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Spike width and frequency alter stability of phase-locking in electrically coupled neurons

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Abstract

The stability of phase-locked states of electrically coupled type-1 phase response curve neurons is studied using piecewise linear formulations for their voltage profile and phase response curves. We find that at low frequency and/or small spike width, synchrony is stable, and anti-synchrony unstable. At high frequency and/or large spike width, these phase-locked states switch their stability. Increasing the ratio of spike width to spike height causes the antisynchronous state to transition into a stable synchronous state. We compute the interaction function and the boundaries of stability of both these phase-locked states, and present analytical expressions for them. We also study the effect of phase response curve skewness on the boundaries of synchrony and antisynchrony.

1. Introduction

Electrical coupling between neurons leads to faster communication than chemical synapses and plays important role in a variety of brain functions [1, 2, 3, 4]. Though electrical synapses are often found along with gabaergic chemical synapses in a number of brain areas [5, 6, 7, 8], electrical synapses themselves can significantly affect synchrony between the coupled neurons [9, 10, 11, 12, 13, 14]. A number of theoretical works dealt with understanding the essential components responsible for such synchrony or failure of synchrony among electrically coupled neurons, some of them using relaxation oscillator models [9, 15, 11], and others using integrate-and-fire model neurons [16, 17, 18, 19]. The latter approach is based on the widely successful theory of weakly coupled oscillators [20, 21, 22, 23, 24] that requires specification of the coupling mechanism including the voltage time course and the neuron's phase response curve (PRC). A knowledge of these two elements is sufficient to completely predict when and why a set of identical neurons synchronize. This approach allows one to generalize the results to not only integrate-and-fire models whose PRCs are heavily tilted toward longer phases (hence skewed) [17, 18] but to other PRCs that are representative of the type of neurons [25].

A requirement for applying the theory of weakly coupled oscillators is that the individual neurons spike regularly. Emergence of regular oscillations in neurons could occur either via a saddle-node like bifurcation that results in infinitely long period orbits or via a Hopf bifurcation that results in sudden emergence of oscillation frequency [25, 26, 27]. Associated with the saddle-node (type-1) bifurcation, it was shown that the phase response curve near the bifurcation has a particularly simple form, or the canonical form [25]. The canonical form of a type-1 phase response curve (PRC) has been extensively used to predict synchronous behavior of coupled spiking neurons when the coupling is synaptically driven [28, 29]. Although the PRC could acquire some shape distortion with frequency, the canonical form provides a test case for studying synchronous behavior of many other type-1 PRC neurons. The quadratic integrate-and-fire (QIF) model is indeed equivalent to the canonical type-1 PRC neuron model [30, 25], and was used to study phase-locked states in the past in some detail [19]. However, just like the leaky integrate-and-fire models, the QIF

models do not possess a spike profile. QIF models may also cause the PRCs acquire discontinuous jumps at the edges with increasing frequency [19].

We wish to investigate the role of voltage shape parameters on the phase-locking of a pair of electrically coupled type-1 PRC neurons. Our aim is to be able to freely alter the spike width, height, and threshold. But in experiments and models, the PRC and the voltage are tightly related [28, 31, 32, 33]. Such interdependence also makes the task of relating each of the shape parameters exclusively to the emerging phase-locked states difficult. Thus we set out to treat them independent by freely varying the shape parameters such that the sensitivity of each shape parameter can be discovered. To achieve this goal we analyze the dynamics based on the interaction function that does not require detailed dynamical evolution equations of the oscillators, but only requires a specification of the voltage and the PRC shapes. Both the PRC and the voltage are formulated with piecewise linear curves. This formulation also allows us to derive explicit formulas for the boundaries of synchrony and antisynchrony.

A symmetric type-1 PRC and the voltage shape are formulated and the method to find the stability of the phase-locked states is described in Section 2. An explicit form of the interaction function and the stability boundaries are derived for both zero spike width and non-zero spike width in Section 3. We discover that a sinusoidal interaction function similar to that employed in studies of Kuramoto phase-coupled oscillators is associated with symmetric PRC when spike width is zero. For non-zero spike width the interaction function becomes more positive that would require more odd Fourier terms. Boundaries of the synchronous and antisynchronous states are obtained from an analysis of the eigenvalues. The effect of the temporal location of maximum phase advancement of the PRC, the skewness is studied in Section 4. The interaction function is derived in the case of zero spike width, and the analytical results for the stability of synchrony and antisynchrony are discussed.

2. Model

The canonical form of type-1 PRC is given by a simple functional form, $1 - \cos(2\pi t/T)$, where T is the period of oscillation, and t is the time [25]. This PRC is positive over its entire period, increasing from 0 (at $t = 0$) to 2 (at $t = T/2$) and then decreasing to 0 at $t = T$. The interaction between the oscillators using this PRC is not much different from a piecewise linear (PWL) form that comprises of two segments $4t/T$ from $t = 0$ to $T/2$, and $4(1 - t/T)$ from $t = T/2$ to T , and in particular the ranges of stability of synchronous and antisynchronous states computed numerically are similar to those obtained from the PWL formulation. This is not surprising because when expanded in Fourier series, the above PWL form becomes $1 - 0.81 \cos(2\pi t/T) - 0.09 \cos(6\pi t/T) - 0.032 \cos(10\pi t/T) - \dots$, and the first cosine term has 98.5% of the Fourier power contained in all the oscillating modes. The PWL form is much simpler to solve analytically and thus we can predict the stability boundaries of synchrony and antisynchrony, and obtain formulas for stability in the parameter planes of spike width, frequency, and spike threshold and height. Thus we formulate the PRC as a combination of two PWL curves as follows:

$$Z(t) = \begin{cases} Z_1(t) = 2t, & 0 \leq t < \frac{T}{2}, \\ Z_2(t) = 2(T - t), & \frac{T}{2} \leq t < T. \end{cases} \quad (1)$$

Scaling the amplitude of the PRC by a positive value will scale the eigenvalues and the interaction function with that value, but will not affect the stability boundaries. The PRC

shape is illustrated in Fig. 1(b). The voltage time course is formulated with three piecewise linear curves [Fig. 1(a)] based on Hodgkin-Huxley (HH) model neuron,

$$V(t) = \begin{cases} V_1(t) = V_p - \frac{a_2}{2W}t, & \text{if } 0 \leq t < 2W, \\ V_2(t) = V_m + \frac{a_3}{T - \frac{5}{2}W}(t - 2W), & \text{if } 2W \leq t < T - \frac{W}{2}, \\ V_3(t) = V_{th} + \frac{a_1}{\frac{W}{2}}(t - T + \frac{W}{2}), & \text{if } T - \frac{W}{2} \leq t < T. \end{cases} \quad (2)$$

where $a_1 = V_p - V_{th}$, $a_2 = V_p - V_m > 0$, and $a_3 = V_{th} - V_m = 0$. Note that $a_1 = a_2 - a_3$, $a_2, W > 0$, and $T > 0$. Examining the spike profile reveals that the actual spike width is $5W/2$, but for ease of terminology, we refer to W the spike width parameter. We will also use the normalized width parameter as $W' = W/T$ which is the product of the spike width and frequency. The levels V_p , V_{th} , and V_m are, respectively, the peak, threshold, and maximum hyperpolarization of the voltage time course. The time for downstroke ($2W$) and the upstroke ($W/2$) together should be smaller than the time period, and hence $W < 2T/5$. The region above this condition is marked as forbidden region in the figures. The voltage profile of the HH model at an applied current of $10\mu\text{A}/\text{cm}^2$ can be modeled quite accurately using $V_p = 35\text{ mV}$, $V_m = -72\text{ mV}$, $V_{th} = -48\text{ mV}$, and $W = 1.1\text{ ms}$ ($T = 14.636\text{ ms}$). These spike amplitude parameters are also used later in Figs. 2 and 3. Given the PRC and the voltage time courses, the interaction of the electrically coupled spiking neurons, each oscillating at a steady frequency of ω , can be written as a set of two coupled phase oscillators (with phase θ_1 and θ_2) with their phase evolutions given by [20, 34, 35, 36, 30, 37, 21, 38]

$$d\theta_1/dt = \omega + \varepsilon H(\theta_2 - \theta_1), \quad (3)$$

$$d\theta_2/dt = \omega + \varepsilon H(\theta_1 - \theta_2), \quad (4)$$

where ε is the coupling strength, and $H(\phi)$ is the interaction function defined by

$$H(\phi) = \frac{1}{T} \int_0^T Z(t) [V(t+\phi) - V(t)] dt. \quad (5)$$

The dependence of $H(\phi)$ on the phase difference and its functional form are a direct consequence of employing the theory of weakly coupled oscillators. We will use this expression along with Eqs. 1 and 2 to compute the interaction function by appropriately choosing the limits of integration to segregate PWL regimes that have constant slopes. In order to simplify the presentation, we will often use $Z(t)$ and $V(t)$ and their segments without explicitly mentioning the time argument, the notation $\tilde{Z}_n^{ab} \equiv Z_n[V_a(t+\phi) - V_b(t)]$, and $V_a^\phi \equiv V_a(t+\phi)$, where a and b assume indices 1, 2, 3, and n assumes 1 and 2. The Z and V segments are as defined in Eq. 1 and 2.

