx}{dt} &= rx \left(1 - \frac{x}{K\alpha}\right) - \sigma \left(\frac{x}{K\alpha} - \frac{y}{K(1-\alpha)}\right) - \frac{mxz}{a+x}, \\ \frac{dy}{dt} &= sy \left(1 - \frac{y}{K(1-\alpha)}\right) + \sigma \left(\frac{x}{K\alpha} - \frac{y}{K(1-\alpha)}\right) - \frac{nyz}{b+y} - \frac{qEy}{uE+vy}, \\ \frac{dz}{dt} &= \omega z \left(1 - \frac{\gamma z}{x+\beta y}\right) - dz.\end{aligned}\quad (2.4)$$

with initial conditions $x(0) > 0$, $y(0) > 0$, $z(0) > 0$ and $0 < \alpha < 1$.

3 Positivity and boundedness of the solution

In this section, we shall derive some conditions for the positivity and boundedness of the system (2.4).

Theorem 3.1 *If $y(t)$ is always nonnegative then all possible solutions of the system (2.4) is positive.*

Proof From the first equation of the system (2.4) we can write,

$$\begin{aligned}\frac{dx}{dt} &= rx \left(1 - \frac{x}{K\alpha}\right) - \sigma \left(\frac{x}{K\alpha} - \frac{y}{K(1-\alpha)}\right) - \frac{mxz}{a+x} \\ &= x \left[r \left(1 - \frac{x}{K\alpha}\right) - \sigma \left(\frac{1}{K\alpha} - \frac{y}{Kx(1-\alpha)}\right) - \frac{mz}{a+x} \right] \\ \frac{dx}{x} &= \left[r \left(1 - \frac{x}{K\alpha}\right) - \sigma \left(\frac{1}{K\alpha} - \frac{y}{Kx(1-\alpha)}\right) - \frac{mz}{a+x} \right] dt \\ &= \varphi(x, y, z) dt\end{aligned}$$

where $\varphi(x, y, z) = \left[r \left(1 - \frac{x}{K\alpha}\right) - \sigma \left(\frac{1}{K\alpha} - \frac{y}{Kx(1-\alpha)}\right) - \frac{mz}{a+x} \right]$.

Integrating in the region $(0, t)$ where $x(0) > 0$, $y(0) > 0$, $z(0) > 0$ we get,

$$x(t) = e^{\int \varphi(x,y,z)dt} > 0 \quad \forall t.$$

Again, from the 3rd equation of the system (2.4) we get

$$\begin{aligned}\frac{dz}{dt} &= \omega z \left(1 - \frac{\gamma z}{x+\beta y}\right) - dz. \\ &= z \left[\omega \left(1 - \frac{\gamma z}{x+\beta y}\right) - d \right]. \\ \frac{dz}{z} &= \left[\omega \left(1 - \frac{\gamma z}{x+\beta y}\right) - d \right] dt. \\ &= \psi(x, y, z) dt\end{aligned}$$

where $\psi(x, y, z) = \omega \left(1 - \frac{\gamma z}{x+\beta y}\right) - d$

On integration in the interval $[0, t]$ we get $z(t) = e^{\int \psi(x,y,z)dt} > 0 \forall t$.

Hence, according to our proposition if $y(t) > 0 \forall t$ then we may conclude that all the solutions of the system (2.4) are always positive.

In the rest part of our analysis, we assume that $y(t)$ is always nonnegative so that the solutions of the system (2.4) are always positive. In the next theorem, we try to find some sufficient condition for which the solutions of the system (2.4) are bounded. $\square$

Theorem 3.2 *If $\sigma > Ks(1 - \alpha)$ then the solution of the system (2.4) are bounded above.*

Proof From the first equation of the system (2.4) we may conclude that,

$$x(t) \leq K\alpha, \forall t$$

Again, from the third equation of the system (2.4) we can write,

$$\gamma z(t) \leq (x(t) + \beta y(t)), \forall t$$

$$\text{or, } z(t) \leq \frac{(K\alpha + \beta y(t))}{\gamma}, \forall t.$$

Using above equation in the second equation of the system (2.4) we obtain

$$\begin{aligned} \frac{dy}{dt} &\leq sy \left(1 - \frac{y}{K(1-\alpha)} \right) + \sigma \left(\frac{x}{K\alpha} - \frac{y}{K(1-\alpha)} \right) \\ &= \sigma \frac{x}{K\alpha} - \left(\frac{\sigma}{K(1-\alpha)} - s \right) y - \frac{sy^2}{K(1-\alpha)} \\ &\leq \sigma - \left(\frac{\sigma}{K(1-\alpha)} - s \right) y \\ &= K_1 - \xi y \text{ where } \xi = \frac{\sigma}{K(1-\alpha)} - s, K_1 = \sigma. \end{aligned}$$

$$\text{or, } \frac{dy}{dt} + \xi y \leq K_1.$$

Now if $\sigma > Ks(1 - \alpha)$ then $\xi > 0$ and then integrating above equation we get,

$$y(t) \leq \frac{\xi y(0) - K_1}{\xi} e^{-\xi t} + K_1$$

$$\text{i.e. } y(t) \leq K_1, \forall t.$$

Using above relation we may write that,

$$z(t) \leq \frac{1}{\gamma} (K\alpha + \beta K_1), \forall t.$$

Hence, all the solutions of the system (2.4) are bounded if $\sigma > Ks(1 - \alpha)$. $\square$

Note The above condition is the sufficient condition for boundedness of the system (2.4) but not necessary. Again, if all the solutions of the system (2.4) are positive then we say that the above condition is the sufficient condition for the solutions of the system (2.4) are bounded (i.e. both bounded below and above).

4 Analysis of the model at its interior equilibrium

To analyze the system at its equilibria, we first try to find all possible non-negative equilibria of the system though we particularly interested about the analysis around the interior equilibrium point $P(x^*, y^*, z^*)$ to the system for its natural importance.

Apart from interior equilibrium, we first consider predator free equilibria as $P_1(x_1, y_1, 0)$ where

$$y_1 = -\frac{x_1(1-\alpha)(rx_1 - rK\alpha + \sigma)}{\sigma\alpha}$$

and x_1 can be obtained from the following equation

$$L_6x_1^6 + L_5x_1^5 + L_4x_1^4 + L_3x_1^3 + L_2x_1^2 + L_1x_1 + L_0 = 0 \quad (4.1)$$

where

$$\begin{aligned} L_6 &= sr^3(1-\alpha)^2, \quad L_5 = -rv(1-\alpha)(\sigma - rK\alpha)(2sr(1-\alpha) + 1), \\ L_4 &= rv(1-\alpha)\left(s(1-\alpha)(rK\sigma\alpha + (\sigma - rK\alpha)^2) - r\sigma^3\right) - 2rvs(1-\alpha)^2(\sigma - rK\alpha)^2, \\ L_3 &= v(1-\alpha)(\sigma - rK\alpha)(r\sigma^3 - s(1-\alpha)(rK\sigma\alpha + (\sigma - rK\alpha)^2)) \\ &\quad - 2rs(1-\alpha)uE\sigma\alpha(\sigma - rK\alpha) - rv(1-\alpha)(sK\sigma\alpha(1-\alpha) - \sigma^2\alpha)(\sigma - rK\alpha), \\ L_2 &= s\sigma avK(1-\alpha)^2(\sigma - rK\alpha)^2 - rv(1-\alpha)\sigma^3\alpha - qE\sigma^2\alpha^2K(1-\alpha) \\ &\quad + uE\sigma\alpha(r\sigma^3 - s(1-\alpha)(rK\sigma\alpha + (\sigma - rK\alpha)^2)), \\ L_1 &= \sigma^3\alpha v(1-\alpha)(\sigma - rK\alpha) - qE\sigma^2\alpha^2K(1-\alpha)(\sigma - rK\alpha) + uE\sigma\alpha(\sigma - rK\alpha) \\ &\quad \times (s\sigma\alpha K(1-\alpha) - \sigma^2\alpha), \\ L_0 &= uE\sigma^4\alpha^2 \end{aligned}$$

