The vortex patch problem for the surface quasi-geostrophic equation

José Luis Rodrigo*

Department of Mathematics, Princeton University, Washington Road, Princeton, NJ 08544-1000

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In this article, the analog of the Euler vortex patch problem for the surface quasi-geostrophic equation is considered.

The surface quasi-geostrophic (QG) equation was originally introduced as a model for atmospheric turbulence. It represents the evolution of the temperature on the 2D boundary of a half space with small Rossby and Ekman numbers. For a more detailed analysis of the geophysical properties of QG see ref. 1. The QG equation is given by

$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + u \cdot \nabla \theta = 0, \quad [1]$$

where

$$u = (u_1, u_2) = \left(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x} \right) \quad [2]$$

and

$$(-\Delta)^{1/2}\psi = \theta. \quad [3]$$

For simplicity we are considering fronts on the cylinder, i.e., we take (x, y) in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$. In this setting we define $-\Delta^{-1/2}$ that comes from inverting the third equation by convolution with the kernel.[†]

$$\frac{\chi(u, v)}{(u^2 + v^2)^{1/2}} + \eta(u, v),$$

where $\chi(x, y) \in C_0^\infty$, $\chi(x, y) = 1$ in $|x - y| \leq r$ and $\text{supp} \chi$ is contained in $\{|x - y| \leq R\}$ with $0 < r < R < 1/2$. Also $\eta \in C_0^\infty$, $\eta(0, 0) = 0$.

The mathematical interest lies in the strong connections with the 3D Euler equation. Constantin, Majda, and Tabak (2, 3) noticed that the vorticity ω of the Euler equation satisfies the same equation as $\nabla^\perp \theta$, precisely

$$\frac{D\nabla^\perp \theta}{Dt} = (\nabla v) \nabla^\perp \theta.$$

There are many other analogies that have been analyzed in refs. 2 and 3. Other analyses of the equation are found in refs. 4–11.[‡]

The question we are going to explore here is an analog of the vortex patch problem for 2D Euler equation, i.e. the evolution of a solution that consists of two regions, where the function θ takes the values 0 and 1, separated by a smooth curve (see Fig. 1). For 2D Euler the global regularity of the vortex patches has been proved by Chemin (12). A simpler proof has been obtained by Bertozzi and Constantin (13) and Majda and Bertozzi (14). See also refs. 15 and 16.[§]

We notice that the question we are addressing here is the local existence of a solution for the QG-vortex patch problem. Deciding whether there is global regularity of the solutions is a very interesting open problem.

I refer the reader to ref. 17 for a discussion of a possible relationship between the problem studied here and a problem arising from 3D Euler.

We need the following definition:

Definition. A bounded function θ is a weak solution of QG if for any $\phi \in C_0^\infty(\mathbb{R}/\mathbb{Z} \times \mathbb{R} \times [0, \varepsilon])$ we have

$$\int_{\mathbb{R}^+ \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}} \theta(x, y, t) \partial_t \phi(x, y, t) dy dx dt + \int_{\mathbb{R}^+ \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}} \theta(x, y, t) u(x, y, t) \cdot \nabla \phi(x, y, t) dy dx dt = 0, \quad [4]$$

where u is determined by Eqs. 2 and 3.

In the case of a sharp front we have the following form for the scalar

$$\theta(x, y, t) = 1 \quad \text{if } y \geq \varphi(x, t) \quad \text{and} \quad \theta = 0 \text{ otherwise.} \quad [5]$$

For a sharp front for the surface QS equation we have the following results.

Theorem 1. If θ is a weak solution of the surface QS (see Definition) of the form described in Eq. 5, then the function φ satisfies the equation,

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(x, t) = & \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial \varphi}{\partial x}(x, t) - \frac{\partial \varphi}{\partial u}(u, t)}{[(x - u)^2 + (\varphi(x, t) - \varphi(u, t))^2]^{1/2}} \\ & \cdot \chi(x - u, \varphi(x, t) - \varphi(u, t)) du \\ & + \int_{\mathbb{R}/\mathbb{Z}} \left[\frac{\partial \varphi}{\partial x}(x, t) - \frac{\partial \varphi}{\partial u}(u, t) \right] \eta(x - u, \varphi(x, t) \\ & - \varphi(u, t)) du. \end{aligned} \quad [6]$$

Theorem 2. Given any periodic, smooth function $\varphi_0(x)$ the initial value problem determined by Eq. 6 with initial data $\varphi(x, 0) = \varphi_0(x)$ has a unique smooth solution for a small time, determined by the initial data φ . Moreover the function θ defined by Eq. 5 is a weak solution of the QG equation.