Conditions for stability can be obtained either using $H(\phi)$ or directly from $V(t)$ and $Z(t)$ using the formulas derived below in Eqs. 7 and 8. We will use the latter method that gives insights into the regimes of V and Z that contribute to the stability. From Eqs. 3–4, we write

$$\dot{\psi} = \varepsilon G(\psi), \quad (6)$$

where $\psi = \theta_2 - \theta_1$, and the $G(\psi) = H(-\psi) - H(\psi)$ represents the instantaneous growth of the phase difference, which may be termed the growth function. The stability of synchronous state is determined by the eigenvalue of the above equation at $\psi^* = 0$, i.e.

$$\lambda = \varepsilon G'(0) = \frac{-2\varepsilon}{T} \int_0^T Z(t) V'(t) dt. \quad (7)$$

$Z(t)$ has discontinuous derivatives at $t = 0$, $T/2$, and $V(t)$ has discontinuous derivatives at $t = 0$, $2W$ and $t = T - W/2$. Case (a): If $2W$ is less than $T/2$, i.e. if the spike minimum occurs before the PRC acquires its maximum, then the case is depicted in Fig. 2(a), and λ is computed by combining the contributions from the four segments: λ_{1a} , λ_{2a} , λ_{3a} , and λ_{4a} . Case (b): If $2W$ is bigger than $T/2$, i.e. if the spike maximum hyperpolarization occurs after the PRC peak position, then another set of four eigenvalue components λ_{1b} , λ_{2b} , λ_{3b} , and λ_{4b} contribute to the eigenvalue. These are illustrated later in Fig. 2(c). Similarly the stability of antisynchrony is given by the eigenvalue of the Eq. 6 at $\psi^* = T/2$, i.e.

$$\gamma = \varepsilon G'(T/2) = \frac{-2\varepsilon}{T} \int_0^T Z(t) V'(t - T/2) dt. \quad (8)$$

To compute γ , it is best to lay $V(t + T/2)$ along with $Z(t)$ on the same time axis as in Fig. 2(b). The two cases described above apply here. In case (a) when $2W$ is less than $T/2$, the eigenvalue components γ_{1a} , γ_{2a} , γ_{3a} , and γ_{4a} constitute γ , and the corresponding segments are illustrated in Fig. 2(b). And in case (b) when $2W$ is bigger than $T/2$, the corresponding eigenvalue components γ_{1b} , γ_{2b} , γ_{3b} , and γ_{4b} that constitute γ can be computed. The shaded regions in Fig. 2(a,b) indicate segments that contribute positively to the corresponding eigenvalue, and thus could potentially cause instability of that state. These components are illustrated in Fig. 2(d).

3. Results

3.1. Zero spike width

First consider the case of zero spike width when $W = 0$. The PRC is positive and symmetric about its half period. The interaction function can be computed directly from Eq. 5 as

$$H(\phi) = \begin{cases} a_3 T (\frac{1}{2} \frac{\phi}{T} - \frac{\phi^2}{T^2}), & 0 \leq \phi < \frac{T}{2}, \\ a_3 T (\frac{1}{2} - \frac{3}{2} \frac{\phi}{T} + \frac{\phi^2}{T^2}), & \frac{T}{2} \leq \phi < T. \end{cases} \quad (9)$$

This expression can also be obtained as a special case from the interaction function discussed in the next section by setting $W = 0$: The coefficients of $h_4(\phi)$ in Eq. 17 become $f_{4a} = 0$, $f_{4b} = a_3/2$, and $f_{4c} = -a_3$ and result in the first expression above. And the coefficients of $h_7(\phi)$ in Eq. 17 become $f_{7a} = a_3/2$, $f_{7b} = -3a_3/2$, and $f_{7c} = a_3$ leading to the second expression above. A Fourier expansion of Eq. 9 results in the first mode retaining most of the power, and thus may be approximated by

$$H(\phi) = a_3 T 0.0645 \sin(2\pi\phi/T). \quad (10)$$

The second largest mode [$a_3 T 0.0024 \sin(6\pi\phi/T)$] is weaker than the first by more than an order of magnitude, and hence the first mode is a good approximation. This functional form is of Kuramoto like interaction function [39] among coupled phase oscillators. The phase-locked states and their stability can be discerned by using the above approximation in Eq. 6 that gives $\dot{\psi} = -2a_3 T 0.0645 \sin(2\pi\psi/T)$. Plotting the right hand side of this equation visually depicts the phase-locked states (the zero-crossings) and their stability (negative slope indicating stability). Thus we see that synchrony ($\psi^* = 0$) is stable and antisynchrony ($\psi^* = T/2$) unstable.

The presence of higher order Fourier terms in an interaction function could also change the stability of the phase-locked states, but in the present case after finding the actual eigenvalues using Eq. 9 we will see that the stability of the synchrony and antisynchrony are as predicted by the above approximation. From Fig. 2(a) we see that in the limit of zero spike width, the contributions of λ_{1a} and λ_{4a} become zero because the PRC itself is zero in that limit. From the symmetry of the PRC and the constancy of the slope of the voltage we can expect that λ_{2a} and λ_{3a} will be identical. In fact from their expressions given in Section 3.2 we obtain (by setting $W = 0$) each of these components as $-a_3/(2T)$, and thus the total eigenvalue (due to the depolarizing phase of the voltage time course) is $-a_3/T$ which is negative. And hence the synchrony is indeed a stable state in the present model, in contrast to the instability of it in leaky integrate-and-fire model if no spike effects are included [18].

In order to find the stability of the antisynchronous state we need to arrange the voltage and the phase response curves with relative phase shift of half period. This gives us three integrals contributing to the eigenvalue components, one between 0 and $T/2$ with an integrand $(-2/T)Z_1(t)V_2'(t)$, the second between $T/2$ and T with an integrand $(-2/T)Z_2(t)V_2'(t)$, and the third integral just due to the discontinuity in the phase shifted voltage at $t = T/2$ with an integrand $(-2/T)(-a_3/\xi)$ and the limits between $T - \xi/2$ and $T + \xi/2$, where ξ must be taken in the limit of $\xi \rightarrow 0$. Each of the first two integrals evaluates to $-\frac{a_3}{2T}$, and the third that is due to the voltage discontinuity evaluates to a positive quantity, $\frac{2a_3}{T}$. Combining the three components and multiplying by e , we arrive at the eigenvalue as $\frac{\epsilon a_3}{T}$ which is positive. Hence the antisynchronous state is unstable. The rate at which the instability grows is proportional to the maximum advancement in the PRC, and the total amount of depolarization (a_3) measured from the spike minimum.

To see mechanistically how the synchronous state becomes stable, note that a slightly leading oscillator will have its voltage higher than the lagging oscillator for most of its oscillation period, and thus helps in driving the lagging oscillator's phase toward spiking earlier than it would in the absence of coupling. But a small reduction in the lagging oscillator's phase growth is suffered due to the reversal of the voltage magnitude ordering in the small window between the leading and the lagging oscillator. Because the response of the PRC is very small for small phases such reduction really becomes small compared to the growth already received during the oscillation cycle, thus leading to synchrony. This synchrony is conditioned on the smallness of the PRC near early phases, and also on the lack of the contribution of the spike downstroke in the small window alluded to above. Thus for $W = 0$, the synchronous state is always stable.

To see how the antisynchronous state becomes unstable, consider the case of the two neurons firing with nearly but less than half phase lag. Let the first neuron fire at $t = 0$ and let the second neuron fire at $T/2 - t_1$ where t_1 is a small positive constant. The voltage of the lagging second neuron is bigger than the first during the period $T/2 - t_1$ (thus causing an advancement of the first neuron's phase), and is smaller than the first during the rest of the time duration $T/2 + t_1$ (causing a delay of the first neuron's phase), and because of the positive PRC shape, the cumulative effect on the first neuron is a slight delay in its phase thus increasing the separation bigger than $T/2 + t_1$ during the next cycle. This chain of events eventually leads the neurons to run away from the antisynchronous state causing an instability.

3.2. Non-zero spike width

The voltage profile has three PWL segments, and we must time shift it by ϕ in order to compute the integrand in Eq. 5. Consequently we will have six regimes of constant slope segments. Since $Z(t)$ also has a discontinuous derivative at half period, it divides the regime that contains the half period point into two. Thus we will have totally seven regimes that contribute to the computation of $H(\phi)$. The boundaries of these regimes depend on the specific value of the phase shift in relation to the spike width. For small ϕ such that $0 < \phi < W/2$, we can identify the boundaries of these regimes as $L_1 = 2W - \phi$, $L_2 = 2W$, $L_3 = T/2$, $L_4 = T - W/2 - \phi$, $L_5 = T - W/2$, and $L_6 = T - \phi$. The phase regime where these limits are valid are bounded by the curves $\phi = 0$ and $\phi = W/2$ [Fig. 1(c)]. As phase shift ϕ is increased, the shifted voltage overlaps with the un-shifted voltage time course at different phases, requiring us to re-identify the intermediate segments that have constant slopes. For example, when $W/2 < \phi < 2W$ two of the above limits change: $L_5 = T - \phi$, and $L_6 = T - W/2$, and all the other limits remain the same. This second regime is bounded by the curves $\phi = W/2$ and $2W$. Thus as ϕ is shifted toward T we encounter a total of ten such regimes bounded by the curves shown in Fig. 1(c). These ten regimes contribute ten nonlinear segments to the interaction function. Each of these segments is computed using the expression in Eq. 17. As we can clearly see, at zero spike width all these segments converge to partition the ϕ range into just two yielding the interaction function displayed in Eq. 9.

The formula in Eq. 17 is plotted in Fig. 1(d) for a set of spike widths, and the corresponding growth functions are plotted in Fig. 1(e). At very small spike width or frequency, the interaction function is like a sine function, and the synchronous state is stable and the antisynchronous state is unstable. As the factor W' becomes large, $H(\phi)$ becomes more like a cosine function. Although it is not representable by a single cosine, its Fourier expansion would have more odd terms than even terms. The stability of synchrony and antisynchrony are swapped. The transition between these two regimes is mediated by the emergence of other non-zero phase-locked states.

Stability of synchrony and antisynchrony—The arrangement of V and Z for determining the stability of synchrony and that of the antisynchrony are shown respectively in Fig. 2(a) and (b). Because the V_2 and V_3 segments of the voltage have positive slopes and Z is positive throughout, applying Eq. 7 reveals that the eigenvalue components contributed by these two segments of the voltage are negative, and thus contribute to stabilizing the synchrony. The only segment that contributes to destabilizing the synchrony is the spike downstroke segment (shaded). Because the PRC is approaching zero near the edge, the contribution of this segment in the limit of zero spike width is zero. Thus we see why synchrony is stable for small spike width. This mechanism is similar in character to the stabilizing effect of spike width in integrate-and-fire models [18, 13].

A similar thing happens for the antisynchronous state [Fig. 2(b)], i.e. the voltage segments V_2 (which is now split across two regimes), and V_3 contribute toward stabilizing the antisynchrony, but the contribution of V_1 segment (shaded) which becomes positive does not go to zero as the spike width goes to zero on account of the fact that the PRC at half period is finite. In fact the contribution of this segment at small W (thus large slope) becomes very large overpowering the stabilizing contributions from the other segments. Hence the antisynchronous state becomes unstable at small W .