The interior equilibrium is $P^*(x^*, y^*, z^*)$ where x^*, y^* and z^* is the positive root of system $\dot{x} = \dot{y} = \dot{z} = 0$. We have

$$y^* = \frac{(1-\alpha)x^*(a_1x^{*2} + a_2x^* + a_3)}{\alpha(a_4x^* + a_5)} \quad (4.2)$$

where $a_1 = -\gamma d$, $a_2 = K\alpha\gamma d - a\gamma d - \gamma d\sigma$, $a_3 = aK\alpha\gamma d - mK(\omega - d)$,

$$a_4 = K\beta m(\omega - d)(1-\alpha) - \sigma\gamma d, \quad a_5 = -a\sigma\gamma d.$$

$$\text{and } z^* = \frac{1}{\gamma d}(\omega - d)\left(x^* + \beta \frac{(1-\alpha)x^*(a_1x^{*2} + a_2x^* + a_3)}{\alpha(a_4x^* + a_5)}\right) \quad (4.3)$$

and x^* is the solution of $f(x^*) = 0$.

where

$$\begin{aligned}
 f(x^*) = & \gamma d \left(a_1 (1 - \alpha) x^{*3} + a_2 (1 - \alpha) x^{*2} + (a_3 (1 - \alpha) - \alpha a_4) x^* + \alpha b a_5 \right) \\
 & \times \left[\left(s \left(\frac{(K \alpha a_4 - a_3) x^* - a_2 x^{*2} - a_1 x^{*3} + \alpha K a_5}{\alpha K (a_4 x^* + a_5)} \right) \right. \right. \\
 & + \sigma \left(\frac{-K a_1 x^{*2} - (K a_2 - K (1 - \alpha) a_4) x^* + (a_5 - K a_3)}{K (1 - \alpha) (a_1 x^{*2} + a_2 x^* + a_3)} \right) \\
 & \left. \left. - \frac{q E \alpha (a_4 x^* + a_5)}{v (1 - \alpha) a_1 x^{*3} + v (1 - \alpha) a_2 x^{*2} + v (1 - \alpha) a_3 x^* + u E \alpha a_4 x^* + u E \alpha a_5} \right) \right] \\
 & - n (\omega - d) [a_1 \beta (1 - \alpha) x^{*3} + (a_2 \beta (1 - \alpha) - \alpha a_4) x^{*2} + (\alpha a_5 + \beta (1 - \alpha) a_3) x^*]
 \end{aligned} \tag{4.4}$$

Therefore, after getting the positive solutions of x^* from the (4.4) it is easy to get the interior positive solutions of y^* and z^* from the (4.2) and (4.3).

Theorem 4.1 *The predator-free equilibrium $P_1(x_1, y_1, 0)$ is unstable or there exists a saddle node bifurcation according as*

$$\begin{aligned}
 & \left(\frac{r s x_1 y_1}{K^2 \alpha (1 - \alpha)} + \frac{s y_1^2 \sigma}{K^2 x_1 (1 - \alpha)^2} + \frac{r x_1^2 \sigma}{K^2 y_1 \alpha^2} \right) \\
 & < \left(\frac{q E r x_1 y_1}{K (q E + v y_1)^2 \alpha} + \frac{q E y_1^2 \sigma}{K x_1 (q E + v y_1)^2 (1 - \alpha)} \right)
 \end{aligned}$$

or,

$$\begin{aligned}
 & \left(\frac{r s x_1 y_1}{K^2 \alpha (1 - \alpha)} + \frac{s y_1^2 \sigma}{K^2 x_1 (1 - \alpha)^2} + \frac{r x_1^2 \sigma}{K^2 y_1 \alpha^2} \right) \\
 & > \left(\frac{q E r x_1 y_1}{K (q E + v y_1)^2 \alpha} + \frac{q E y_1^2 \sigma}{K x_1 (q E + v y_1)^2 (1 - \alpha)} \right)
 \end{aligned}$$

Proof At $P_1(x_1, y_1, 0)$, the characteristics equation of the system (2.4) can be written as

$$\begin{aligned}
 & \lambda \left(\lambda^2 + \lambda \left(\frac{s y_1}{K (1 - \alpha)} + \frac{r x_1}{K \alpha} + \frac{x_1 \sigma}{K y_1 \alpha} + \frac{y_1 \sigma}{K x_1 (1 - \alpha)} - \frac{q E y_1}{(q E + v y_1)^2} \right) \right. \\
 & + \left(\frac{r s x_1 y_1}{K^2 \alpha (1 - \alpha)} + \frac{s y_1^2 \sigma}{K^2 x_1 (1 - \alpha)^2} + \frac{r x_1^2 \sigma}{K^2 y_1 \alpha^2} - \frac{q E r x_1 y_1}{K (q E + v y_1)^2 \alpha} \right. \\
 & \left. \left. - \frac{q E y_1^2 \sigma}{K x_1 (q E + v y_1)^2 (1 - \alpha)} \right) \right) = 0
 \end{aligned}$$

It is clear that 0 is a root of the characteristic equation. Let other two roots are λ_1 and λ_2 .

Then, we have $\lambda_1 + \lambda_2 = -\left(\frac{sy_1}{K(1-\alpha)} + \frac{rx_1}{K\alpha} + \frac{x_1\sigma}{Ky_1\alpha} + \frac{y_1\sigma}{Kx_1(1-\alpha)} - \frac{qEy_1}{(qE+vy_1)^2}\right)$ and

$$\lambda_1\lambda_2 = \frac{rsx_1y_1}{K^2\alpha(1-\alpha)} + \frac{sy_1^2\sigma}{K^2x_1(1-\alpha)^2} + \frac{rx_1^2\sigma}{K^2y_1\alpha^2} - \frac{qEr x_1y_1}{K(qE+vy_1)^2\alpha} - \frac{qEy_1^2\sigma}{Kx_1(qE+vy_1)^2(1-\alpha)}.$$

If $\lambda_1\lambda_2 < 0$ then there must exist a positive and a negative root simultaneously and then we must say that the system is unstable, which exhibits the first condition.

If $\lambda_1\lambda_2 > 0$ then the both roots might be positive or negative according to the positive or the negative nature of $\lambda_1 + \lambda_2$.

It is to be noted that the saddle-node equilibrium occurs in nonlinear systems with one zero eigenvalue when the system undergoes the saddle-node bifurcation, where a saddle and a node approach each other, coalesce into a single equilibrium and then disappear.

It is also evident that saddle-nodes are always unstable.