Abbreviation: QC, quasi-geostrophic.

*E-mail: jrodrigo@math.princeton.edu.

[†]To avoid irrelevant considerations at ∞ , we will take η to be compactly supported.

[‡]Constantin, P., Proceedings of the Workshop on Earth Climate as a Dynamical System, September 25–26, 1992, Argonne, IL; ANL/MCSTM-170.

[§]Krasny, R., Proceedings of the International Congress of Mathematics, August 21–29, 1990, Kyoto, Japan.

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Fig. 1. Sharp front.

To prove *Theorem 1* we substitute the above expression for θ in Eq. 4 and try to obtain an equation for the evolution of the curve φ .

Notice that by using Eqs. 2 and 3 we obtain $u(x, y, t) = K * \theta(x, y, t)$, where K looks locally like the orthogonal of the Riesz transform. More details about the Riesz transform can be found in refs. 18 and 19.

$$u = \nabla^\perp (\Delta^{-1/2} \theta),$$

we obtain that u is in BMO (see refs. 19 and 20 for more details) and hence the integrals below will make perfect sense.

$$\begin{aligned} \int_{\mathbb{R}^+ \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}} \theta \partial_t \phi \, dy dx dt &= \int_{y > \varphi(x, t)} \partial_t \phi \, dx dy dt \\ &= \int_{y = \varphi(x, t)} \phi \frac{\partial_t \varphi}{(1 + (\partial_x \varphi)^2 + (\partial_t \varphi)^2)^{1/2}} \\ &\quad \cdot (1 + (\partial_x \varphi)^2 + (\partial_t \varphi)^2)^{1/2} \, dx dt \\ &= \int_{y = \varphi(x, t)} \phi \partial_t \varphi \, dx dt. \end{aligned}$$

As for the other term in Eq. 4 (considering only the space integration)

$$\begin{aligned} \int_{\mathbb{R} \times \mathbb{R}/\mathbb{Z}} \theta u \cdot \nabla \phi \, dy dx &= \lim_{\delta \rightarrow 0} \int_{y > \varphi(x, t) + \delta} u \cdot \nabla \phi \, dx dy \\ &= \lim_{\delta \rightarrow 0} \int_{y = \varphi(x, t) + \delta} u \phi \left(\frac{\partial \varphi}{\partial x'} - 1 \right) dx. \end{aligned}$$

Now we look more closely to the integrand of the above expression. We have

$$\begin{aligned} u \phi \left(\frac{\partial \varphi}{\partial x'} - 1 \right) &= \phi(x, y, t) \int_{u > \varphi(u, t)} K(x - u, y - v) \\ &\quad \cdot \left(\frac{\partial \varphi}{\partial x'} - 1 \right) dudv. \end{aligned}$$

The precise expression of K is given by

$$K(u, v) = \nabla^\perp \left\{ \frac{\chi(u, v)}{(u^2 + v^2)^{1/2}} + \eta(u, v) \right\},$$

where $\chi(u, v)$ is supported in $|u - v| \leq 1/4$ and η is compactly supported.