In order to obtain precise formulas for synchrony, we must examine the eigenvalue Eq. 7. From Fig. 2(a), we see that there are four components [$\lambda_{1a} = 4\epsilon a_2 W'/T$, $\lambda_{2a} = -\epsilon a_3(1 - 16W'^2)/[T(2 - 5W')]$, $\lambda_{3a} = -\epsilon a_3(1 - W'^2)/[T(2 - 5W')]$, $\lambda_{4a} = -\epsilon(a_2 - a_3)W'/T$] [Fig. 2(c)] that contribute to the eigenvalue, λ_a as long as the spike minimum occurs before the PRC maximum, i.e. $2W < T/2$ or $W < T/4$. λ_{1a} corresponds to the rate of spike downstroke/

repolarization, λ_{4a} corresponds to the rate of spike upstroke, and the other two components correspond to the depolarizing phase of the voltage time course before and after the half period point. When the spike minimum occurs after the PRC maximum (i.e. when $W < T/4$), then another set of four components (λ_{1b} that is computed by integrating from 0 to $T/2$, λ_{2b} computed from $T/2$ to $2W$, λ_{3b} computed from $2W$ to $T - W/2$, and finally λ_{4b} computed from $T - W/2$ to T) [Fig. 2(d)] contribute to the eigenvalue, λ_b . In this parameter regime, the spike is very broad, and the downstroke effect is carried by both λ_{1b} and λ_{2b} whose positive/destabilizing effect dominates the negative/stabilizing effect of the spike upstroke λ_{4b} and the depolarization λ_{3b} . We first compute the eigenvalue when $W < T/4$. Combining the eigenvalue components, we obtain the eigenvalue as $\lambda_a = \epsilon(x_1 - x_2)x_3$ where $x_1 = 3a_2 W(2T - 5W)$, $x_2 = 2a_3(T^2 - TW - 6W^2)$, and $x_3 = 1/(T^2(2T - 5W))$. We will now show that synchronous state is stable if and only if

$$1 \geq \frac{a_3}{a_2} > \left(\frac{a_3}{a_2}\right)^* \equiv \rho_1 = \frac{3W'(2 - 5W')}{2(1 - W' - 6W'^2)}, \text{ for } W' < \frac{1}{4}. \quad (11)$$

At a non-zero spike width, when $a_3 = 0$, the eigenvalue is clearly positive, because $\lambda|_{a_3=0} = 3\epsilon a_2 W/T^2 > 0$. (Small a_3 implies large spike amplitude.) Thus the synchronous state is unstable at $a_3 = 0$ for non-zero spike width. The eigenvalue becomes zero on the critical curve ρ_1 that is obtained by solving the equation $\lambda_a = 0$. Across this critical curve, the eigenvalue transitions to a negative value indicating stability because $d\lambda_a/da_3|_{a_3=a_2\rho_1} = -2\epsilon(T - 3W)(T + 2W)x_3 < 0$, thus the condition in Eq. 11 specifies the stability of synchrony. The curve ρ_1 is shown in Fig. 2(f).

Next consider the case of $W > T/4$. Then computing the four components [$\lambda_{1b} = \epsilon a_2/(4TW')$, $\lambda_{2b} = \epsilon a_2(4W' - 1)(3 - 4W')/(4TW')$, $\lambda_{3b} = -\epsilon a_3(2 - 3W')/T$, $\lambda_{4b} = -\epsilon(a_2 - a_3)W'/T$] that contribute to the eigenvalue, we obtain $\lambda_b = -\epsilon(x_4 + x_5)x_6$, where $x_4 = 4a_3(T - 2W)W$, $x_5 = a_2(T^2 - 8TW + 10W^2)$, and $x_6 = 1/(2T^2W)$. We will now show that the synchronous state is stable if and only if

$$1 \geq \frac{a_3}{a_2} > \left(\frac{a_3}{a_2}\right)^* \equiv \rho_2 = \frac{8W' - 1 - 10W'^2}{4W'(1 - 2W')} \text{ for } W' \geq \frac{1}{4}. \quad (12)$$

As before, we find that the eigenvalue is positive when $a_3 = 0$ because $\lambda_b|_{a_3=0} = -\epsilon x_5 x_6 = \epsilon a_2 [T(4W - T) + 2W(2T - 5W)] x_6 > 0$. This inequality is obtained by using the condition $W > T/4$ and the constraint on W , $W < 2T/5$. Thus the synchronous state is unstable as before when $a_3 = 0$. The eigenvalue becomes zero on the critical curve ρ_2 which is obtained by solving $\lambda_b = 0$. Across this curve, we see that the eigenvalue transitions to a negative value because $d\lambda_b/da_3|_{a_3=a_2\rho_2} = -2\epsilon(T - 2W)/T^2 < 0$, and thus the synchronous state becomes stable when the condition in Eq. 12 is satisfied. The curve ρ_2 is shown in Fig. 2(f).

Although the conditions derived above are exact, a simple polynomial relation between spike width, frequency, and the spike height parameters may be desirable to better understand the interdependence. A visual examination of the curve ρ_1 in Fig. 2(c) reveals a linear dependence of a_3/a_2 on the normalized spike width. To reveal this we invert the curves ρ_1 and ρ_2 to express them as a function of W' , and then expand ρ_1 around $W' = 0$ and ρ_2 around $W' = 3/4$ in Taylor series. This results in simple approximations for stable synchrony:

$$W' < \begin{cases} \frac{1}{3}\frac{a_3}{a_2} + \frac{1}{6}\left(\frac{a_3}{a_2}\right)^2 + \mathcal{O}(4), & \text{if } \frac{a_3}{a_2} < \frac{3}{4}, \\ \frac{1}{4} + \frac{1}{6}\left(\frac{a_3}{a_2} - \frac{3}{4}\right) + \frac{1}{27}\left(\frac{a_3}{a_2} - \frac{3}{4}\right)^2 + \mathcal{O}(3), & \text{if } \frac{a_3}{a_2} \geq \frac{3}{4}. \end{cases} \quad (13)$$

A similar analysis can be carried out for the stability of antisynchrony using Eq. 8 and Fig. 2(b) for the two conditions, $W < T/4$ and $W > T/4$ that yield respectively γ_a and γ_b . When $W < T/4$, owing to the symmetry of the PRC, we obtain $\gamma_a = -\lambda_a$. Thus the critical curve obtained from the eigenvalue γ_a [Fig. 2(b)] is identical to ρ_1 and the stability criterion is unchanged from Eq. 11 such that when the synchrony becomes unstable, the antisynchrony becomes stable. Similarly, when $W > T/4$, we get $\gamma_b = -\lambda_b$, and thus the critical curve for stability remains the same as ρ_2 , but when the synchrony becomes unstable, antisynchrony becomes stable, and vice versa. A numerically computed bifurcation diagram depicting stable and unstable phase-locked states by solving Eqs. 5 directly is shown in Fig. 2(e), and confirms the boundaries of synchrony and antisynchrony computed analytically. However, if the PRC acquires finite skewness, the symmetry enjoyed by the eigenvalues will not hold, and there may be narrow regimes in the parameter space where neither synchrony nor antisynchrony become stable, or they may exhibit bistability.

In order to find differences in results between the PWL approach and the canonical type-1 PRC that motivated our study, the boundary between synchrony and antisynchrony is computed numerically for the type-1 canonical PRC with the PWL voltage form, and is shown in Fig. 2(f, dashed curve) for comparison. Synchrony is stable at low frequencies, and becomes unstable at high frequencies. When synchrony loses stability across this boundary, the antisynchrony gains stability, and vice versa, similar to the transitions on ρ_1 and ρ_2 . But unlike when the PRC was piecewise linear, the canonical model produced no other phase-locked solutions emerging from the synchronous or antisynchronous solutions. The disparity between the two boundaries at small frequencies is due to the reduced effect of spike downstroke in the canonical model owing to the comparatively smaller slope of it near early and late phases. As the spike threshold increases, the combined contribution from the depolarizing segments and the spike upstroke becomes comparable in magnitude to that of the spike downstroke, and the difference between the two boundaries is less disparate.

4. Effect of PRC skewness

Working under the purview of type-1 PRCs and continuing with the simple two PWL form for the PRC studied in the previous section, we can introduce skewness to the shape of the PRC by altering the lengths of the segments $Z_1(t)$ and $Z_2(t)$, and then study its influence on the phase-locked states. We formulate and study such a shape in this section. We provide the interaction function in the case of zero spike width only, but present the stability diagram computed analytically in the plane of PRC skewness and the spike width; The results are summarized in Fig. 3. We define the new PRC shape by

$$Z(t) = \begin{cases} Z_1 = \frac{2}{T+A}t, & 0 \leq t < \frac{T}{2} + \frac{A}{2}, \\ Z_2 = \frac{2}{T-A}(T-t) & \frac{T}{2} + \frac{A}{2} \leq t < T, \end{cases} \quad (14)$$

which is a generalized version of Eq. 1. T is as defined in Section 2. A is the shape factor which we term 'skewness'. The PRC is illustrated for a few discrete values of skewness in Fig. 3(a). By sweeping the parameter A from $-T$ to T we would have qualitatively swept the spectrum of most common type-1 PRCs in a qualitative manner. For $A = T$, the shape represents PRCs with large skewness such as those of leaky integrate-and-fire neuron models [40]. A positive skewness is common in many neuronal models [28, 41, 42] and real spiking neurons [43, 22, 12, 44]. But the PRCs in experiments and models can also exhibit different skewness levels and shapes that may be beyond the piecewise linear formalism. The model of the voltage time course considered here is the same as the three piecewise linear functions introduced in Section 2.