Now, the characteristic equation of the system around its interior equilibrium point $P^*(x^*, y^*, z^*)$ is given by,

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \quad (4.5)$$

where, $b_1 = c_1 - \sigma c_2$, $b_2 = d_1 - \sigma d_2$ and $b_3 = e_1 - \sigma e_2$,

$$c_1 = \frac{rx^*}{K} + \frac{sy^*}{K(1-\alpha)} - \frac{mx^*z^*}{(a+x^*)^2} - \frac{ny^*z^*}{(b+y^*)^2} - \frac{\omega\gamma z^*}{(x^*+\beta y^*)} - \frac{qEy^*}{(uE+vy^*)^2},$$

$$c_2 = -\frac{\gamma\sigma}{Kx^*(1-\alpha)} - \frac{x^*\sigma}{Ky^*\alpha},$$

$$\begin{aligned} d_1 = & \frac{rsx^*y^*}{K^2(1-\alpha)} + \frac{sy^{*2}}{K^2x^*(1-\alpha)^2} + \frac{ny^*z^{*2}\omega\gamma}{(b+y^*)^2(x^*+\beta y^*)} \\ & + \frac{qEy^*z^*(mx^*(x^*+\beta y^*)+\omega\gamma(a+x^*)^2)}{(a+x^*)^2(x^*+\beta y^*)(uE+\beta y^*)^2} + \frac{mx^*z^{*2}}{(a+x^*)^2} \left(\frac{ny^*}{(b+y^*)^2} + \frac{\omega\gamma}{x^*+\beta y^*} \right) \\ & - \frac{qEr x^*y^*}{K(uE+vy^*)^2} - \frac{nr x^*y^*z^*}{K(b+y^*)^2} - \frac{msx^*y^*z^*}{K(1-\alpha)(a+x^*)^2} - \frac{msx^*z^{*2}\gamma\omega}{(x^*+\beta y^*)^2(a+x^*)} \\ & - \frac{ny^*z^{*2}\beta\gamma\omega}{(x^*+\beta y^*)^2(b+y^*)} - \frac{rx^*y^*z^*\omega\gamma}{K(x^*+\beta y^*)} - \frac{sy^*z^*\gamma\omega}{K(1-\alpha)(x^*+\beta y^*)} \end{aligned}$$

$$\begin{aligned} d_2 = & \frac{Eqy^{*2}}{Kx^*(1-\alpha)(uE+vy^*)^2} + \frac{ny^{*2}z^*}{Kx^*(1-\alpha)(b+y^*)^2} + \frac{mx^{*2}z^*}{K(a+x^*)y^*\alpha} \\ & + \frac{y^*z^*\omega\gamma}{Kx^*(1-\alpha)(x^*+\beta y^*)} + \frac{x^*z^*\omega\gamma}{Ky^*\alpha(x^*+\beta y^*)} - \frac{sy^{*2}}{K^2x^*(1-\alpha)} - \frac{rx^{*2}}{K^2y^*\alpha} \end{aligned}$$

$$\begin{aligned}
e_1 = & \frac{mEx^*y^*z^{*2}\omega\gamma}{(a+x^*)(uE+vy^*)^2(x^*+\beta y^*)^2} + \frac{mnx^*y^*z^{*3}\beta\omega\gamma}{(a+x^*)^2(b+y^*)^2(x^*+\beta y^*)^2} \\
& + \frac{qrEx^*y^*z^*\omega\gamma}{K(uE+vy^*)^2(x^*+\beta y^*)} + \frac{nrx^*y^*z^{*2}\omega\gamma}{K(b+y^*)^2(x^*+\beta y^*)} \\
& + \frac{qmEx^*y^*z^{*2}\omega\gamma}{(a+x^*)^2(uE+vy^*)^2(x^*+\beta y^*)} + \frac{msx^*y^*z^{*2}\omega\gamma}{K(1-\alpha)(a+x^*)^2(x^*+\beta y^*)} \\
& - \frac{msx^*y^*z^{*2}\omega\gamma}{K(1-\alpha)(a+x^*)(x^*+\beta y^*)^2} - \frac{nrx^*y^*z^{*2}\beta\omega\gamma}{K(b+y^*)(x^*+\beta y^*)^2} \\
& - \frac{mnx^*y^*z^{*2}\omega\gamma}{(a+x^*)^2(b+y^*)^2(x^*+\beta y^*)} - \frac{srx^*y^*z^*\omega\gamma}{K^2(1-\alpha)(x^*+\beta y^*)} \\
e_2 = & \frac{ny^*z^{*2}\omega\gamma}{K(1-\alpha)(b+y^*)^2(x^*+\beta y^*)^2} + \frac{mx^{*2}z^{*2}\omega\gamma}{Ky^*\alpha(a+x^*)(x^*+\beta y^*)^2} \\
& + \frac{mx^*z^{*2}\beta\omega\gamma}{K\alpha(a+x^*)(x^*+\beta y^*)^2} + \frac{ny^{*2}z^{*2}\beta\omega\gamma}{Kx^*(1-\alpha)(b+y^*)(x^*+\beta y^*)^2} \\
& + \frac{rx^{*2}z^*\omega\gamma}{K^2y^*\alpha(x^*+\beta y^*)} - \frac{qEy^{*2}z^*\omega\gamma}{Kx^*(1-\alpha)(uE+vy^*)^2(x^*+\beta y^*)} \\
& - \frac{ny^{*2}z^{*2}\omega\gamma}{Kx^*(1-\alpha)(b+y^*)^2(x^*+\beta y^*)} - \frac{mx^{*3}z^{*3}\omega\gamma}{Ky^*\alpha(a+x^*)^2(x^*+\beta y^*)}
\end{aligned}$$

It is to be noted that $b_1 > 0$ if $\sigma < \frac{c_1}{c_2}$ and $b_3 > 0$ if $\sigma < \frac{e_1}{e_2}$.

Now, $b_1b_2 - b_3 = f_1\sigma^2 + f_2\sigma + f_3$ where $f_1 = c_2d_2$, $f_2 = e_2 - c_1d_2 - c_2d_1$, $f_3 = c_1d_1$. $\square$

Theorem 4.2 *The sufficient condition for the system (2.4) is locally asymptotically stable around its interior equilibrium point $P^*(x^*, y^*, z^*)$ are*

$$\sigma < \min \left\{ \frac{c_1}{c_2}, \frac{e_1}{e_2} \right\} \text{ and } f_1\sigma^2 + f_2\sigma + f_3 > 0.$$

Proof If the interior equilibrium point $P^*(x^*, y^*, z^*)$ of the system (2.4) exists then its characteristics equation at the interior equilibrium point is given by the (4.5).

Therefore, it is clear that $b_1 > 0$ if $\sigma < \frac{c_1}{c_2}$ and $b_3 > 0$ if $\sigma < \frac{e_1}{e_2}$.

Thus, for $b_1 > 0$ and $b_3 > 0$ we must have $\sigma < \min \left\{ \frac{c_1}{c_2}, \frac{e_1}{e_2} \right\}$.

Again, we know that $b_1b_2 - b_3 > 0$ if $f_1\sigma^2 + f_2\sigma + f_3 > 0$.

Hence, the theorem is proved. $\square$

5 Global stability

In this section, we will use geometric approach to derive the sufficient conditions for global stability of the system at the positive equilibrium. For detailed calculations one can see Chakraborty et al. [34], Li and Muldowney [35], Bunomo et al. [36], Martin [37] etc.