And so[†]

$$\begin{aligned} &u \phi \left(\frac{\partial \varphi}{\partial x'} - 1 \right) \\ &= \phi(x, y, t) \int_{v > \varphi(u, t)} - \left(1, \frac{\partial \varphi}{\partial x} \right) \\ &\quad \cdot \nabla_{u, v} \left\{ \frac{\chi(x - u, y - v)}{((x - u)^2 + (y - v)^2)^{1/2}} + \eta(x - u, y - v) \right\} dudv \\ &= \phi(x, y, t) \int_{v > \varphi(u, t)} - \operatorname{div}_{u, v} \left(\frac{\chi(x - u, y - v)}{((x - u)^2 + (y - v)^2)^{1/2}} \right) \\ &\quad + \eta(x - u, y - v), \\ &\quad \frac{\partial \varphi}{\partial x} \left(\frac{\chi(x - u, y - v)}{((x - u)^2 + (y - v)^2)^{1/2}} + \eta(x - u, y - v) \right) dudv \\ &= \phi(x, y, t) \int_{v = \varphi(u, t)} - \left\{ \frac{\chi(x - u, y - v)}{((x - u)^2 + (y - v)^2)^{1/2}} \right. \\ &\quad \left. + \eta(x - u, y - v) \right\} \left(1, \frac{\partial \varphi}{\partial x} \right) \cdot \left(\frac{\partial \varphi}{\partial u'} - 1 \right) du \\ &= \phi(x, y, t) \left\{ \int_{v = \varphi(u, t)} - \frac{\frac{\partial \varphi}{\partial u} - \frac{\partial \varphi}{\partial x}}{((x - u)^2 + (y - v)^2)^{1/2}} \right. \\ &\quad \chi(x - u, y - v) \\ &\quad \left. - \left[\frac{\partial \varphi}{\partial u} - \frac{\partial \varphi}{\partial x} \right] \eta(x - u, y - v) du \right\}. \end{aligned}$$

Hence we have

$$\begin{aligned} &\lim_{\delta \rightarrow 0} \int_{y = \varphi(x, t) + \delta} u \phi \left(\frac{\partial \varphi}{\partial x'} - 1 \right) dx \\ &= \int_{y = \varphi(x, t)} - \phi(x, y, t) \int_{v = \varphi(u, t)} \frac{\frac{\partial \varphi}{\partial u} - \frac{\partial \varphi}{\partial x}}{((x - u)^2 + (y - v)^2)^{1/2}} \\ &\quad \chi(x - u, y - v) dudx \\ &\quad + \int_{y = \varphi(x, t)} - \phi(x, y, t) \int_{v = \varphi(u, t)} \left[\frac{\partial \varphi}{\partial u} - \frac{\partial \varphi}{\partial x} \right] \\ &\quad \eta(x - u, y - v) dudx. \end{aligned}$$

Putting these two equalities together we have

[†]We move the $\perp$ that appears in K to the factor $(\partial \varphi / \partial x, -1)$.

$$\begin{aligned}
& \int_{y=\varphi(x,t)} \phi(x,y,t) \frac{\partial \varphi}{\partial t}(x,t) dx dt \\
&= \int_{y=\varphi(x,t)} -\phi(x,y,t) \int_{v=\varphi(u,t)} \frac{\frac{\partial \varphi}{\partial u} - \frac{\partial \varphi}{\partial x}}{((x-u)^2 + (y-v)^2)^{1/2}} \\
&\quad \chi(x-u, y-v) du dx dt \\
&\quad + \int_{y=\varphi(x,t)} -\phi(x,y,t) \int_{v=\varphi(u,t)} \left[\frac{\partial \varphi}{\partial u} - \frac{\partial \varphi}{\partial x} \right] \\
&\quad \eta(x-u, y-v) du dx dt.
\end{aligned}$$

From that equality we obtain the equation we were looking for,

$$\begin{aligned}
\frac{\partial \varphi}{\partial t}(x,t) &= \int_{\mathbb{R}/\mathbb{Z}} \frac{\frac{\partial \varphi}{\partial x}(x,t) - \frac{\partial \varphi}{\partial u}(u,t)}{[(x-u)^2 + (\varphi(x,t) - \varphi(u,t))^2]^{1/2}} \\
&\quad \chi(x-u, \varphi(x,t) - \varphi(u,t)) du \\
&\quad + \int_{\mathbb{R}/\mathbb{Z}} \left[\frac{\partial \varphi}{\partial x}(x,t) - \frac{\partial \varphi}{\partial u}(u,t) \right] \eta(x-u, \varphi(x,t) - \varphi(u,t)) du.
\end{aligned}$$

[7]

This completes the proof of *Theorem 1*.

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