4.1. Zero spike width

When $W=0$, the voltage time course is given by $V_2(t)$ in the entire duration from 0 to T . Using the formula in Eq. 5, we can compute the interaction function in two steps: when

$0 \leq \phi < \frac{(T-A)}{2}$, and when $\frac{(T-A)}{2} \leq \phi < T$. Simplifying the corresponding integrals, we get very simple analytical expressions when the skewness is between $-T$ and T (excluding the boundaries):

$$H(\phi) = \begin{cases} \frac{a_3 T^2}{2(T-A)} \left(1 - \frac{A}{T} - 2\frac{\phi}{T}\right) \frac{\phi}{T}, & 0 \leq \phi < \frac{T-A}{2}, \\ \frac{a_3 T^2}{2(T+A)} \left(1 - \frac{A}{T} - 2\frac{\phi}{T}\right) \left(1 - \frac{\phi}{T}\right), & \frac{T-A}{2} \leq \phi < T. \end{cases} \quad (15)$$

This expression is plotted in Fig. 3(b, curves ii, iii, iv) for negative, zero, and positive skewness levels. If we set $A=0$ we recover the interaction function in Eq. 9. The above expressions also give interaction functions at $A/T=-1$ and $A/T=1$. From the first

expression we obtain $H(\phi) = \frac{a_3 T}{2} \left(1 - \frac{\phi}{T}\right) \frac{\phi}{T}$ when $A/T=-1$, and from the second expression

we get $H(\phi) = \frac{a_3 T}{2} \left(\frac{\phi}{T} - 1\right) \frac{\phi}{T}$ when $A/T=1$. These two expressions are completely even functions, and are plotted in Fig. 3(b, curves i and v). Because the PRC is zero at the edges, the discontinuity in V does not contribute any extra terms. Thus as the PRC begins to tilt its peak value toward longer phases (A/T approaching 1), or tilt toward smaller phases (A/T approaching -1), the level of oddness in the interaction function decreases, and its evenness increases such that in the limiting case of $A/T \rightarrow 1$ and $A/T \rightarrow -1$, $H(\phi)$ becomes completely even. This transition from odd to even function is confirmed by a numerical computation (not shown) of the Fourier coefficients of the odd (sine) and even (cosine) terms.

Except perhaps when $A/T=-1$ and 1, when $W/T=0$ it is clear from observing Figs. 2(a,b) that varying skewness will not alter the signs of the segments from what they are at $A/T=0$, and specifically the signs of the terms contributing to Eq. 7 or Eq. 8 are unchanged with skewness. In other words, skewness cannot change the stability of synchrony or antisynchrony at zero spike width. In particular the two segments of the PRC contribute respectively the eigenvalues, $-a_3(T+A)/(2T^2)$ and $-a_3(T-A)/(2T^2)$ toward synchrony. Thus λ becomes $-a_3/T$ and is thus negative resulting in stable synchrony. For the antisynchrony the stability is computed in two cases: $A/T=0$, and $A/T>0$. Owing to the fact that the voltage is a single linear segment, in either case γ becomes $a_3(T-A)/[T(T+A)]$ which is positive leading to an unstable antisynchrony. The growth functions plotted in Fig. 3(c) confirm this conclusion. However, when the PRC has jumps at the edges, as in the quadratic integrate-and-fire model [19], the stability is affected by the skewness. In the limit of $A/T \rightarrow -1$ and $A/T \rightarrow 1$, the antisynchronous state continues to be described by the growth function derived from $H(\phi)$ as depicted in Fig. 3(c, curves i and v) indicating that it is neutrally stable. But the stability of synchrony poses a special problem because in the limit of zero spike width $V(t)$ becomes discontinuous at the time when $Z(t)$ is non-zero. Including the contributions from these edges, we find that the eigenvalue for synchronous

state at $A/T=-1$ is $\frac{1}{T}(2a_2 - a_3) > 0$, making it unstable, and that at $A/T=1$ is $\frac{-1}{T}(2a_2 - a_3) < 0$ making it stable.

4.2. Non-zero spike width

From Section 3.2, we know that the interaction function quickly becomes distorted for finite spike width even in the absence of PRC skewness. Skewness imparts more complexity in the

functional forms of $H(\phi)$. For W that is significantly smaller than the spike period, the profiles of $H(\phi)$ at various skewness levels are similar to those computed at $W = 0$. At $W/T = 0$, $H(\phi)$ becomes zero at $\phi = 0.5$ because when the voltage profiles are lagged with that phase (0.5), the leading segment of the voltage has exactly the opposite effect as the lagging segment of the voltage. But for finite W/T , the difference in the voltage is magnified when the PRC is maximal, and is diminished near the ends of the PRC. Thus when the second neuron leads by half period placing its spike near $t = T/2$, the interaction function becomes non-zero thus shifting the zero crossing of $H(\phi)$ to longer phases. Thus the $H(\phi)$ for symmetric PRC [Fig. 3(b, curve iii)] is not completely sinusoidal, but the first mode of the sine component still dominates. As the skewness is increased toward 1, the sine coefficient decreases and the cosine coefficient increases as in the case of $W = 0$, but the overall relationship between the skewness and the functional form of $H(\phi)$ is not altered significantly. But as the spike width is further increased, we can expect larger deviations from sinusoidal even in the absence of skewness, and it is also possible to encounter predominantly even interaction functions.

Stability of the synchrony and antisynchrony are determined, respectively, by Eqs. 7 and 8. We saw earlier [see Fig. 1(e)] that when PRC is symmetric, large enough spike width can destabilize synchrony. The principal reason for such instability is the spike downstroke regime that has the destabilizing effect increasingly becoming bigger with spike width. Another way of increasing the effect of the spike downstroke is to let the PRC assume higher levels in this regime. This can be achieved by tilting the PRC toward left (i.e. for $A < 0$). Thus decreasing A could destabilize the synchronous state more easily. This is seen in the example illustrated in Fig. 3(d). On the other hand the converse effect can be expected if the level of PRC is decreased near small phases; the stabilizing effect of the rest of the voltage segments dominates. Thus increasing A to positive values enhances synchrony [Fig. 3(d)]. A numerical one-parameter bifurcation diagram as a function of skewness [Fig. 3(e)] confirms these observations.

We also saw earlier [see Fig. 1(e)] that the antisynchronous state becomes stable with increasing spike width or frequency because of the spike downstroke having diminished effect at $t = T/2$ [Fig. 2(b)]. If the PRC is tilted toward left, then the neuron is affected less during spike downstroke than during the upstroke because of the PRC segment having a negative slope during the spike. Thus the stabilizing effect of the spike upstroke dominates resulting in a more negative eigenvalue. Thus we can expect stable antisynchrony for negative A [Fig. 3(d)]. The converse may happen for positive A because now the PRC segment has a positive slope causing the spike downstroke playing a weaker role. These observations are confirmed by the one-parameter bifurcation diagram in Fig. 3(e). Boundaries of synchrony and antisynchrony in the parameter space of skewness and spike width and frequency are computed analytically and are presented in Fig. 3(f) for $a_3/a_2 = 0.2234$ and in Fig. 3(g) for $a_3/a_2 = 0$. The symmetry of the PRC in the absence of skewness resulted in a single boundary between synchrony and antisynchrony. Such a simplicity is lost with finite skewness. But not very tall spikes, the critical curves still remain in close proximity for very small skewness, and differ substantially at large skewness and frequency. The method of finding the critical curves is the same as that presented in the previous section. For synchronous state, the curve α_1 is obtained by considering the case $4W - T - A < T - W$, α_2 from the case $A < 4W - T$. And for the antisynchronous state, the curve β_1 is obtained by considering the case $A < -W$, β_2 from the case $-W - A < 0$, β_3 is obtained by considering the case $0 - A < 4W$ and $W < T/4$, and finally β_4 from the case $0 - A$ and $W > T/4$. The curves are enlisted in the Appendix. At low frequencies and/or when the spike width is very small, we expect predominantly synchronous state, and as the frequency or spike width increases, the antisynchronous state becomes stable such that at very large W/T the antisynchronous state is stable for all levels of skewness. As the spike becomes taller

(i.e. as a_3/a_2 becomes smaller and approaches zero), the stable synchrony is given by $A/T > 3/5$, whereas the antisynchronous state becomes stable for all negative skewness as well as the parameter region that falls below and right of the curves β_3 and β_4 .

If the PRC model displayed a subdued response at early phases, the effect of the downstroke and thus the positive eigenvalue component of λ would get reduced. Consequently synchrony might become unstable at even higher levels of frequency or spike width. However the effect on γ would be less substantial [see Fig. 2(b)] because of the fact that the segment of the PRC at early phases gets convolved with the slope of the depolarizing phase of $V(t)$ that has far less magnitude than that of the spike downstroke. Thus the reduction in the negative eigenvalue component of γ would be small. However, if the PRC response at early phases is small and the spike is very thin, then anti synchrony could become stable at large skewness in a small range of the spike width.

5. Discussion and conclusions

Symmetric type-1 PRC

We have introduced a piecewise linear representation for typical type-1 phase response curves, and another such representation for the voltage time course based on the HH model, and computed the interaction function and the stability regions [Fig. 2(f)] for synchrony and antisynchrony. Synchrony is stable (and antisynchrony is unstable) for small spike width and/or frequency. As the spike width becomes long or when the frequency becomes large, synchrony loses stability giving rise to stable antisynchrony. The converse takes place if the ratio of the spike threshold and the spike peak is increased at any combination of spike width and frequency. When the spikes have sufficiently small thresholds, it is the antisynchrony that becomes stable. Increasing the spike threshold to more depolarized levels makes synchrony stable and antisynchrony unstable. The stability curves are summarized in Eqs. 11, 12, and 13. Bistability between synchrony and antisynchrony does not occur in this model owing to the shape symmetry of the PRC. Qualitatively similar regions of stability for synchronous and antisynchronous states are obtained when the above piecewise linear PRC is replaced by the canonical type-1 PRC as displayed in Fig. 2(f), but the non-zero phase-locked branches found in the PWL model are absent in the canonical PRC model. The stability of the phase-locked solutions in the weak coupling limit under study does not directly depend on the voltage levels, but only depends on the time derivative of the voltage segments, and thus the specific values of V_p , V_m , and V_{th} do not directly affect the stability boundaries as is also evident from the expressions for ρ_1 and ρ_2 .

PRC with skewness

We have also examined the role of PRC skewness on the phase-locked states, and found [Fig. 3(f)] that at not very high levels of W/T , synchrony is usually favored as skewness is increased, and antisynchrony is favored at W/T is increased. Thus skewness countered the frequency and spike width. As the ratio a_3/a_2 (i.e. when the height of the depolarizing voltage segment) becomes zero, the boundary of synchrony α_1 extends from $W/T = 0$ to 0.4 and the stability region is given by $\alpha_1 \equiv A/T > 0.6$. The stable antisynchronous state persists for all $A/T < 0$ and its boundary is given by β_3 (that now extends left to the point $(W/T, A/T) = (0, 0)$).