The autonomous system (2.1) can be written as, $\frac{dx}{dt} = f(x)$

$$\text{where } f(x) = \begin{pmatrix} rx \left(1 - \frac{x}{K\alpha}\right) - \sigma \left(\frac{x}{K\alpha} - \frac{y}{K(1-\alpha)}\right) - \frac{mxz}{a+x} \\ sy \left(1 - \frac{y}{K(1-\alpha)}\right) + \sigma \left(\frac{x}{K\alpha} - \frac{y}{K(1-\alpha)}\right) - \frac{nyz}{b+y} - \frac{qEy}{uE+vy} \\ \omega z \left(1 - \frac{\gamma z}{x+\beta y}\right) - dz \end{pmatrix};$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The variational matrix, $V(x)$, of the system (2.1) can be written as:

$$V = \begin{pmatrix} -\frac{rx}{K\alpha} - \frac{\sigma y}{Kx(1-\alpha)} + \frac{mxz}{(a+x)^2} & \frac{\sigma}{K(1-\alpha)} & -\frac{mx}{(a+x)} \\ \frac{\sigma}{K\alpha} & -\frac{sy}{K(1-\alpha)} - \frac{\sigma x}{K\gamma\alpha} + \frac{nyz}{(b+y)^2} + \frac{qEy}{(uE+vy)^2} & -\frac{ny}{b+y} \\ \frac{\omega\gamma z^2}{(x+\beta y)^2} & -\frac{\beta\omega\gamma z^2}{(x+\beta y)^2} & -\frac{\omega\gamma z}{x+\beta y} \end{pmatrix}$$

If $V^{[2]}$ is the second additive compound matrix of V due to Bunomo et al. [36], we can write as

$$V^{[2]} = \frac{df^{[2]}}{dx} = \begin{pmatrix} \frac{mxz}{(a+x)^2} - \frac{rx}{K\alpha} - \frac{\sigma y}{Kx(1-\alpha)} & \frac{\sigma}{K(1-\alpha)} & -\frac{mx}{(a+x)} \\ \frac{nyz}{(b+y)^2} - \frac{sy}{K(1-\alpha)} - \frac{\sigma x}{K\gamma\alpha} & -\frac{\omega\gamma z}{x+\beta y} - \frac{\sigma y}{Kx(1-\alpha)} & -\frac{ny}{b+y} \\ +\frac{qEy}{(uE+vy)^2} & +\frac{mxz}{(a+x)^2} - \frac{rx}{K\alpha} & \\ \frac{\omega\gamma z^2}{(x+\beta y)^2} & -\frac{\beta\omega\gamma z^2}{(x+\beta y)^2} & \frac{nyz}{(b+y)^2} - \frac{sy}{K(1-\alpha)} - \frac{\sigma x}{K\gamma\alpha} \\ & +\frac{qEy}{(uE+vy)^2} - \frac{\omega\gamma z}{x+\beta y} \end{pmatrix}$$

We consider $M(x)$ in $C^1(D)$ in a way that $M = \text{diag} \left\{ \frac{x}{z}, \frac{x}{z}, \frac{x}{z} \right\}$. Then we can write

$$M^{-1} = \text{diag} \left\{ \frac{z}{x}, \frac{z}{x}, \frac{z}{x} \right\} \text{ and}$$

$$M_f = \frac{dM}{dx} = \text{diag} \left\{ \frac{\dot{x}}{z} - \left(\frac{x}{z^2} \right) \dot{z}, \frac{\dot{x}}{z} - \left(\frac{x}{z^2} \right) \dot{z}, \frac{\dot{x}}{z} - \left(\frac{x}{z^2} \right) \dot{z} \right\}$$

Thus easily we can show that,

$$M_f M^{-1} = \text{diag} \left\{ \frac{\dot{x}}{x} - \frac{\dot{z}}{z}, \frac{\dot{x}}{x} - \frac{\dot{z}}{z}, \frac{\dot{x}}{x} - \frac{\dot{z}}{z} \right\}$$

and $MV^{[2]}M^{-1} = V^{[2]}.$

So calculating we get,

$$\begin{aligned}
 B &= M_f M^{-1} + M V^{[2]} M^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \\
 B_{11} &= \left(\frac{\dot{x}}{x} - \frac{\dot{z}}{z} - \frac{rx}{K\alpha} - \frac{\sigma y}{Kx(1-\alpha)} + \frac{mxz}{(a+x)^2} - \frac{sy}{K(1-\alpha)} \right. \\
 &\quad \left. - \frac{\sigma x}{Ky\alpha} + \frac{nyz}{(b+y)^2} + \frac{qEy}{(uE+vy)^2} \right) \\
 B_{21} &= \left(\frac{\sigma}{K\alpha} - \frac{\omega\gamma z^2}{(x+\beta y)^2} \right)^t \\
 B_{12} &= \left(\frac{\sigma}{K(1-\alpha)} - \frac{mx}{a+x} \right) \\
 B_{22} &= \begin{pmatrix} \frac{\dot{x}}{x} - \frac{\dot{z}}{z} - \frac{\omega\gamma z}{x+\beta y} - \frac{rx}{K\alpha} - \frac{\sigma y}{Kx(1-\alpha)} + \frac{mxz}{(a+x)^2} & -\frac{ny}{b+y} \\ -\frac{\beta\omega\gamma z^2}{(x+\beta y)^2} & \frac{\dot{x}}{x} - \frac{\dot{z}}{z} - \frac{sy}{K(1-\alpha)} - \frac{\sigma x}{Ky\alpha} + \frac{nyz}{(b+y)^2} \\ & + \frac{qEy}{(uE+vy)^2} - \frac{\omega\gamma z}{x+\beta y} \end{pmatrix}
 \end{aligned}$$

Now we set to define the following vector norm in R^3 $|< u, v, w >| = \max\{|u|, |v| + |w|\}$ where (u, v, w) is the vector is the vector norm in R^3 and it is denoted by Γ

$$\begin{aligned}
 \Gamma(B) &\leq \{p_1, p_2\}; \quad p_i = \Gamma_1(B_{ii}) + |B_{ij}|. \\
 \Gamma(B_{11}) &= \frac{\dot{x}}{x} - \frac{\dot{z}}{z} - \frac{rx}{K\alpha} - \frac{\sigma y}{Kx(1-\alpha)} + \frac{mxz}{(a+x)^2} - \frac{sy}{K(1-\alpha)} - \frac{\sigma x}{Ky\alpha} \\
 &\quad + \frac{nyz}{(b+y)^2} + \frac{qEy}{(uE+vy)^2} \\
 B_{12} &= \max \left\{ \frac{\sigma}{K(1-\alpha)}, \left| -\frac{mx}{a+x} \right| \right\} \\
 B_{21} &= \max \left\{ \frac{\sigma}{K\alpha}, \left| -\frac{\omega\gamma z^2}{(x+\beta y)^2} \right| \right\}. \\
 \Gamma(B_{22}) &= \frac{\dot{x}}{x} - \frac{\dot{z}}{z} - \frac{\omega\gamma z}{x+\beta y} + \max \left\{ -\frac{rx}{K\alpha} - \frac{\sigma y}{Kx(1-\alpha)} + \frac{mxz}{(a+x)^2}, \right. \\
 &\quad \left. -\frac{sy}{K(1-\alpha)} - \frac{\sigma x}{Ky\alpha} + \frac{nyz}{(b+y)^2} + \frac{qEy}{(uE+vy)^2} \right\}
 \end{aligned}$$