Relation to previous works

Unlike the behavior of neurons coupled with mutual synaptic excitation [45, 27, 25, 46], electrically coupled neurons promote synchrony [12, 47, 48]. A number of theoretical works addressed the role of electrical coupling in synchronization mechanism. Sherman and Rinzel [9] reported an interesting range of patterns that emerge between neurons from weak

coupling mechanism. Cymbalyuk et al. [15] studied the effect of an external polarizing current drive on a pair of nonlinear two dimensional model neurons. They predicted an unstable synchrony at small current drives, but synchrony becomes stable for intermediate to large values of the current. They also predicted the existence of other non-zero phase-locked states. Alvarez et al. [49] investigated the role of frequency and coupling strength on synchrony in locus ceruleus neurons both experimentally and computationally using a conductance based model. Nomura et al. [50] studied fast-spiking interneuron model and showed that synchrony can be stable for a range of oscillation frequencies. Effect of noise on electrically coupled neurons was investigated by Ostojic et al. [51].

However, an exhaustive investigation of the underlying mechanisms that lead the electrically coupled neurons to synchrony was made by Lewis and Rinzel [18] and Chow and Kopell [17] using leaky integrate-and-fire (LIF) models after appropriate modifications to include spike effects. In contrast to the linear PRC profiles considered here, the integrate-and-fire neuron has an exponential PRC that is highly skewed and exhibits discontinuous jumps at the ends. The voltage profile of the LIF model considered by Lewis and Rinzel [18] is equivalent to the voltage time course in the current study [see Fig. 2(a)] with the widths of the spike up and downstrokes set to zero; With increasing frequency their spike width did not increase and hence the destabilizing effect of the spike downstroke (the shaded region) on synchrony remained low (proportional to the small PRC jump at zero phase). Hence, unlike in Fig. 2(f), synchrony continued to be stable in their model at all frequencies. Also the antisynchrony was found to be stable at low frequencies in the LIF model. This is because of the fact that the PRC is highly skewed (hence large positive at later phases) in the LIF model and thus the stabilizing contribution of the depolarizing phase of the voltage time course [to the right of the spike downstroke in Fig. 2(b)] is much larger than the destabilizing contribution of the zero spike-width downstroke (i.e. the discontinuous jump proportional to the total depolarizing level). The PRC level of the LIF is proportional to the frequency of oscillation, and hence at high frequencies its level (at the discontinuity) increases resulting in larger contribution of the discontinuity that destabilizes antisynchrony.

Chow and Kopell's [17] study covered both weak and strong coupling regimes of a pair of coupled LIF neurons. In the weak coupling regime (their Fig. 4), the synchronous state is stable at low frequencies, and unstable at high frequencies. This is consistent with the results of the current study. As a consequence of their requirement that the assumed spike profile be consistent with the evolutionary dynamics of the LIF model, the antisynchronous state does not exist at very low frequencies, but when it starts existing it becomes stable and then loses stability as the frequency is gradually increased. This stability transition is similar to the Lewis and Rinzel study. Chow and Kopell also found a patch of stable antisynchrony at very high frequencies.

In contrast to both these formulations, the voltage time course in the current study has both spike upstroke as well as spike downstroke, and their widths are related as in a Hodgkin-Huxley model voltage time course. Increasing the spike width not only increased the spike upstroke width (as in Chow and Kopell) but also increased the spike downstroke width. Increasing the spike downstroke width decreases its effectiveness in destabilizing the antisynchrony, and hence the antisynchrony acquires stability. In the current formulation the issue of consistency between the shapes and the evolutionary dynamics does not arise because we did not base the stability on the evolutionary dynamics. Our results show that spike width and frequency have identical role in the stability of both synchrony and antisynchrony.

The QIF's PRC was parametrized by Pfeuty et al. [19] by altering the spike threshold and reset that resulted in different PRC shapes. Except for a special choice of parameters, those

shapes acquire finite discontinuities at the edges of the PRCs; The present results can be related to the case when there are no PRC discontinuities. For the reset voltage level that is equal in magnitude but opposite in sign to the threshold voltage level and at low frequencies, the PRC shape becomes identical to that of the canonical PRC with no discontinuities. The model predicts a stable synchrony [13] and our results [Fig. 2(f)] are consistent with it. However the QIF model does not have a full spike profile, but the spike upstroke effect can be emulated to a limited extent by the sharp gradient of the voltage at the end of its oscillation cycle. Inclusion of a spike downstroke could potentially destabilize synchrony. Our results could be seen as extending the QIF model (without PRC discontinuities) when finite spike profile is added. Pfeuty et al. focused on the stability of the antisynchronous state. They showed that the antisynchrony was unstable at all frequencies. Again because of the finite spike width in our model, we found that the antisynchrony is only unstable at low W/T [Fig. 2(f), 3(f)], but at large W/T it could acquire stability. The QIF model produces more PRC shapes than the two PWL model presented here. The effect of skewness and the finite PRC jumps at the edges could together be contributing to the change of stability of both synchrony and antisynchrony [19, 13].

Pairs of electrically coupled fast-spiking and low-threshold spiking neocortical layer IV interneurons were studied by Mancilla et al. [12] to investigate phase-locking among them. The PRCs of fast-spikers, for example, are of type-1 in nature like in the current study, but are additionally characterized by relatively small response at early phases; Synchrony was observed at all the recorded frequencies, and there was no evidence of antisynchrony. The PRCs in the current study do not really possess the property of small amplitude at early phases, but we may expect the results to be applicable to such studies because of the general relations we have laid out between PRC skewness, spike width and the phase-locked states. The recordings of Mancilla et al. clearly showed positive skewness (A/T above 0.2) and the product of spike width and the frequency can be estimated to be no more than 0.05. This parameter regime can be spotted in Figs. 3(f,g) to be clearly far from antisynchrony, in consistency with the experimental observations.

Emergence of antisynchrony

How prevalent is an antisynchronous state (see [52, 53]) between electrically coupled neurons? Many theoretical and computational works predicted stable antisynchronous state in a network of two coupled neurons using a variety of models including excitable neuron models [9, 15], two dimensional neural relaxation oscillator models [54], leaky integrate-and-fire models [17, 18], quadratic integrate-and-fire models [19], and stomatogastric nervous system models [11], but very few experimental results demonstrated it in the brain neurons. As theorized by Sherman and Rinzel [9] phase-shifted voltage time courses could inhibit each other because of the voltage differences that could have opposite signs during the cycle. In the current study we could further see that while the spike downstroke contributes to destabilizing, all other segments of the voltage help stabilizing antisynchronous state [Fig. 2(a,b)]. But the present study asserts that although large spike width and/or frequency could make the antisynchrony stable, large PRC skewness [Fig. 3(f,g)] has the opposite effect by making it really hard to achieve it. PRCs of most neurons indeed exhibit a positive skewness.

Non-zero phase-locked states

The transition between synchrony and anti-synchrony in our model is mediated by other non-zero phase-locked states [Figs. 2(e), 3(e)] even if the boundary between those two states is defined by a single curve [Fig. 2(f)]. These states emerge in a pitchfork bifurcation occurring either at the in-phase or anti-phase states. For finite skewness there in fact is a region where only non-zero phase-locked states are stable [Fig. 3(f), white region].

Expanding the Eq. 6, we clearly see that synchrony ($\psi = 0$) and anti-synchrony ($\psi = T/2$) always lead to zeros of $G(\psi)$, but the prediction of other phase-locked states is not easy. Because the interaction function is highly nonlinear [Fig. 1(d)] a number of Fourier modes may need to be specified in order to accurately represent it. And thus it becomes extremely difficult to predict the existence and magnitude of non-zero phase-locked states. Such states are also found in other neuronal models [15, 17, 18].

Projections to more realistic PRC shapes

The applicability of the model results are only limited by the shape of the voltage time course and the assumptions of two-PWL PRC. We modeled the voltage using the classic spiking HH model. Our aim was to minimize the number of free parameters in defining it. Only one time parameter, the spike width, is introduced, and the rest define the spike height, hyperpolarization and threshold. The rise time is considered to be $W/2$ and the downstroke time $2W$. When fitting such a model to any other model or experimental system, it is easy to fit the downstroke, but the upstroke time may differ from our assumption. But notice from Figs. 2(a,b) that the upstroke contributes to stabilizing both synchronous and antisynchronous states. Altering the width of the upstroke does not diminish the effect of the downstroke at all, but either adds to or removes some of the stabilizing effect from the neighboring segment. Thus we may expect little qualitative changes to the stability switches shown in Fig. 2(f) when the width of the upstroke is altered.

Many realistic PRCs may be expected to have positive skewness, i.e. $A > 0$. But quantitative changes in the stability structures may be expected if the shape of the PRC is altered such that its level is either zero or very small near early phases. The shape of the PRC we considered was aimed at understanding the role of typical type-1 PRCs and the skewness in such PRCs. In both Chow and Kopell and Lewis and Rinzel's works the PRC is exponential due to the underlying leaky integrate-and-fire model, and thus the contribution of the PRC at early phases in comparison with that at later phases is significantly smaller than a similar contribution in our linear formulation. The effect of the downstroke and thus the positive eigenvalue component that it imparts to λ will get reduced by such a profile. Consequently synchrony may become unstable at even higher levels of frequency or spike width. However the effect on γ would depend sensitively on the level of skewness. This is because of the fact that at large but not very large skewness the segment of the PRC at early phases gets convolved with the slope of the depolarizing phase of $V(t + T/2)$ that has far less magnitude than that of the spike downstroke. And at very large skewness the spike effect may be less substantial due to very small PRC level during the upstroke and downstroke of $V(t + T/2)$, leading to a stable antisynchrony at low frequencies, like in integrate-and-fire model [13]. A thorough investigation of a more realistic PRC shapes is needed to understand the effect of PRC skewness comprehensively.

Carrying the results of two oscillator network over to a large network is challenging. If all-to-all connections are assumed, a large network of identical oscillators could also show synchrony in the regime where a two-oscillator network showed. But slight amount of heterogeneity among frequencies, coupling non-uniformity, PRC characteristics at the edges along with its skewness would all play important role in determining the collective state.

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Appendix 1. Interaction function

From Eq. 5 we see that in order to obtain $H(\phi)$ we must carry out the integral on the right hand side over time. This obviously results in multiple sub-integrals due to the piecewise nature of $V(t)$. These sub-integrals are obtained by first choosing a value for ϕ , illustrating $V(t + \phi)$, $V(t)$ and $Z(t)$ along a single time axis, and then determining the points along the time axis at which there is a single change of slope in the piecewise linear segments. As explained in Section 3.2, it is easy to see that there will be six such points between 0 and 1. These six points along with the end points of 0 and T form the limits $(L_1, \dots, L_6)$ of the sub-integrals. Once these limits are identified, we can directly read off the voltage and PRC segments that form the corresponding integrands: $I_1, \dots, I_7$. Thus the interaction function discussed in Section 3.2 is computed using the formula

$$H(\phi) = \frac{1}{T} \int_0^{L_1} I_1 dt + \frac{1}{T} \int_{L_1}^{L_2} I_2 dt + \frac{1}{T} \int_{L_2}^{L_3} I_3 dt + \frac{1}{T} \int_{L_3}^{L_4} I_4 dt + \frac{1}{T} \int_{L_4}^{L_5} I_5 dt + \frac{1}{T} \int_{L_5}^{L_6} I_6 dt + \frac{1}{T} \int_{L_6}^T I_7 dt, \quad (16)$$

Note however that ϕ is nothing but time itself. To compute $H(\phi)$ for ϕ between 0 and T , the above integral must be computed by continually re-illustrating $V(t + \phi)$ from $\phi = 0$ to T . This leads to ten or some times nine consecutive segments along ϕ that are separated by the boundaries shown in Fig. 1(c). We obtain these boundaries by simply examining the temporal relationships between the consecutive voltage and PRC segments. The limits of the integrals and the integrands undergo systematic changes as ϕ is traversed from 0 to T . For example, when $0 < W < T/9$ [the range of thick lines in Fig. 1(c)], the interaction function is given by piecewise combination of polynomials obtained in the ten consecutive segments. As W is increased, some consecutive segments may have their boundaries swapped or altered. The first two such alterations occur at $W = T/8$ and $T/6$. In the following we compute $H(\phi)$ for W up to $T/6$ by computing it in three ranges: Case (i) $0 < W < T/9$, Case (ii) $T/9 < W < T/8$, and Case (ii) $T/8 < W < T/6$.

Case (i) Consider the range $0 < W < T/9$. In the first segment $0 < \phi < W/2$. By following the procedure laid out above, we can write the integrands as

$I_1=\tilde{Z}_1^{11}$, $I_2=\tilde{Z}_1^{21}$, $I_3=\tilde{Z}_1^{22}$, $I_4=\tilde{Z}_2^{22}$, $I_5=\tilde{Z}_2^{32}$, and $I_6=\tilde{Z}_2^{33}$. The functional form of $\tilde{Z}_n^{ab}$ is defined in Section 2. The limits of the integrals for this case are listed in Section 3.2. In the second segment, $W/2 - \phi < 2W$, and the seven integrands are given by $\tilde{Z}_1^{11}$, $\tilde{Z}_1^{21}$, $\tilde{Z}_1^{22}$, $\tilde{Z}_2^{22}$, $\tilde{Z}_2^{32}$, $\tilde{Z}_2^{12}$, and $\tilde{Z}_2^{13}$, and the limits L_1 to L_6 are $2W - \phi$, $2W$, $T/2$, $T - \phi - W/2$, $T - \phi$, $T - W/2$. In the third segment, $2W - \phi < 5W/2$, and the seven integrands are $\tilde{Z}_1^{21}$, $\tilde{Z}_1^{22}$, $\tilde{Z}_2^{22}$, $\tilde{Z}_2^{32}$, $\tilde{Z}_2^{12}$, $\tilde{Z}_2^{13}$, and $\tilde{Z}_2^{23}$, and the limits L_1 to L_6 are $2W$, $T/2$, $T - \phi - W/2$, $T - \phi$, $T - W/2$, and $T - \phi + 2W$. In the fourth segment, $5W/2 - \phi < (T - W)/2$, and the seven integrands are $\tilde{Z}_1^{21}$, $\tilde{Z}_1^{22}$, $\tilde{Z}_2^{22}$, $\tilde{Z}_2^{32}$, $\tilde{Z}_2^{12}$, $\tilde{Z}_2^{23}$, and $\tilde{Z}_2^{33}$, and the limits L_1 to L_6 are $2W$, $T/2$, $T - \phi - W/2$, $T - \phi$, $T - \phi + 2W$, and $T - W/2$. In the fifth segment, $(T - W)/2 - \phi < T/2$, and the seven integrands are $\tilde{Z}_1^{21}$, $\tilde{Z}_1^{22}$, $\tilde{Z}_2^{32}$, $\tilde{Z}_2^{33}$, $\tilde{Z}_2^{12}$, $\tilde{Z}_2^{22}$, and $\tilde{Z}_2^{23}$, and the limits L_1 to L_6 are $2W$, $T - \phi - W/2$, $T/2$, $T - \phi$, $T - \phi + 2W$, and $T - W/2$. In the sixth segment, $T/2 - \phi < T/2 + 2W$, and the seven integrands are $\tilde{Z}_1^{21}$, $\tilde{Z}_1^{22}$, $\tilde{Z}_2^{32}$, $\tilde{Z}_2^{12}$, $\tilde{Z}_2^{13}$, $\tilde{Z}_2^{22}$, and $\tilde{Z}_2^{23}$, and the limits L_1 to L_6 are $2W$, $T - \phi - W/2$, $T - \phi$, $T/2$, $T - \phi + 2W$, and $T - W/2$. In the seventh segment, $T/2 + 2W - \phi < T - 5W/2$, and the seven integrands are $\tilde{Z}_1^{21}$, $\tilde{Z}_1^{22}$, $\tilde{Z}_2^{32}$, $\tilde{Z}_2^{12}$, $\tilde{Z}_2^{13}$, $\tilde{Z}_2^{22}$, and $\tilde{Z}_2^{23}$, and the limits L_1 to L_6 are $2W$, $T - \phi - W/2$, $T - \phi$, $T - \phi + 2W$, $T/2$, and $T - W/2$. In the eighth segment, $T - 5W/2 - \phi < T - 2W$, and the seven integrands are $\tilde{Z}_1^{21}$, $\tilde{Z}_1^{31}$, $\tilde{Z}_2^{32}$, $\tilde{Z}_2^{12}$, $\tilde{Z}_2^{13}$, $\tilde{Z}_2^{22}$, and $\tilde{Z}_2^{23}$, and the limits L_1 to L_6 are $T - \phi - W/2$, $2W$, $T - \phi$, $T - \phi + 2W$, $T/2$, and $T - W/2$. In the ninth segment, $T - 2W - \phi < T - W/2$, and the seven integrands are $\tilde{Z}_1^{21}$, $\tilde{Z}_1^{31}$, $\tilde{Z}_2^{11}$, $\tilde{Z}_2^{12}$, $\tilde{Z}_2^{13}$, $\tilde{Z}_2^{22}$, and $\tilde{Z}_2^{23}$, and the limits L_1 to L_6 are $T - \phi - W/2$, $T - \phi$, $2W$, $T - \phi + 2W$, $T/2$, and $T - W/2$. And finally in the tenth segment, $T - W/2 - \phi < T$, and the seven integrands are $\tilde{Z}_1^{31}$, $\tilde{Z}_1^{11}$, $\tilde{Z}_2^{12}$, $\tilde{Z}_2^{22}$, $\tilde{Z}_2^{23}$, and $\tilde{Z}_2^{33}$, and the limits L_1 to L_6 are $T - \phi$, $2W$, $T - \phi - 2W$, $T/2$, $T - W/2$, and $2T - \phi - W/2$. Using these integrands and limits in Eq. 16, we obtain the interaction function as a set of ten polynomial segments:

$$H(\phi) = \begin{cases} h_1(\phi) = T f_{1a} + f_{1b}\phi + \frac{1}{T} f_{1c}\phi^2 + \frac{1}{T^2} f_{1d}\phi^3, & 0 \leq \phi < W/2, \\ h_2(\phi) = T f_{2a} + f_{2b}\phi + \frac{1}{T} f_{2c}\phi^2 + \frac{1}{T^2} f_{2d}\phi^3, & W/2 \leq \phi < 2W, \\ h_3(\phi) = T f_{3a} + f_{3b}\phi + \frac{1}{T} f_{3c}\phi^2, & 2W \leq \phi < 5W/2, \\ h_4(\phi) = T f_{4a} + f_{4b}\phi + \frac{1}{T} f_{4c}\phi^2, & 5W/2 \leq \phi < (T - W)/2, \\ h_5(\phi) = T f_{5a} + f_{5b}\phi + \frac{1}{T} f_{5c}\phi^2 + \frac{1}{T^2} f_{5d}\phi^3, & (T - W)/2 \leq \phi < T/2, \\ h_6(\phi) = T f_{6a} + f_{6b}\phi + \frac{1}{T} f_{6c}\phi^2 + \frac{1}{T^2} f_{6d}\phi^3, & T/2 \leq \phi < T/2 + 2W, \\ h_7(\phi) = T f_{7a} + f_{7b}\phi + \frac{1}{T} f_{7c}\phi^2, & T/2 + 2W \leq \phi < T - 5W/2, \\ h_8(\phi) = T f_{8a} + f_{8b}\phi + \frac{1}{T} f_{8c}\phi^2, & T - 5W/2 \leq \phi < T - 2W, \\ h_9(\phi) = T f_{9a} + f_{9b}\phi + \frac{1}{T} f_{9c}\phi^2, & T - 2W \leq \phi < T - W/2, \\ h_{10}(\phi) = T f_{10a} + f_{10b}\phi + \frac{1}{T} f_{10c}\phi^2 + \frac{1}{T^2} f_{10d}\phi^3, & T - W/2 \leq \phi < T. \end{cases} \quad (17)$$

The coefficients f_{1a} , ..., f_{10d} are dependent of W , a_2 , and a_3 , and are listed later in this section.

Case (ii) Consider the case $T/9 - W < T/8$. The interaction function will retain most of the terms that contributed to $H(\phi)$ in Case (i) above, but will have the boundaries of the 6th and 8th regimes altered [as can be seen from Fig. 1(c)], and that of the 7th redefined. In the seventh regime the altered integrands from the seventh regime defined in the previous case are $I_2=\tilde{Z}_1^{31}$ and $I_5=\tilde{Z}_2^{12}$, and the altered limits from those of the seventh regime defined in the previous case are $L_1 = T - \phi - W/2$, $L_2 = 2W$, $L_4 = T/2$, and $L_5 = T - \phi + 2W$. Computing the corresponding integral, we obtain the interaction function as

$$H_1(\phi) = \begin{cases} h_6(\phi), & T/2 \leq \phi < T - 5W/2, \\ j_7(\phi) = T j_{7a} + j_{7b}\phi + \frac{1}{T} j_{7c}\phi^2 + \frac{1}{T^2} j_{7d}\phi^3, & T - 5W/2 \leq \phi < T/2 + 2W, \\ h_8(\phi), & T/2 + 2W \leq \phi < T - 2W, \\ H(\phi), & \text{otherwise,} \end{cases} \quad (18)$$

where the coefficients $j_{7a}, \dots, j_{7d}$ are dependent of W and are listed later in this section.