$$\begin{aligned}
p_1 &= \Gamma_1(B_{11}) + B_{12} \\
&= \frac{\dot{x}}{x} - \frac{\dot{z}}{z} - \frac{rx}{K\alpha} - \frac{\sigma y}{Kx(1-\alpha)} + \frac{mxz}{(a+x)^2} - \frac{sy}{K(1-\alpha)} - \frac{\sigma x}{K\gamma\alpha} \\
&\quad + \frac{nyz}{(b+y)^2} + \frac{qEy}{(uE+vy)^2} + \max \left\{ \frac{\sigma}{K(1-\alpha)}, \left| -\frac{mx}{a+x} \right| \right\} \\
&= \frac{\dot{x}}{x} - \omega + d + \frac{\omega\gamma z}{(x+\beta y)} - \frac{rx}{K\alpha} - \frac{\sigma y}{Kx(1-\alpha)} + \frac{mxz}{(a+x)^2} - \frac{sy}{K(1-\alpha)} \\
&\quad - \frac{\sigma x}{K\gamma\alpha} + \frac{nyz}{(b+y)^2} + \frac{qEy}{(uE+vy)^2} + \max \left\{ \frac{\sigma}{K(1-\alpha)}, \left| -\frac{mx}{a+x} \right| \right\}
\end{aligned}$$

$$\begin{aligned}
p_2 &= \Gamma_2(B_{22}) + B_{21} \\
&= \frac{\dot{x}}{x} - \omega + d + \max \left\{ -\frac{rx}{K\alpha} - \frac{\sigma y}{Kx(1-\alpha)} + \frac{mxz}{(a+x)^2}, -\frac{sy}{K(1-\alpha)} - \frac{\sigma x}{K\gamma\alpha} \right. \\
&\quad \left. + \frac{nyz}{(b+y)^2} + \frac{qEy}{(uE+vy)^2} \right\} + \max \left\{ \frac{\sigma}{K\alpha}, \left| -\frac{\omega\gamma z^2}{(x+\beta y)^2} \right| \right\} \\
&= \frac{\dot{x}}{x} - \omega + d + \max \left\{ -\frac{rx}{K\alpha} - \frac{\sigma y}{Kx(1-\alpha)} + \frac{mxz}{(a+x)^2}, -\frac{sy}{K(1-\alpha)} - \frac{\sigma x}{K\gamma\alpha} + \frac{nyz}{(b+y)^2} \right\} \\
&\quad \left\{ +\frac{qEy}{(uE+vy)^2}, \frac{\sigma}{K\alpha}, \left| -\frac{\omega\gamma z^2}{(x+\beta y)^2} \right| \right\}
\end{aligned}$$

$$\begin{aligned}
\Gamma(B) &= \frac{\dot{x}}{x} - \omega + d \\
&\quad - \min \left\{ \frac{rx}{K\alpha} + \frac{\sigma y}{Kx(1-\alpha)} + \frac{sy}{K(1-\alpha)} + \frac{\sigma x}{K\gamma\alpha} \right. \\
&\quad \left. - \left(\frac{\omega\gamma z}{(x+\beta y)} + \frac{mxz}{(a+x)^2} + \frac{nyz}{(b+y)^2} + \frac{qEy}{(uE+vy)^2} \right) - \max \left\{ \frac{\sigma}{K(1-\alpha)}, \frac{mx}{a+x} \right\}, \right. \\
&\quad \left. \max \left\{ -\frac{rx}{K\alpha} - \frac{\sigma y}{Kx(1-\alpha)} + \frac{mxz}{(a+x)^2}, \frac{nyz}{(b+y)^2} + \frac{qEy}{(uE+vy)^2} - \frac{sy}{K(1-\alpha)} - \frac{\sigma x}{K\gamma\alpha}, \right. \right. \\
&\quad \left. \left. \frac{\sigma}{K\alpha}, \left| -\frac{\omega\gamma z^2}{(x+\beta y)^2} \right| \right\} \right\}
\end{aligned}$$

Now, we assume that there exists a positive $\mu_1 \in \Re$ and $t_1 > 0$ such that $\mu_1 = \inf \{x(t), y(t), z(t)\}$ whenever $t > t_1$. Also we take,

$$\begin{aligned}
\mu_2 &= \max \left\{ \frac{\sigma}{K(1-\alpha)}, \frac{m\mu_1}{a+\mu_1} \right\} \\
\mu_3 &= \max \left\{ -\frac{r\mu_1}{K\alpha} - \frac{\sigma}{K(1-\alpha)} + \frac{m\mu_1^2}{(a+\mu_1)^2}, -\frac{s\mu_1}{K(1-\alpha)} - \frac{\sigma}{K\alpha} \right. \\
&\quad \left. + \frac{n\mu_1}{(b+\mu_1)^2} + \frac{qE\mu_1}{(uE+v\mu_1)^2}, \frac{\sigma}{K\alpha}, \left| -\frac{\omega\gamma}{(1+\beta)^2} \right| \right\} \\
\mu_4 &= \min \left\{ \frac{r\mu_1}{K\alpha} + \frac{\sigma}{K(1-\alpha)} + \frac{s}{K(1-\alpha)} + \frac{\sigma}{K\alpha} \right. \\
&\quad \left. - \left(\frac{\omega\mu_1}{(1+\beta)} + \frac{m\mu_1^2}{(a+\mu_1)^2} + \frac{n\mu_1^2}{(b+\mu_1)^2} + \frac{qE\mu_1}{(uE+v\mu_1)^2} \right) - \mu_2, \mu_3 \right\}
\end{aligned}$$

$$\Gamma(B) \leq \frac{\dot{x}}{x} - \omega + d - \min \left\{ \frac{r\mu_1}{K\alpha} + \frac{\sigma}{K(1-\alpha)} + \frac{s}{K(1-\alpha)} + \frac{\sigma}{K\alpha} - \left(\frac{\omega\mu_1}{(1+\beta)} + \frac{m\mu_1^2}{(a+\mu_1)^2} + \frac{n\mu_1^2}{(b+\mu_1)^2} + \frac{qE\mu_1}{(uE+v\mu_1)^2} \right) - \mu_2, \mu_3 \right\}$$

$$= \frac{\dot{x}}{x} - \omega + d - \mu_4$$

$$\text{i.e.}; \Gamma(B) \leq \frac{\dot{x}}{x} - (\omega - d + \mu_4)$$

$$\text{i.e.}, \frac{1}{t} \int_0^t \Gamma(B) ds \leq \frac{1}{t} \ln \left| \frac{x(t)}{x(0)} \right| - (\omega - d + \mu_4)$$

$$\limsup_{t \rightarrow \infty} \sup \frac{1}{t} \int_0^t \Gamma(B(s, x_0)) ds < -(\omega - d + \mu_4) < 0$$

Now, we assert the following theorem to existence of global stability around its interior equilibrium.

Theorem 5.1 *The system (2.4) is globally asymptotically stable around its interior equilibrium if $\omega + \mu_4 > d$ where*

$$\mu_4 = \min \left\{ \frac{r\mu_1}{K\alpha} + \frac{\sigma}{K(1-\alpha)} + \frac{s}{K(1-\alpha)} + \frac{\sigma}{K\alpha} - \left(\frac{\omega\mu_1}{(1+\beta)} + \frac{m\mu_1^2}{(a+\mu_1)^2} + \frac{n\mu_1^2}{(b+\mu_1)^2} + \frac{qE\mu_1}{(uE+v\mu_1)^2} \right) - \mu_2, \mu_3 \right\}$$

With $\mu_1 \in R$ such that for $t_1 > 0$ we have $\mu_1 = \inf \{x(t), y(t), z(t)\}$ whenever $t > t_1$.

6 The optimal control theory

The management of renewable resources can be viewed as a dynamic allocation problem. How much of a resource should be harvested today and how much should be left for tomorrow?