Case (iii) Finally consider the case $T/8 < W < T/6$. The interaction function will retain most of the terms that contributed to $H(\phi)$ in Case (ii) above case, but will have the boundaries of the 6th, 7th, and 9th regimes altered [as can be seen from Fig. 1(c)], and that of the 8th redefined. In this eighth regime the altered integrands from that of the first case are $I_3 = \tilde{Z}_1^{11}$ and $I_5 = \tilde{Z}_2^{12}$, and the altered limits are $L_2 = T - \phi$, $L_3 = 2W$, $L_4 = T/2$, and $L_5 = T - \phi + 2W$. Evaluating the integrals with these integrands and limits, the following expressions for the interaction function are obtained.

$$H_2(\phi) = \begin{cases} h_6(\phi), & T/2 \leq \phi < T - 5W/2, \\ j_7(\phi), & T - 5W/2 \leq \phi < T - 2W, \\ j_8(\phi) = T j_{8a} + j_{8b}\phi + \frac{1}{T} j_{8c}\phi^2 + \frac{1}{T^2} j_{8d}\phi^3, & T - 2W \leq \phi < T/2 + 2W, \\ h_9(\phi), & T/2 + 2W \leq \phi < T - W/2, \\ H(\phi), & \text{otherwise,} \end{cases} \quad (19)$$

where the coefficients $j_{8a}, \dots, j_{8d}$ are dependent of W and are listed below.

Now the coefficients $f_{1a}, \dots, f_{10d}$ that appear in Eq. 17, $j_{7a}, \dots, j_{7d}$ that appear in Eq. 18, and $j_{8a}, \dots, j_{8d}$ that appear in Eq. 19 are listed below. We represent W/T by w for readability.

$$f_{1a} = 0, f_{1b} = (-2a_3 - 10a_2w + 6a_3w - 12V_mw + 16V_pw - 4V_{th}w + 25a_2w^2 + 2a_3w^2 + 30V_mw^2 - 40V_pw^2 + 10V_{th}w^2)/(2(-2 + 5w)), f_{1c} = (-2(-a_3 - 2V_m + 2V_p + 4a_3w + 5V_mw - 5V_pw))/(-2 + 5w), f_{1d} = (-(-2a_2 + 5a_2w - 4a_3w)/(3w(-2 + 5w))).$$

$$f_{2a} = 0, f_{2b} = (-2a_3 - 10a_2w + 6a_3w - 12V_mw + 16V_pw - 4V_{th}w + 25a_2w^2 + 2a_3w^2 + 30V_mw^2 - 40V_pw^2 + 10V_{th}w^2)/(2(-2 + 5w)), f_{2c} = (-2(-a_3 - 2V_m + 2V_p + 4a_3w + 5V_mw - 5V_pw))/(-2 + 5w), f_{2d} = (-(-2a_2 + 5a_2w - 4a_3w)/(3w(-2 + 5w))).$$

$$f_{3a} = (8w^2(-4a_2 - 6V_m + 6V_p + 10a_2w + 4a_3w + 15V_mw - 15V_pw))/(3(-2 + 5w)), f_{3b} = (-2a_3 - 6a_2w - 6a_3w - 20V_mw + 16V_pw + 4V_{th}w + 15a_2w^2 + 30a_3w^2 + 50V_mw^2 - 40V_pw^2 - 10V_{th}w^2)/(2(-2 + 5w)), f_{3c} = (2a_3)/(-2 + 5w).$$

$$f_{4a} = (8w^2(-4a_2 - 6V_m + 6V_p + 10a_2w + 4a_3w + 15V_mw - 15V_pw))/(3(-2 + 5w)), f_{4b} = (-2a_3 - 6a_2w - 6a_3w - 20V_mw + 16V_pw + 4V_{th}w + 15a_2w^2 + 30a_3w^2 + 50V_mw^2 - 40V_pw^2 - 10V_{th}w^2)/(2(-2 + 5w)), f_{4c} = (2a_3)/(-2 + 5w).$$

$$f_{5a} = (-2a_2 + 2a_3 + 11a_2w - 16a_3w - 6V_mw + 6V_{th}w - 21a_2w^2 + 45a_3w^2 + 27V_mw^2 - 27V_{th}w^2 - 47a_2w^3 - 50a_3w^3 - 132V_mw^3 + 96V_pw^3 + 36V_{th}w^3 + 155a_2w^4 + 83a_3w^4 + 255V_mw^4 - 240V_pw^4 - 15V_{th}w^4)/(6w(-2 + 5w)), f_{5b} = (-(-4a_2 + 4a_3 + 18a_2w - 22a_3w - 8V_mw + 8V_{th}w - 30a_2w^2 + 42a_3w^2 + 8V_mw^2 + 16V_pw^2 - 24V_{th}w^2 + 25a_2w^3 + 2a_3w^3 + 30V_mw^3 - 40V_pw^3 + 10V_{th}w^3)/(2w(-2 + 5w)), f_{5c} = (-2(2a_2 - 2a_3 - 7a_2w + 7a_3w + 2V_mw - 2V_{th}w + 5a_2w^2 - 9a_3w^2 - 5V_mw^2 + 5V_{th}w^2))/(w(-2 + 5w)), f_{5d} = (-4(-2a_2 + 2a_3 + 5a_2w - 4a_3w))/(3w(-2 + 5w)).$$

$$f_{6a} = (2a_2 - 5a_2w - 20a_3w - 24V_mw + 24V_pw - 24a_2w^2 + 120a_3w^2 + 108V_mw^2 - 60V_pw^2 - 48V_{th}w^2 - 188a_2w^3 - 200a_3w^3 - 528V_mw^3 + 384V_pw^3 + 144V_{th}w^3 + 620a_2w^4 +$$

$$332a_3w^4 + 1020V_mw^4 - 960V_pw^4 - 60V_{th}w^4)/(24w(-2+5w)), f_{6b} = -(2a_2 - 5a_2w - 8a_3w - 16V_mw + 16V_pw - 20a_2w^2 + 44a_3w^2 + 16V_mw^2 - 8V_pw^2 - 8V_{th}w^2 + 50a_2w^3 + 4a_3w^3 + 60V_mw^3 - 80V_pw^3 + 20V_{th}w^3)/(4w(-2+5w)), f_{6c} = (2a_2 - 5a_2w - 8V_mw + 8V_pw + 16a_3w^2 + 20V_mw^2 - 20V_pw^2)/(2w(-2+5w)), f_{6d} = (-2a_2 + 5a_2w - 4a_3w)/(3w(-2+5w)).$$

$$f_{7a} = -(6a_3 - 18a_2w - 18a_3w - 60V_mw + 48V_pw + 12V_{th}w + 43a_2w^2 + 98a_3w^2 + 156V_mw^2 - 120V_pw^2 - 36V_{th}w^2 + 5a_2w^3 - 19a_3w^3 - 15V_mw^3 + 15V_{th}w^3)/(6(-2+5w)), f_{7b} = (6a_3 - 6a_2w - 6a_3w - 20V_mw + 16V_pw + 4V_{th}w + 15a_2w^2 + 30a_3w^2 + 50V_mw^2 - 40V_pw^2 - 10V_{th}w^2)/(2(-2+5w)), f_{7c} = (-2a_3)/(-2+5w).$$

$$f_{8a} = -(6a_3 - 18a_2w - 18a_3w - 60V_mw + 48V_pw + 12V_{th}w + 43a_2w^2 + 98a_3w^2 + 156V_mw^2 - 120V_pw^2 - 36V_{th}w^2 + 5a_2w^3 - 19a_3w^3 - 15V_mw^3 + 15V_{th}w^3)/(6(-2+5w)), f_{8b} = (6a_3 - 6a_2w - 6a_3w - 20V_mw + 16V_pw + 4V_{th}w + 15a_2w^2 + 30a_3w^2 + 50V_mw^2 - 40V_pw^2 - 10V_{th}w^2)/(2(-2+5w)), f_{8c} = (-2a_3)/(-2+5w).$$

$$f_{9a} = -(6a_3 - 18a_2w - 18a_3w - 60V_mw + 48V_pw + 12V_{th}w + 43a_2w^2 + 98a_3w^2 + 156V_mw^2 - 120V_pw^2 - 36V_{th}w^2 + 5a_2w^3 - 19a_3w^3 - 15V_mw^3 + 15V_{th}w^3)/(6(-2+5w)), f_{9b} = (6a_3 - 6a_2w - 6a_3w - 20V_mw + 16V_pw + 4V_{th}w + 15a_2w^2 + 30a_3w^2 + 50V_mw^2 - 40V_pw^2 - 10V_{th}w^2)/(2(-2+5w)), f_{9c} = (-2a_3)/(-2+5w).$$

$$f_{10a} = -(-16a_2 + 16a_3 + 64a_2w - 74a_3w - 24V_mw + 24V_{th}w - 90a_2w^2 + 126a_3w^2 + 24V_mw^2 + 48V_pw^2 - 72V_{th}w^2 + 75a_2w^3 + 6a_3w^3 + 90V_mw^3 - 120V_pw^3 + 30V_{th}w^3)/(6w(-2+5w)), f_{10b} = (-16a_2 + 16a_3 + 56a_2w - 58a_3w - 16V_mw + 16V_{th}w - 50a_2w^2 + 78a_3w^2 + 28V_mw^2 + 16V_pw^2 - 44V_{th}w^2 + 25a_2w^3 + 2a_3w^3 + 30V_mw^3 - 40V_pw^3 + 10V_{th}w^3)/(2w(-2+5w)), f_{10c} = (2(4a_2 - 4a_3 - 12a_2w + 11a_3w + 2V_mw - 2V_{th}w + 5a_2w^2 - 9a_3w^2 - 5V_mw^2 + 5V_{th}w^2))/(w(-2+5w)), f_{10d} = (4(-2a_2 + 2a_3 + 5a_2w - 4a_3w))/(3w(-2+5w)).$$