This is a dynamic optimization of the exploitation policy connected with the considered system and our objective is to optimize the total discounted net revenues from the fishery. Symbolically, our strategy is to maximize the present value J which can be formulated as follows:

$$J(E) = \int_{t_0}^{t_f} e^{-\delta t} \left(\frac{pqy}{uE + vy} - c \right) E dt. \quad (6.1)$$

where δ is instantaneous annual discount rate, c is constant fishing cost per unit effort and p is constant price per unit biomass of landed fish harvested from the unreserved area.

Suppose α_δ is an optimal reserve with corresponding states x_δ , y_δ and z_δ .

We are seeking to derive the optimal control α_δ such that,

$$J(\alpha_\delta) = \max \{J(\alpha) : \alpha \in U\}$$

where U is the control set defined by,

$$U = \{\alpha : [t_0, t_f] \rightarrow [0, 1] \mid \alpha \text{ is Lebesgue Measurable}\}.$$

The Hamiltonian of this control problem can be expressed as

$$\begin{aligned} H = & \left\{ \frac{pqy}{uE + vy} - c \right\} E + \lambda_1 \left\{ rx \left(1 - \frac{x}{K\alpha} \right) - \sigma \left(\frac{x}{K\alpha} - \frac{y}{K(1-\alpha)} \right) - \frac{mxz}{a+x} \right\} \\ & + \lambda_2 \left\{ sy \left(1 - \frac{y}{K(1-\alpha)} \right) + \sigma \left(\frac{x}{K\alpha} - \frac{y}{K(1-\alpha)} \right) - \frac{nyz}{b+y} - \frac{qEy}{uE + vy} \right\} \\ & + \lambda_3 \left\{ \omega z \left(1 - \frac{\gamma z}{x + \beta y} \right) - dz \right\} \end{aligned} \quad (6.2)$$

where $\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$ are the adjoint variables.

The transversality conditions can be obtained from $\lambda_i(t_f) = 0$, $i = 1, 2, 3$.

Now it is possible to find the characterization of the optimal control α_δ .

On the set $\{t \mid 0 < \alpha_\delta(t) < 1\}$, we have

$$\begin{aligned} \frac{\partial H}{\partial \alpha} = & \lambda_1 \left(\frac{rx^2}{K\alpha^2} + \left(\frac{y}{K(1-\alpha)^2} + \frac{x}{K\alpha^2} \right) \sigma \right) \\ & - \lambda_2 \left(\frac{sy^2}{K(1-\alpha)^2} + \left(\frac{y}{K(1-\alpha)^2} + \frac{x}{K\alpha^2} \right) \sigma \right) = 0 \text{ at } \alpha_\delta. \end{aligned}$$

Consequently, $\alpha_\delta = \frac{-l_2 - \sqrt{l_2^2 - 4l_1l_3}}{2l_1}$ where

$$\begin{aligned} l_1 = & (Epqy + rx^2\lambda_2 - EqyEpqy + rx^2\lambda_1 - sy^2Epqy + rx^2\lambda_1 + rx\lambda_2 \\ & + y\sigma\lambda_2 - x\lambda_1\sigma - y\lambda_1\sigma), \\ l_2 = & -2x(rx\lambda_2 + \sigma\lambda_2 - \sigma\lambda_1), \\ l_3 = & x(rx\lambda_2 + \sigma\lambda_2 - \sigma\lambda_1) \end{aligned}$$

Finally, the adjoint equations are

$$\begin{aligned} \frac{d\lambda_1}{dt} = \delta\lambda_1 - \frac{\partial H}{\partial x} = \delta\lambda_1 - \lambda_1 \left\{ r \left(1 - \frac{2x}{K\alpha} \right) - \frac{\sigma}{K\alpha} - \frac{mz}{a+x} + \frac{mxz}{(a+x)^2} \right\} \\ + \lambda_2 \frac{\sigma}{K\alpha} + \lambda_3 \frac{\omega\gamma z^2}{(x+\beta y)^2}, \end{aligned} \quad (6.3)$$

$$\begin{aligned} \frac{d\lambda_2}{dt} = \delta\lambda_2 - \frac{\partial H}{\partial y} = \delta\lambda_2 - \frac{pqE}{uE+vy} \left(1 + \frac{vy}{uE+vy} \right) - \frac{\lambda_1\sigma}{K(1-\alpha)} \\ - \lambda_2 \left(\frac{qE}{uE+vy} \left(1 - \frac{vy}{uE+vy} \right) + \frac{nyz}{(b+y)^2} - \frac{nz}{(b+y)} \right) \\ + s \left(1 - \frac{2y}{K(1-\alpha)} \right) - \frac{\sigma}{K(1-\alpha)} - \lambda_3 \frac{\beta\omega\gamma z^2}{(x+\beta y)^2}, \end{aligned} \quad (6.4)$$

$$\frac{d\lambda_3}{dt} = \delta\lambda_3 - \frac{\partial H}{\partial z} = \delta\lambda_3 + \lambda_1 \frac{mx}{a+x} + \lambda_2 \frac{ny}{b+y} + \lambda_3 \left(d - \omega \left(1 - \frac{2\gamma z}{x+\beta y} \right) \right). \quad (6.5)$$

Therefore, we summarize the above analysis by the following theorem:

Theorem 6.1 *There exists an optimal reserve area α_δ and corresponding solutions x_δ, y_δ and $z_\delta J(\alpha)$ over U . Furthermore, there exist adjoint functions λ_1, λ_2 and λ_3 satisfying (6.3), (6.4) and (6.5) with transversality conditions $\lambda_i(t_f) = 0, i = 1, 2, 3$. Moreover; the optimal reserve is given by, $\alpha_\delta = \frac{-l_2 - \sqrt{l_2^2 - 4l_1l_3}}{2l_1}$.*

7 Numerical simulation to study optimal control problem

The numerical simulation of optimal control [38] under various parameters set can be done using fourth-order Runge–Kutta forward backward sweep method [39]. The convergence of this iterative method is based on Hackbush [40]. Let x_0, y_0 and z_0 be the initial value of the respective prey and predator populations.

Firstly, we take the discrete interval $[t_0, t_n]$ at the points $t_i = t_0 + ih (i = 0, 1, 2, \dots, n)$ where h is the time step such that $t_n = t_f$. Now a combination of forward and backward difference approximation is considered to solve the system. The time derivative of state variables can be expressed by their first order forward difference as follows:

$$\begin{aligned} \frac{x_{i+1} - x_i}{h} &= rx_{i+1} \left(1 - \frac{x_{i+1}}{K\alpha} \right) - \sigma \left(\frac{x_{i+1}}{K\alpha} - \frac{y_i}{K(1-\alpha)} \right) - \frac{mx_{i+1}z_i}{a+x_i}, \\ \frac{y_{i+1} - y_i}{h} &= sy_{i+1} \left(1 - \frac{y_{i+1}}{K(1-\alpha)} \right) + \sigma \left(\frac{x_{i+1}}{K\alpha} - \frac{y_{i+1}}{K(1-\alpha)} \right) - \frac{ny_{i+1}z_i}{b+y_{i+1}} - \frac{qEy_{i+1}}{uE+vy_{i+1}}, \\ \frac{z_{i+1} - z_i}{h} &= \omega z_{i+1} \left(1 - \frac{\gamma z_{i+1}}{x_{i+1} + \beta y_{i+1}} \right) - dz_{i+1}. \end{aligned}$$

By using a similar technique, we approximate the time derivative of the adjoint variables by their first order backward difference and we use the approximate scheme as follows:

$$\begin{aligned}\frac{\lambda_1^{n-i} - \lambda_1^{n-i-1}}{h} &= \delta \lambda_1^{n-i-1} - \lambda_1^{n-i-1} \left\{ r \left(1 - \frac{2x_{i+1}}{K\alpha} \right) - \frac{\sigma}{K\alpha} - \frac{mz_{i+1}}{a + x_{i+1}} + \frac{mx_{i+1}z_{i+1}}{(a + x_{i+1})^2} \right\} \\ &\quad + \lambda_2^{n-i} \frac{\sigma}{K\alpha} + \lambda_3^{n-i} \frac{\omega \gamma z_{i+1}^2}{(x_{i+1} + \beta y_{i+1})^2}, \\ \frac{\lambda_2^{n-i} - \lambda_2^{n-i-1}}{h} &= \delta \lambda_2^{n-i-1} - \frac{pqE}{uE + vy_{i+1}} \left(1 + \frac{vy_{i+1}}{uE + vy_{i+1}} \right) - \frac{\lambda_1^{n-i-1}\sigma}{K(1-\alpha)} \\ &\quad - \lambda_2^{n-i-1} \left(\frac{qE}{uE + vy_{i+1}} \left(1 - \frac{vy_{i+1}}{uE + vy_{i+1}} \right) + \frac{ny_{i+1}z_{i+1}}{(b + y_{i+1})^2} \right. \\ &\quad \left. - \frac{nz_{i+1}}{(b + y_{i+1})} + s \left(1 - \frac{2y_{i+1}}{K(1-\alpha)} \right) - \frac{\sigma}{K(1-\alpha)} \right) \\ &\quad - \lambda_3^{n-i-1} \frac{\beta \omega \gamma z_{i+1}^2}{(x_{i+1} + \beta y_{i+1})^2}, \\ \frac{\lambda_3^{n-i} - \lambda_3^{n-i-1}}{h} &= \delta \lambda_3^{n-i-1} + \lambda_1^{n-i-1} \frac{mx_{i+1}}{a + x_{i+1}} + \lambda_2^{n-i-1} \frac{ny_{i+1}}{b + y_{i+1}} \\ &\quad + \lambda_3^{n-i-1} \left(d - \omega \left(1 - \frac{2\gamma z_{i+1}}{x_{i+1} + \beta y_{i+1}} \right) \right)\end{aligned}$$

It may be noted that it is quite difficult to have numerical value of the parameters of the system based on real world observations. On the other hand, it is necessary to have some idea regarding the sensitivity of the parameters in connection to the observed real system. Therefore, the major results described by the simulations presented should be considered from a qualitative, rather than a quantitative point of view. However, numerous scenarios covering the breadth of the biological feasible parameter space were conducted and the results shown above display the breadth of dynamical results collected from all the scenarios tested. MATLAB and Mathematica are the main software used for the purpose of simulation experiments. We, therefore, take here some hypothetical data in order to simulate the system numerically,

$$\begin{aligned}r &= 1.5, \quad K = 100, \quad \sigma = 0.25, \quad m = 0.4, \quad a = 5, \quad n = 0.4, \quad b = 3, \quad u = 0.5, \quad v = 0.4, \\ \beta &= 0.5, \quad \omega = 1.2, \quad \gamma = 0.3, \quad d = 0.001, \quad q = 2, \quad p = 1.2, \quad \delta = 0.01, \quad s = 1.2, \quad E = 5, \\ x_0 &= 2.5, \quad y_0 = 2.6, \quad z_0 = 0.8, \quad c = 0.5, \quad t = 100.\end{aligned}$$

Figure 1 depicts the variation of optimal size of the reserve and species specific population density with the increasing time. It is to be noted that optimal size of the reserve is increased with the increasing time. It is also observed from the figure that though optimal size of the reserve is continuously increasing with time but the populations are getting protection from extinction up to some extent of time if we initially create a 30–40% reserve area. However, after some extent of time optimal size of the reserve is exponentially increased with the increasing time. This may be explained due to the

Fig. 1 Variation of optimal size of the reserve, protected and unprotected prey population and predator population with the increasing time

effect of over exploitation of the resource in unprotected area and increasing rate of migration of the resource from protected to unprotected area. It is evident that protected prey population increases with the increasing time. Subsequently, it is also expected that unprotected prey population decreases with the increasing time. It is interesting to observe that predator population increases with the increasing time. This is due to availability of food (prey population) in the system since predator population consumes both protected and unprotected prey population for their survival. Figure 1 clearly describe these results.

Migration of the populations is primarily dependent on the density of the resource and the carrying capacity of the populations. If the density of two adjacent populations are significantly different then migration is generally taking place from high density population towards the low density population. Figure 2 shows the variation of migration with increasing time in the presence of reserve area. It is clearly observed that initially migration increases with the increasing time but after some extent of time migration decreases. This is due to the exponential increase of optimal size of the reserve with increasing time which is clearly explained in Fig. 1. In other words, as the optimal size of the reserve area increases, carrying capacity of the protected prey population is also increased and eventually migration from protected area to unprotected area decreases.

It is to be noted that predator population consumes both protected and unprotected prey population. Again, it is ensured that prey population is always sustained in the system due

Fig. 2 Variation of migration from protected prey to unprotected prey population, total functional response of predators to prey density, harvesting from unprotected prey population and present value function of the system with the increasing time

to the direct effect of reserve area. Therefore, the rate of predation is getting increased due to the availability of enough food resource in the system. Subsequently, total functional response of predators to prey density increases with the increasing time. Figure 2 illustrates the result.

The optimal revenue earned from the system is entirely dependent on the harvesting and availability of the resource. It is clearly observed from Fig. 2 that harvesting of the resource is decreased with the increasing time. We may recall Fig. 1 in this respect since optimal size of the reserve exponentially increased after some extent of time and the availability of the resource in the unprotected area automatically decreased. In consequence to this, it is expected that present value function of the system should be decreased with the increasing time and ultimately tend to zero which is clearly shown in Fig. 2. However, it may also be noted that marine reserve can control the optimality of the system in economic sense.

Harvesting has a strong impact on the dynamic evaluation of a population subjected to it. First of all, depending on the nature of the applied harvesting strategy, the long run stationary density of population may be significantly smaller than the long run stationary density of a population in the absence of harvesting. Therefore, while a population can in the absence of harvesting be free of extinction risk, harvesting can lead to the incorporation

Fig. 3 Variation of protected prey population with the optimal size of the reserve area. The *solid line* corresponds to $E = 5.0$, the *dotted line* to $E = 3.5$ and *dashed line* to $E = 7.0$

of a positive extinction probability and therefore, to potential extinction in finite time. Secondly, if a population is subjected to a positive extinction rate then harvesting can drive the population density to a dangerously low level at which extinction becomes sure no matter how the harvester affects the population afterwards. In this regard, the simultaneous effects of a regulatory mechanism like creation of optimal reserve area and applied harvesting strategy can lead a system to its optimal stage. This optimal system can achieve the sustainable development of the resource through keeping the economic interest of harvesting at an ideal level.