$$j_{7a} = (31a_2w^2)/6 + (a_2w)/2 - a_2/(24w) - (a_3((50-83w)w-30)w)/(6(5w-2)) - (5a_3)/(6(5w-2)) - 1/2w^2(-17V_m + 16V_p + V_{th}) - V_mw + V_m/2 - (V_p)/2 + V_{th}w, j_{7b} = -(5a_2w)/2 + a_2/(4w) - (a_3(w(w+11)-2))/(5w-2) + 2(V_p - V_m) - 3V_mw + 4V_pw - V_{th}w, j_{7c} = -a_2/(2w) + (8a_3w)/(5w-2) - 2(V_p - V_m), j_{7d} = a_2/(3w) - (4a_3)/(3(5w-2)).$$

$$j_{8a} = (31a_2w^2)/6 + (a_2w)/2 - a_2/(24w) - (a_3((50-83w)w-30)w)/(6(5w-2)) - (5a_3)/(6(5w-2)) - 1/2w^2(-17V_m + 16V_p + V_{th}) - V_mw + V_m/2 - (V_p)/2 + V_{th}w, j_{8b} = -(5a_2w)/2 + (a_2)/(4w) - (a_3(w(w+11)-2))/(5w-2) + 2(V_p - V_m) - 3V_mw + 4V_pw - V_{th}w, j_{8c} = -(a_2)/(2w) + (8a_3w)/(5w-2) - 2(V_p - V_m), j_{8d} = (a_2)/(3w) - (4a_3)/(3(5w-2)).$$

Appendix 2. Stability boundaries for skewed PRCs

In Section 4 the effect of PRC skewness is discussed, and the corresponding results are presented in Fig. 3. The eigenvalues for stability of synchrony and antisynchrony are computed as in Section 3 but using the PRC shape given by Eq. 14. The analysis for obtaining the critical curves that define the boundaries of synchrony (α_1 , α_2) and antisynchrony (β_1 , β_2 , and β_3) is identical to that presented in Section 3 but is slightly more involved. Here we present the critical curves [plotted in Figs. 3(f,g)] obtained by such analysis. The results are verified by actual computation of the eigenvalues in the parameter space as in Fig. 3(e). In the following expressions $a'_3 = a_3/a_2$, and $W' = W/T$. The regions of stable synchrony are specified by two conditions, and are given by

$$\frac{A}{T} > \left(\frac{A}{T}\right)^* \equiv \alpha_1 = \frac{y_1 - \sqrt{y_2 + y_3}}{4a'_3T^2}, \text{ if } 4W' - 1 \leq A/T < 1 - W'$$

where $y_1=2W'(5-a'_3)+5W'^2(4a'_3-5)$,
 $y_2=8a'_3[3W'(5W'-2)+2a'_3(1-3W')(1+2W')]$, $y_3=W'^2(10-25W'-2a'_3+20a'_3W')^2$, and

$$\frac{A}{T} > \left(\frac{A}{T}\right)^* \equiv \alpha_2 = 4a'_3(2W'-1)W' - (10W'^2 - 8W' + 1), \text{ if } -1 \leq A/T < 4W' - 1.$$

When a_3 itself is zero, only α_1 forms the stability boundary. The eigenvalue contributions from the depolarizing phase become zero, and the total eigenvalue is the sum of the contributions from the spike downstroke $[(-2/T) \int_0^{2W} Z_1 V'_1 dt]$ and the spike upstroke $[(-2/T) \int_{T-W/2}^T Z_2 V'_3 dt]$ that becomes $a_2 W' (3T - 5A)/(T^2 - A^2)$. This gives the boundary of synchrony as: $A/T > (A/T)^* \equiv \alpha_1 = 3/5$ [shown in Fig. 3(g)].

The regions of stable antisynchrony are specified by four conditions, and they are given by

$$\frac{A}{T} < \left(\frac{A}{T}\right)^* \equiv \beta_1 = \frac{5W'^2(4a'_3-5)}{2a'_3} + W' \left(\frac{5}{a'_3} - 1 \right) - 1, \text{ if } -1 \leq A/T < -W',$$

$$\frac{A}{T} < \left(\frac{A}{T}\right)^* \equiv \beta_2 = \frac{y_4 + [W'(2-5W')(y_5+y_6+y_7+y_8+y_9)]^{1/2}}{4(1-a'_3)(2-5W')}, \text{ if } -W' \leq A < 0,$$

where $y_4=W'(5W'-2)(4-5W'-2a'_3+4a'_3W')$, $y_5=-8W'(-10+8a'_3+a_3'^2)$,
 $y_6=-4W'^2(70-100a'_3+37a_3'^2)$, $y_7=16a'_3(a'_3-1)$, $y_8=2W'^3(25-14a'_3)(5-4a'_3)$,

$y_9=-5W'^4(5-4a'_3)^2$, $\frac{A}{T} < \left(\frac{A}{T}\right)^* \equiv \beta_3 = \frac{y_{10} - [W'(2-5W')(y_{11}+y_{12}+y_{13})]^{1/2}}{2-5W'}$, if $0 < A/T < 4W'$
 and $W' > 1/4$, where $y_{10}=W'(2-5W')(4-5W'-2a'_3+4a'_3W')$,
 $y_{11}=4a_3'^2W'(2-5W')(1-2W')^2$, $y_{12}=5W'(2-5W')(2-8W'+5W'^2)$, and
 $y_{13}=4a_3(1-9W'+40W'^2-85W'^3+50W'^4)$, and finally

$\frac{A}{T} < \left(\frac{A}{T}\right)^* \equiv \beta_4 = 8a'_3W'^2 - 4a'_3W' - 10W'^2 + 8W' - 1$, if $0 \leq A/T$ and $W' > 1/4$. As a_3 becomes smaller (i.e. as the spike becomes taller), the curve β_1 becomes steeper, and β_2 spans shorter range along W/T axis. At $a_3/a_2 = 0$, the curves β_3 and β_4 together define the upper boundary for antisynchrony, and are simplified to:

$A/T < (A/T)^* \equiv \beta_3 = (4-5W')W' - W' \sqrt{5} \sqrt{W'(5W'-8)+2}$, and $A/T < (A/T)^* \equiv \beta_4 = -10W'^2 + 8W' - 1$ [shown in Fig. 3(g)].

Figure 1.

(a) Profile of spike time course with finite width, and (b) the PRC shape considered. (c) Boundary curves of ϕ delineating the components that contribute to $H(\phi)$. Different regimes along W/T are delineated by intersections of these curves. The first regime $0 < W/T < 1/9$ is marked by thick curves, and the corresponding interaction function is in Eq. 17. Interaction functions up to $W/T = 1/6$ are computed in the Appendix. (d) Interaction function as the parameter W/T is increased (using formulas in Eqs. 17, 18, and 19). (e) The growth functions corresponding to the curves in (d). Synchrony is stable and antisynchrony is unstable at small spike width or frequency. As the spike width or frequency is increased, an exchange of stability takes places between these two states. The nature of this exchange is shown numerically in Fig. 2(e). Further increase of W/T produces no change in the stability of the synchronous and antisynchronous states. In this and later figures, when unspecified, the voltage shape parameters are of the HH model discussed in Section 2.

Figure 2.

The contributions of various segments of $V(t)$ and $Z(t)$ toward stability of synchrony (a) and antisynchrony (b) are visually represented. The arrangement of V and Z satisfies the condition $0 < W/T < 1/4$. The shaded regions contribute to destabilizing the corresponding phase-locked state. In the limit of W becoming zero, λ_{1a} and not γ_{3a} becomes zero, and hence synchrony is stable, and antisynchrony unstable. When W is large such that $1/4 < W/T < 2/5$, the corresponding eigenvalue components are represented by $\lambda_{1b}, \dots$, and $\gamma_{1b}, \dots$. (c,d) The eigenvalue components and the total eigenvalue for synchronous (c) and antisynchronous (d) states as W' is increased at $a_3/a_2 = 0.2234$, $T = 14.636$ (HH model parameters as discussed in Section 2). λ_{1a} is zero at $W' = 0$, but grows to become largest component with W' , but γ_{3a} is the dominant component for small W' , and is overpowered by the combined effect of other components at large W' . (e) A numerically found bifurcation diagram for the parameters of (c) and (d) as a function of W' confirming the prediction of stability switches of synchrony and antisynchrony as the product of spike width and frequency increases. Other stable phase-locked solutions - that are not studied here - also emerged near the transition point. (f) Stability diagram of synchronous and antisynchronous states computed analytically in Eqs. 11 and 12 and defined by ρ_1 and ρ_2 . Numerically computed boundary using the canonical type 1 PRC shape and the PWL voltage time course is shown by the dashed curve. Note that the ratio a_3/a_2 is related to the relative spike amplitude $1 - a_3/a_2$.

Figure 3.

Effect of PRC skewness on phase-locking. (a) PRCs with skewness $A/T = -1$ (i), -0.5 (ii), 0 (iii), 0.5 (iv), and 1 (v) are illustrated. (b) The interaction functions corresponding to the five PRCs illustrated in (a) in the absence of spike width ($W/T = 0$). (c) The growth functions for the curves in (b) indicating stable synchrony and unstable antisynchrony for A/T between -1 and 1 . (d) Effect of spike width ($W/T = 0.2$) is shown on the growth functions at different levels of skewness. The non-zero spike width caused the symmetric and negative skewed PRCs impart instability to synchrony, but large positive skewness countered it. (e) One-parameter bifurcation diagram as a function of skewness, showing stable antisynchrony for left leaning or nearly symmetric PRCs, and stable synchrony for right leaning PRCs. For

the value of spike width ($W/T = 0.2$) used, these states are replaced by non-zero phase-locked states for a small range of positive but intermediate skewness. (f,g) Stability diagrams in the plane of normalized skewness and normalized spike width depicting stable synchrony (shaded) and stable antisynchrony (hatched) regions at two spike heights. The white space holds non-zero phase-locked states. For d-f the spike amplitude parameters are chosen from time course of HH model with an applied current of $10 \mu\text{A}/\text{cm}^2$ that results in $a_3/a_2 = 0.2234$. However, only the slopes and not the actual values of the voltage segments affect the stability.