Figures 3, 4 and 5 depict the variation of the optimal prey and predator population with increasing size of the optimal reserve area. It is observed from Fig. 3 that protected prey

Fig. 4 Variation of unprotected prey population with optimal size of the reserve area. The *solid line* corresponds to $E = 5.0$, the *dotted line* to $E = 3.5$ and *dashed line* to $E = 7.0$

Fig. 5 Variation of predator population with optimal size of the reserve area. The *solid line* corresponds to $E = 5.0$, the *dotted line* to $E = 3.5$ and *dashed line* to $E = 7.0$

population increases with the increasing size of the optimal reserve area. It may be noted that density of the protected prey population decreases with increasing effort used to harvest the resource. It is also clearly observed from Fig. 4 that unprotected prey population decreases with increasing size of the optimal reserve area. Moreover, it is clear from the figure that unprotected prey population may get some protection if harvesting effort is simultaneously controlled with the optimal size of the reserve area. Figure 5 shows that predator population increases with increasing size of the optimal reserve area.

Fig. 6 Variation of migration from protected prey population to unprotected prey population with optimal size of the reserve area. The *solid line* corresponds to $E = 5.0$, the *dotted line* to $E = 3.5$ and *dashed line* to $E = 7.0$

Fig. 7 Variation of total functional response of predators to prey density with optimal size of the reserve area. The *solid line* corresponds to $E = 5.0$, the *dotted line* to $E = 3.5$ and *dashed line* to $E = 7.0$

It is interesting to observe from Fig. 6 that migration of the population from protected area to unprotected area initially exponentially increases with simultaneous increase of harvesting effort and optimal reserve area but after a certain size of reserve area it is decreased and finally tends to zero. However, it is clear from the figure that migration is inversely proportional with harvesting effort. It is also noted from the figure that migration of the population is exponentially increased up to 30–40% of the reserve area. This

Fig. 8 Variation of harvesting of unprotected prey population with optimal size of the reserve area. The *solid line* corresponds to $E = 5.0$, the *dotted line* to $E = 3.5$ and *dashed line* to $E = 7.0$

behaviour can be explained due to exponential increase of reserve area after some extent of time which is clearly observed in Fig. 1.

It is evident from Fig. 7 that total functional response of predators to prey density increases with increasing size of the optimal reserve area. It may also be observed that harvesting effort has a strong effect on functional response of predators to prey density. Functional response of predators to prey density is inversely proportional to the harvesting effort which is quite natural.

Harvesting has great impact on the dynamics of this system since harvesting rate exhibits solution effects with respect to the stock and effort. It is interesting to observe that fishing effort is controlled in the system through the dynamics of the optimal reserve area which ultimately control the co-existence of the populations. Figures 8 and 9 respectively depict the variation of harvesting and present value function of the system with simultaneous increase of harvesting effort and optimal reserve area. It is clear from the figures that both harvesting and present value function are directly proportional to the harvesting effort. Solution effects with respect to both the stock and effort levels on harvesting rate are clearly reflected in the figures.

The sensitivity of intrinsic growth parameter of prey population on the optimal prey and predator population is described in Figs. 10 and 11. The intrinsic growth rate of the population plays an important role on the dynamics of the system. It is clearly observed from Figs. 10 and 11 that, in presence of harvesting, the density of the populations are directly proportional to the intrinsic growth rate of populations. It is evident from the figures that optimal reserve size increases with the increasing time. The interesting fact of the system is that though the variability in intrinsic growth rate of prey populations has no direct effect on predator population but it is clear from the figure that predator population gets increased with the increasing intrinsic growth rate of prey population. This phenomenon can be easily described using the functional dependency of the populations.

Fig. 9 Variation of present value function of the system with optimal size of the reserve area. The solid line corresponds to $E = 5.0$, the dotted line to $E = 3.5$ and dashed line to $E = 7.0$

Fig. 10 Variation of optimal size of the reserve area, protected prey, unprotected prey and predator population with the increasing time. The *solid line* corresponds to $r = 1.5$, the *dotted line* to $r = 0.5$ and *dashed line* to $r = 2.5$

Economic parameters in a prey–predator type system play an important role to assess the effective consequences of the system towards societal benefits. Thus to achieve economic efficiency, we need to incorporate some measures towards harvesting cost and net economic revenue which we earn from the harvested resource. The formulation of the optimal allocation problem is entirely dependent on economic parameters such as price per unit biomass of catch, fishing cost per unit effort and discount rate. Subsequently, these parameters determine the stock level maximizing the present value of the flow of resource rent over time. Figure 12 reflects the variation of present value function with the simultaneous increase in discount rate and optimal size of the reserve. It is observed that present value function is inversely proportional with instantaneous discount rate. As a result, the analysis presented here comes closer to reflecting the economic concerns and realities of the principal resource users in the system, and as such our objective is to represent an advance over previous efforts to analyze and provide policy recommendations in order to conserve the resource keeping harvesting at an ideal level.

Fig. 11 Variation of optimal size of the reserve area, protected prey, unprotected prey and predator population with the increasing time. The *solid line* corresponds to $s = 1.2$, the *dotted line* to $s = 0.6$ and *dashed line* to $s = 1.8$

Fig. 12 Variation of present value function of the system with optimal size of the reserve area. The *solid line* corresponds to $\delta = 0.01$, the *dotted line* to $\delta = 0.02$ and *dashed line* to $\delta = 0.03$

8 Concluding remarks

The present study deals with a prey–predator system with a fractional reserve zone for prey species. The predator population grows as per the logistic law with an intrinsic growth rate, s and carrying capacity proportional to the total population size of prey. Prey population, available in unreserved area, is only harvested and a more realistic functional form of harvest is taken into consideration.

This paper has mainly analyzed the use of protected areas as a tool towards building optimal strategy for sustainable use of natural resources like fisheries. We have also focused on the economic and biological conditions where net economic rent of the fishery can be improved through the use of a protected area. However, both use and non-use values from other users of the marine environment is not included in the analysis. It is observed that marine protected areas have been shown to yield positive benefits to fisheries in terms of improved resource rent under a number of circumstances. Moreover, the benefit from protected area establishment in a fishery is derived from three different scenarios. Foremost, it may be concluded from the obtained results that, in non-optimally managed fisheries, protected areas potentially reduce the level of effort expended in the fishery; thus, shift the fishery towards optimal exploitation. Second, protected areas can cause a shift in biomass towards the optimal level. Third, the establishment of a marine protected area in multi-species fisheries influences the resource base (changing population proportions). Protected areas can shift the balance of stocks towards more optimal proportions, improving rent and also having distributional effects on the adjacent fisheries.

It is observed that the migration of the resource, from protected area to unprotected area and vice versa, is playing an important role towards the standing stock assessment in both the areas which ultimately control the harvesting efficiency and enhance the fishing stock to reach its extinct limit. We have also directed the utilities of marine protected area towards the ecosystem functioning where ecosystem are easily disrupted by fishing efforts. The optimal size of reserve may be incorporated in the system based on the level of effort expended towards exploitation. In general, it may also be concluded that marine protected areas enables us to (i) protect and restore the ecosystems in multispecies system, (ii) ensure that the species and habitats found there can grow and are not threatened or damaged, (iii) provide the sustainable use of natural resources. Hence, the results obtained in this paper can be used to identify the benefits and disadvantages of using MPAs as sole management tool, as well as when it can be used in combination with more traditional management strategies such as effort control.

Moreover, the biological measures can improve the discounted value of the fishery through optimal exploitation. It is therefore expected that the outcomes in the form of improved management tool will enhance the livelihood prospects and socioeconomic aspects of the fishery. In this regard, it may be concluded that the creation of reserve incorporate several positive effects on the stock of the resource as well as on the market economics of the fishery. Hence, marine protected areas can be used as an effective management tool to improve resource rent under a number of circumstances. However, the effectiveness of MPAs in relation to risk and uncertainty is still required to be further studied.

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